Math Circles

Quadratic and cubic equations

Problem sheet 3 (05.07.2024)

1. Quadratic equations

1.1. **The quadratic equation.** We warm up by recalling a few basic facts about quadratic equations. A *quadratic polynomial* is a polynomial of the following form,

$$p(X) = a_0 + a_1 X + a_2 X^2 \in \mathbb{C}[X]$$

where $a_2 \neq 0$. And the corresponding equation $p(X) = 0$ is called a *quadratic equation*.

1.2. **Problems.**

(1) Show that the quadratic equation $a_0 + a_1 X + a_2 X^2 = 0$ has two solutions given by

$$S_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2 a_0}}{2a_2} \quad \text{and} \quad S_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

(2) If we assume that $a_0 \neq 0$, then show that we can also obtain the following expressions for the solutions

$$S_1 = \frac{2a_0}{-a_1 - \sqrt{a_1^2 - 4a_2 a_0}} \quad \text{and} \quad S_2 = \frac{2a_0}{-a_1 + \sqrt{a_1^2 - 4a_2 a_0}}$$

(3) Show that for a monic ($a_2 = 1$) quadratic equation $a_0 + a_1 X + x^2$, the coefficients a_1 and a_2 are polynomial functions of the roots S_1 and S_2. In particular,

$$a_1(S_1, S_2) = S_1 + S_2 \quad \text{and} \quad a_0(S_1, S_2) = S_1 S_2$$

Note that the coefficients are *symmetric polynomials* of the roots, i.e. rearranging the roots does not change the polynomial.

(4) For a general degree n polynomial equation show that the coefficients can always be expressed as symmetric polynomials of the roots.

1.3. **Paths.** Before we go any further, we make a small digression to study a certain geometric concept. A continuous *path* in $\mathbb{C}^n$ is a continuous function $\gamma : [a, b] \to \mathbb{C}^n$, where $[a, b]$ is any closed interval in the real line. In this case we say γ joins the starting point $Z_1 = \gamma(a) \in \mathbb{C}$ to the end point $Z_2 = \gamma(b) \in \mathbb{C}$. The word 'continuous' implies that if we are 'drawing' the path we would not be needing to lift the pen (you can imagine this easily on $\mathbb{C}^1$). At the moment we avoid any rigorous definition of continuity, since an intuitive understanding would suffice.

1.4. **Problems.** In the following problems, the word 'distinct' will mean that the paths have distinct images.

(1) Find two distinct paths joining $-i$ and i in $\mathbb{C}^1$.

(2) Find two distinct paths joining -1 and 1 in $\mathbb{C}^1$.

(3) Find two distinct paths in $\mathbb{C}^1$ whose start and end points are at the origin. Such paths with same start and end points are called *loops* based at that point.

(4) Find two distinct paths joining $(0, 0)$ and $(1 + i, 1 - i)$ in $\mathbb{C}^2$.

(5) Find three distinct loops based at $(0, 0, 0)$ in $\mathbb{C}^3$.

(6) Suppose $\gamma_1 : [a_1, b_1] \to \mathbb{C}^n$ and $\gamma_2 : [a_2, b_2] \to \mathbb{C}^n$ are two continuous paths satisfying $\gamma_2[a_2] = \gamma_1[b_1]$, then show that there exists a continuous path $\gamma_3 : [a_3, b_3] \to \mathbb{C}^n$ such that $\gamma_3[a_3, b_3] = \gamma_1[a_1, b_1] \cup \gamma_2[a_2, b_2]$ and $\gamma_3[a_3] = \gamma_1[a_1]$ and $\gamma_3[b_3] = \gamma_2[b_2]$. This process will be called *concatenation.*

(7) Show that there are infinitely many continuous paths between any two points in $\mathbb{C}^n$.

We end this section by stating two facts without any proof:

- FACT 1: If $f : \mathbb{C}^n \to \mathbb{C}^m$ is any continuous function and $\gamma : [a, b] \to \mathbb{C}^n$ is any continuous path then so is $f \circ \gamma : [0, 1] \to \mathbb{C}^m$.
- FACT 2: Any rational function (the ratio of two polynomials) is a continuous function wherever the denominator does not vanish.

1.5. **General form of the solution.** Given a polynomial $p(X) = a_0 + a_1X + \ldots + a_{n-1}X^{n-1} + a_nX^n \in \mathbb{C}[X]$, we expect to write a solution S of $p(X) = 0$ as a function of the coefficients i.e.

$$S = R(a_0, a_1, \ldots, a_{n-1}, a_n)$$

where R is a function involving the coefficients and the operations $+, -, \times, \div, \sqrt[k]{}$. We have already seen how such a function can be obtained for quadratic equations, and we observed that we can obtain different expressions and all the expressions we obtain involves the $\sqrt{}$. Naively one can ask if a formula for a solution of the quadratic equation is possible using only the four basic operations $+, -, \times, \div$.

