Math Circles

Problems for Complex numbers and Polynomials

Problem sheet 2 (20.06.2024)

1. Complex numbers and Polynomials

1.1. **Basic Definitions.** Let $\mathbb{R}$ denote the set of real numbers, and set $i := \sqrt{-1}$. Let

$$\mathbb{C} := \left\{ x + iy \middle| \, x, y \in \mathbb{R} \right\}$$

denote the set of complex numbers.

For any $z = x + iy \in \mathbb{C}$, x is called the real-part of z, and y is called the imaginary part of z. Let Re(z) and Im(z) denote the real and imaginary parts of z.

We can identify the set of complex numbers with the set of 2-tuples

$$\mathbb{R}^2 := \left\{ (x, y) \middle| \, x, y \in \mathbb{R} \right\},$$

via the following bijective map

$$f : \mathbb{C} \longrightarrow \mathbb{R}^2, \; x + iy \mapsto (x, y). \tag{1}$$

From the above bijection, we identify $\mathbb{C}$ with the 2-dimensional real plane, where the traditional X-axis represents the real part of a complex number, and the traditional Y-axis represents the imaginary part of complex number.

For example the point $z = 2 + i3 \in \mathbb{C}$ represents the coordinate $(2, 3)$ on the real plane $\mathbb{R}^2$.

Binary Operations. For $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, we define

$$\begin{aligned} &\textbf{Addition} : z_1 + z_2 := x_1 + x_2 + i(y_1 + y_2), \text{ i.e.,} \\ &\mathrm{Re}(z_1 + z_2) := x_1 + iy_1, \text{ and}\, \mathrm{Im}(z_1 + z_2) := y_1 + y_2; \end{aligned} \tag{2}$$

$$\begin{aligned} &\textbf{Multiplication} : z_1 \cdot z_2 := (x_1 + iy_1) \times (x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1), \\ &\text{i.e., } \mathrm{Re}(z_1 \cdot z_2) := x_1x_2 - y_1y_2, \text{ and}\, \mathrm{Im}(z_1 \cdot z_2) := x_1y_2 + x_2y_1. \end{aligned} \tag{3}$$

Absolute value and distance function. For $z = x + iy \in \mathbb{C}$, the absolute value of z is defined as

$$|z| := \sqrt{x^2 + y^2}.$$

From the identification of $\mathbb{C}$ with the real-plane $\mathbb{R}^2$ (as in (1)), $|z|$ represents is nothing but the distance of the point (x, y) from the origin $(0, 0)$.

From the above formula, the Euclidean distance between the points $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2 \in \mathbb{C}$ is given by the following formula

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{4}$$

Problems

(1) Find all the complex numbers z such that the coordinates represented by the complex numbers $z, z_1 := 2 + i$, $z_2 := 2 - i$ represent the vertices of an equilateral triangle on the real plane $\mathbb{R}^2$.

(2) Prove the triangular inequalities

$$|z_1 + z_2| \leq |z_1| + |z_2|;$$
$$\big||z_1| - |z_2|\big| \leq |z_1 - z_2|.$$

(3) Find all the complex numbers, which satisfy the equations:

$$|z - 1 + 2i| = |z + 1 - i|;$$
$$\big||z - 1| - |z + 1|\big| = 1.$$

1.2. **Euler's formula.** As discussed in the previous section, any $z = x + iy \in \mathbb{C}$ can be represented by a point (x, y) in the real plane $\mathbb{R}^2$. We now describe a famous formula from one of the great masters, Leonhard Euler, which will make the identification of $\mathbb{C}$ with the real-plane $\mathbb{R}^2$ (as in (1)), geometrically and intuitively concrete.

For any given $z \in \mathbb{C}$, define

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The above series is called a power series, and it is possible to show that the above series converges for all $z \in \mathbb{C}$ and is never zero.

For any $x \in \mathbb{R}$, Euler's formula is

$$e^{ix} = \cos(x) + i\sin(x). \tag{5}$$

We now describe a proof for equation (5). Consider the function

$$f(x) := \frac{\cos(x) + i\sin(x)}{e^{ix}}.$$

It is easy to see that

$$\frac{df(x)}{dx} = \frac{e^{ix}\big(-\sin(x) + i\cos(x)\big) - ie^{ix}\big(\cos(x) + i\sin(x)\big)}{e^{2ix}} =$$
$$\frac{-ie^{ix}\big(\cos(x) + i\sin(x)\big) + ie^{ix}\big(\cos(x) + i\sin(x)\big)}{e^{2ix}} = 0 \implies f \text{ is a constant function on } \mathbb{R}.$$

Furthermore

$$f(0) = 1 \implies \frac{\cos(x) + i\sin(x)}{e^{ix}} = 1 \implies e^{ix} = \cos(x) + i\sin(x).$$

Hence, any $z = x + iy \in \mathbb{C}$ can be expressed as

$$z = re^{i\theta}, \text{ where } r = |z| = \sqrt{x^2 + y^2}, \text{ and } \theta := \tan^{-1}(\mathrm{Im}(z)/\mathrm{Re}(z)) = \tan^{-1}(y/x).$$

Problems

(1) For any $r > 0$, is the following inequality true?

$$\left|r^i\right| < 1.$$

(2) Compute the limit

$$\lim_{n\to\infty} i^{in}.$$

(3) For any $z = x + iy \in \mathbb{C}$, define

$$\cosh(z) := \frac{e^z + e^{-z}}{2}, \ \sinh(z) := \frac{e^z - e^{-z}}{2}.$$

Compute $\cosh(i)$.

