Math Circles

Problems for Complex numbers and Polynomials

Problem Sheet 4 (19.07.2024)

1. Arnold's approach to the Abel-Ruffini Theorem

1.1. **Summary of cubics.** With notation as in Problem Sheet 3, we now summarize our observations for cubic polynomials. Let

$$\mathcal{F}_3 := \{ f(X) = a_0 + a_1X + a_2X^2 + X^3 \,|\, a_0, a_1, a_2 \in \mathbb{C}, \text{ and roots of } f \text{ are all distinct} \}$$

denote the set of polynomials with coefficients in $\mathbb{C}$ with distinct roots. We identify $\mathcal{F}_3$ with $\mathbb{C}^3$, i.e., any $f = a_0 + a_1X + a_2X^2 + X^3$ identified with the point $(a_0, a_1, a_2) \in \mathbb{C}^3$.

Fix a polynomial $f \in \mathcal{F}_3$, and let $\{s_1, s_2, s_3\}$ denote the set of roots of f. Let

$$S_3 := \{ f : \{s_1, s_2, s_3\} \longrightarrow \{s_1, s_2, s_3\} \,|\, f \text{ is a bijection} \}.$$

denote the set of bijections from the set $\{s_1, s_2, s_3\}$ to itself. The set S_3 can be identified with the set

$$\{ f : \{1, 2, 3\} \longrightarrow \{1, 2, 3\} \,|\, f \text{ is a bijection} \}, \tag{1}$$

and here after we refer to S_3 as the set defined in (1).

Any $\gamma \in S_3$ can be represented by a cycle of the form (i, j, k), which represents the bijection

$$\gamma(i) = j, \ \gamma(j) = k, \ \gamma(k) = i,$$

where $\{i, j, k\} = \{1, 2, 3\}$.

Similarly the cycle (i, j) represents the function

$$\gamma(i) = j, \ \gamma(j) = i, \ \gamma(k) = k.$$

(i) Let γ_1 denote a continuous path from the root s_1 to s_2, and let γ_2 denote a continuous path from the root s_2 to s_3. Corresponding to γ_1 and γ_2, there exists loops, Γ_1 and Γ_2 in $\mathbb{C}^3$ centered at the polynomial f, respectively.

(ii) We can identify the paths γ_1 with the function (12), and γ_2 with the cycle $(2, 3)$. Then, the composition of loops $\Gamma = \Gamma_1 \circ \Gamma_2 \circ \Gamma_1^{-1} \circ \Gamma_2^{-1}$ in $\mathbb{C}^3$ corresponds to the function

$$\gamma := (1, 2) \circ (2, 3) \circ (1, 2)^{-1} \circ (2, 3)^{-1}.$$

Since $(1, 2)^{-1} = (1, 2)$ and $(2, 3)^{-1} = (2, 3)$, unravelling definitions, we find that

$$\gamma = (1, 2, 3).$$

(ii) From Problem 2.6 from Problem Sheet-3, it is clear that the functions

$$R_0(a_0(\Gamma(t)), a_1(\Gamma(t)), a_2(\Gamma(t))) \text{ and } R_1(a_0(\Gamma(t)), a_1(\Gamma(t)), a_2(\Gamma(t)))$$

also follow a loop, when the function goes through the loop Γ. However the roots undergo a permutation $s_1 \mapsto s_2$, $s_2 \mapsto s_3$, and $s_3 \mapsto s_1$. Thus, we can conclude that there is no formula for roots coming from a single radical.

(iv) As seen in Problem Sheet-3, the formula for the root of a cubic polynomial with three distinct roots involves a nested sequence of radicals upto second order.