1.6. **Problems.** Suppose there exists a function $R_0 : \mathbb{C}^2 \to \mathbb{C}$ involving only the four basic operations $+, -, \times, \div$, such that $R_0(a_0, a_1)$ is always a solution of the quadratic equation $p(X) = a_0 + a_1X + X^2 = 0$. Notice that for the rest of the section we assume $a_2 = 1$ for simplicity.

(1) Let S_1 and S_2 be the two solutions of the equation $p(X) = 0$. Let $\gamma_1 : [0, 1] \to \mathbb{C}$ be a continuous path satisfying $\gamma_1(0) = S_1$ and $\gamma_1(1) = S_2$. Let $\gamma_2 : [0, 1] \to \mathbb{C}$ be a continuous path satisfying $\gamma_2(0) = S_2$ and $\gamma_2(1) = S_1$. The coefficients $a_0(S_1, S_2)$ and $a_1(S_1, S_2)$ being functions of the solutions, becomes paths in $\mathbb{C}$ by replacing S_1 with γ_1 and S_2 with γ_2. So, we have two paths $a_i : [0, 1] \to \mathbb{C}$ defined by $a_i(t) = a_i(\gamma_1(t), \gamma_2(t))$ for $i = 0, 1$. Show that both $a_0(t)$ and $a_1(t)$ defines loops based at $a_0(0) = a_0$ and $a_1(0) = a_1$ respectively.

(2) Define $\Gamma(t) : [0, 1] \to \mathbb{C}$ by $\Gamma(t) = R_0(a_0(t), a_1(t))$. Show that $\Gamma(t)$ is a continuous loop based at $\Gamma(0)$.

(3) Conclude that the above problem leads to a contradictions and hence we can not have such a formula for a solution of a quadratic equation.

(Hint: Since we expect that formula R_0 should hold in general for all choices of S_1 and S_2 in $\mathbb{C}$, we assume w.l.o.g. $S_1 \neq S_2$. Also choose the paths γ_1 and γ_2 in such a way that they only intersect at the end points. If $\Gamma(0) = S_1 = \gamma_1(0)$, then continuity implies $\Gamma(t) = \gamma_1(t)$. Deduce that this leads to a contradiction.)

2. Cubic equations

2.1. **The cubic equation.** A cubic polynomial is a polynomial of the form,

$$p(X) = a_0 + a_1X + a_2X^2 + a_3X^3 \in \mathbb{C}[X]$$

where $a_3 \neq 0$. And the corresponding equation $p(X) = 0$ is called a *cubic equation.*

2.2. **Problems.**

(1) Find all the solutions of $1 - X^3 = 0$. Verify that there is a solution ω such that ω, ω^2 and $\omega^3 = 1$ are all the solutions.

(2) Any complex number c has three 'cube roots' (see last session) which are defined to be the solution of the equation $c - X^3 = 0$. If $\sqrt[3]{c}$ denote any one of the cube roots, check that $\omega \sqrt[3]{c}$ and $\omega^2 \sqrt[3]{c}$ gives the other two cube roots.

2.3. **An expression for the solution.** Now we consider the general cubic equation $p(X) = a_0 + a_1 X + a_2 X^2 + a_3 X^3 = 0$ and we define the following three quantities:

$$\begin{aligned} \Delta_0 &= a_2^2 - 3a_3a_1, \\ \Delta_1 &= 2a_2^3 - 9a_3a_2a_1 + 27a_3^2a_0 \\ C &= \sqrt[3]{\frac{\Delta_1 \pm \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}} \end{aligned}$$

In the expression for C, we can choose any fixed cube root and any fixed square root. The ambiguity that comes with the $\pm$ sign is resolved as follows: Choose the sign that makes $C \neq 0$. However if both choices gives $C = 0$ then we assign 0 to C.

If $C \neq 0$, then the solutions of $p(X) = 0$ are given by,

$$\begin{aligned} S_1 &= -\frac{1}{3a_3}\left(a_2 + C + \frac{\Delta_0}{C}\right) \\ S_2 &= -\frac{1}{3a_3}\left(a_2 + \omega C + \frac{\Delta_0}{\omega C}\right) \\ S_3 &= -\frac{1}{3a_3}\left(a_2 + \omega^2 C + \frac{\Delta_0}{\omega^2 C}\right) \end{aligned}$$

We close this section after pointing out that the above expression for the solution involves a nested use of radicals: a square root inside a cube root.

2.4. **Problems.**

(1) Show that if $C = 0$ then all three roots are equal to $\frac{-b}{3a}$.

2.5. **General form of the solution.** Now we investigate if it is possible to avoid the nesting of roots in the general expression for the solution of the cubic. To avoid unnecessary complexity we only look at monic cubic polynomials $p(X) = a_0 + a_1 X + a_2 X^2 + X^3 = 0$. There are two possibilities we would like to explore here:

(1) Is it possible to obtain an expression for the solution involving only the four basic operations $+, -, \times, \div$ avoiding any radicals? If this is the case, then we can always write a solution as

$$R_0(a_0, a_1, a_2)$$

where R_0 only involves $+, -, \times, \div$ but no radicals.