(4) For any $z = x + iy \in \mathbb{C}$, define

$$\cos(z) := \frac{e^{iz} + e^{-iz}}{2}, \ \sin(z) := \frac{e^{iz} - e^{-iz}}{2i}, \ \tan(z) := \frac{\sin(z)}{\cos(z)}.$$

Find all the solutions of the equation

$$\tan(z) = z.$$

1.3. **Polynomials.** Let

$$\mathbb{R}[X] := \left\{ f(X) = a_0 + a_1 X + \cdots + a_n X^n \middle| \, a_0, \ldots, a_n \in \mathbb{R} \right\}$$

denote the set of polynomials with coefficients in $\mathbb{R}$. For example consider the polynomials

$$f_1(X) := X^2 - 1; \ f_2(X) := X^4 - 1 = 0; \ f_3(X) := X^3 - X^2 - X + 1.$$

An $\alpha \in \mathbb{R}$ is called a real root of $f(X) \in \mathbb{R}[X]$, if $f(\alpha) = 0$. If $\alpha \in \mathbb{R}$ is a root of $f(X)$ if and only if

$$(x - \alpha) | f(X) \implies \frac{f(X)}{X - \alpha} \in \mathbb{R}[X].$$

We now state the Fundamental Theorem of Algebra, and we will not be able to give an elementary proof of it, at this point of time.

Fundamental Theorem of Algebra. Any non-constant $f \in \mathbb{C}[X]$ admits a root in $\mathbb{C}$, where

$$\mathbb{C}[X] := \left\{ f(X) = a_0 + a_1 X + \cdots + a_n X^n \middle| \, a_0, \ldots, a_n \in \mathbb{C} \right\}$$

denotes the set of polynomials with coefficients in $\mathbb{C}$. Let

$$f(X) := a_0 + a_1 X + \cdots + a_n X^n \in \mathbb{R}[X],$$

where $a_n \neq 0$. Then f is said to be a polynomial of degree n.

Since $\mathbb{R} \subset \mathbb{C}$, we have $\mathbb{R}[X] \subset \mathbb{C}[X]$. Hence, any $f \in \mathbb{R}[X]$ has a root in complex numbers. For example, the polynomial

$$f(X) := X^2 + 1 \text{ has no real roots.}$$

The only roots of f are $\pm i$, and it is easy to see that

$$(X + i)(X - i) = X^2 - i^2 = X^2 + 1.$$

For any $z = x + iy \in \mathbb{C}$, the complex conjugate of z is given by the following formula

$$\overline{z} := x - iy.$$

If for any $\alpha \in \mathbb{C}$ is a root of $f \in \mathbb{R}[X]$, then observe that

$$f(\alpha) = \sum_{j=0}^{n} a_j \alpha^j = 0 \implies \overline{f(\alpha)} = 0 \implies \sum_{j=0}^{n} \overline{a_j \alpha^j} = \sum_{j=0}^{n} a_j \overline{\alpha^j} = \sum_{j=0}^{n} a_j \overline{\alpha}^j = 0 \implies f(\overline{\alpha}) = 0.$$

Hence, we can conclude that complex roots occur in pairs.

For any $n \geq 1$, consider the polynomial

$$f(X) := X^n - 1 \in \mathbb{R}[X] \implies X^n = 1 = e^{2m\pi i}, \text{ for, any m} \in \mathbb{Z}$$
$$\implies X = e^{2\pi mi/n}, \ 0 \leq m \leq n-1.$$

Hence,

$$\left\{1, e^{2\pi i/n}, \cdots, e^{2\pi i(n-1)/n}\right\}$$

are all the roots of $f(X)$.

Problems

(1) Let $f \in \mathbb{R}[X]$ be a polynomial of degree n. Then, show that f has exactly n-roots (not necessarily unique/distinct).

(2) Let $f \in \mathbb{R}[X]$ be a polynomial of degree $2n+1$. Then, show that f has atleast one real root.

(3) Find all the solutions of the polynomial

$$f(X) := X^7 - 2.$$

(4) Show that

$$\cos(2\pi/7)\cos(4\pi/7)\cos(6\pi/7) = \frac{1}{8}.$$