(2) Is it possible to obtain an expression for the solution involving only the four basic operations $+, -, \times, \div$ and also one level of radicals $\sqrt[k]{}$? If this is the case, then we can always write a solution as

$$R_1(a_0, a_1, a_2)$$

where R_1 only involves $+, -, \times, \div, \sqrt[k]{}$ but no nested radicals.

2.6. **Problems.**

(1) Show that the expressions $R_i(a_0, a_1, a_2)$ for $i = 0, 1$ above must be symmetric functions of (a_0, a_1, a_2), i.e. all permutations will result in the same value of the function.

(2) Show by example that if $R_1(a_0, a_1, a_2)$ is a symmetric function involving only one level of radicals, and $a_i : [0, 1] \to \mathbb{C}$ are continuous loops for $i = 0, 1, 2$, then $R_1 : [0, 1] \to \mathbb{C}$ defined by $t \mapsto R_1(a_0(t), a_1(t), a_2(t))$ need not be a continuous loop. Also show that $R_0(a_0(t), a_1(t), a_2(t))$ must be a continuous loop.

(3) Consider a set consisting of n elements $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Let (ij) denote the permutation of the elements where the i-th and j-th elements are interchanged while the remaining elements are kept fixed. Let (ijk) denote the permutation where the i-th element is put in the j-th place, the j-th element is put in the k-th place and the k-th element is put in the i-th place. Show that

$$(12)(23)(12)^{-1}(23)^{-1} = (123)$$

The left hand side is called the *commutator* of (12) and (23) and denoted by $[(12), (23)]$.

(4) Let $\gamma_1 : [0, 1] \to \mathbb{C}$ be a continuous path joining S_1 to S_2, and $\gamma_2 : [0, 1] \to \mathbb{C}$ be another continuous path joining S_2 to S_3. If $(1, 2)$ stands for the interchange of S_1 and S_2, while $(2, 3)$ stands for the interchange of S_2 and S_3, then argue that we could continuously realize both interchanges using γ_1 and γ_2. If we wish to realize the permutation $(1, 2, 3)$ using γ_1 and γ_2, how should we proceed?

(5) Let γ_1 and γ_2 be as in the previous problem. Consider the continuous paths

$\xi : [0, 1] \to \mathbb{C}$ defined by

$$t \mapsto R_0(a_0(\gamma_1(t), \gamma_1^{-1}(t), S_3), a_1(\gamma_1(t), \gamma_1^{-1}(t), S_3), a_2(\gamma_1(t), \gamma_1^{-1}(t), S_3))$$

$\zeta : [0, 1] \to \mathbb{C}$ defined by

$$t \mapsto R_0(a_0(S_1, \gamma_2(t), \gamma_2^{-1}(t)), a_1(S_1, \gamma_2(t), \gamma_2^{-1}(t)), a_2(S_1, \gamma_2(t), \gamma_2^{-1}(t)))$$

Show that both ξ and ζ are continuous loops and conclude that the commutator $[\xi, \zeta]$ is also a continuous loop. Here we can interpret the commutator $[\xi, \zeta] = \xi\zeta\xi^{-1}\zeta^{-1}$ as a concatenation of paths obtained by travelling via look ξ first followed by ζ followed by ξ^{-1} followed by ζ^{-1}.

(6) Argue that if Γ_1 and Γ_2 are loops then $[\Gamma_1, \Gamma_2]$ is also a loop and so is $\sqrt[k]{[\Gamma_1, \Gamma_2]}$. Is $\sqrt[k]{\Gamma_1}$ a loop? Compare with problem (2).

(7) Let γ_1 and γ_2 be as in the previous problem. Consider the continuous loops

$$a_{i,(1,2)} : [0, 1] \to \mathbb{C} \text{ defined by } t \mapsto a_i(\gamma_1(t), \gamma_1^{-1}(t), S_3) \qquad i = 0, 1, 2$$

$$a_{i,(2,3)} : [0, 1] \to \mathbb{C} \text{ defined by } t \mapsto a_i(\gamma_1(t), \gamma_1^{-1}(t), S_3) \qquad i = 0, 1, 2$$

And corresponding to the above loops we define the following continuous path

$\Gamma : [0, 1] \to \mathbb{C}$ defined by $t \mapsto R_1([a_{0,(1,2)}, a_{0,(2,3)}], [a_{1,(1,2)}, a_{1,(2,3)}], [a_{2,(1,2)}, a_{2,(2,3)}])$

Argue that Γ must be a continuous loop.

(8) Conclude that the above problem leads to a contradictions and hence we can not have such a formula (i.e. one without nested radicals) for a solution of a cubic equation.