Problem 1

As above, let

$$\mathcal{F}_n := \{ f(X) = a_0 + a_1 X + \cdots a_{n-1}X^{n-1} + X^n \,|\, a_0, a_1, \ldots, a_{n-1} \in \mathbb{C}, \text{ and roots of } f \text{ are all distinct} \}$$

denote the set of polynomials of degree n, with coefficients in $\mathbb{C}$ and with distinct roots. Let γ_1 and γ_2 are two loops based at a fixed polynomial $f \in \mathcal{F}_n$, i.e., $\gamma_i[0,1] \longrightarrow \mathbb{C}$ is a continuous map, for $i = 1, 2$, and its acts on the polynomial f in the following sense:

$$f(\gamma_i(t)) := a_0(\gamma_i(t)) + a_1(\gamma_i(t))X + \cdots + a_{n-1}(\gamma_i(t))X^{n-1} + X^n; \;\; f(\gamma_i(0)) = f(\gamma_i(1)) = f,$$

thus defining a loop at f in $\mathbb{C}^n$ (here f is identified with the point $(a_0, \ldots, a_{n-1})^t \in \mathbb{C}^n$).

(i) Let $\varphi : \mathcal{F}_3 \longrightarrow \mathbb{C}$ be a continuous function. Observe that f traverses the loop $\gamma := [\gamma_1, \gamma_2]$, where $[\gamma_1, \gamma_2] := \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$. Prove that $\varphi(f)^{1/3}$ also traverses a loop (although cube-root not is well-defined, choose any of the three branches).

(ii) Let $\varphi : \mathcal{F}^n \longrightarrow \mathbb{C}$ be a a continuous function, and let $\alpha \in \mathbb{Q}$. Show that as a fixed $f \in \mathcal{F}_n$ traverses the loop $[\gamma_1, \gamma_2]$, $\varphi(f)^\alpha$ also traverses a loop (as in (i), choose an image which is a continuous path).

(iii) Consider the following subset

$$A_3 := \{\gamma, \gamma^2, \gamma^3\} \subset S_3, \quad \text{where}\, \gamma = (1,2,3), \text{ (i.e., } \gamma(1) = 2,\ \gamma(2) = 3,\ \gamma(3) = 1);$$
$$\gamma^2 = \gamma \circ \gamma = (132), \text{ (i.e., } \gamma^2(1) = 3,\ \gamma^2(2) = 1,\ \gamma^2(3) = 2); \;\; \gamma^3 = \gamma \circ \gamma \circ \gamma = \mathrm{Id},$$

where Id denotes the identity function on the set $\{1, 2, 3\}$. Let

$$\mathcal{C}(A_3) := \{[\gamma_1, \gamma_2] \,|\, \gamma_1, \gamma_2 \in A_3\}$$

denote the commutator of A_3. Show that $\mathcal{C}(A_3) = \{\mathrm{Id}\}$.

Observations

(i) The loop Γ on $\mathcal{F}_3$, which is obtained from the commutator $[\cdot, \cdot]$ corresponds to a permutation on the roots. This corresponds to the fact that the formula for roots cannot be obtained from the elementary operations and taking radicals alone. So each commutator corresponds to a radical.

(ii) However, if we take any element of A_3 via the commutator $[\gamma, \cdot]$, we end up with the identity element, and hence a loop. So we have a formula for the cube-root coming from the application of the commutator operator twice. Hence, the formula corresponds to a nested sequence of radicals of degree 2.

1.2. **Quartics.** We first describe a method to solve a quartic.

(i) Let

$$f(X) := a_0 + a_1X + a_2X^2 + a_3X^3 + X^4 = 0.$$

be a generic quartic. Make the substitution $X = Y - a_3/4$, we find that the quartic now changes to

$$f(X) = Y^4 + b_2Y^2 + b_1Y + b_0 = 0, \text{ where } b_2 = -\frac{3a_3^2}{8} + a_2;$$

(2) $$b_1 = \frac{a_3^3}{8} - \frac{a_3a_2}{2} + a_1; \;\; b_0 = -\frac{3a_3^4}{256} + \frac{a_3^2a_2}{16} - \frac{a_3a_1}{4} + a_0.$$

(ii) Assuming $b_1 \neq 0$ (when $b_1 = 0$, it reduces to solving a quadratic), we now proceed to solve the cubic

(3) $$2Z^3 - b_2Z^2 - 2b_0Z + \left(b_0b_2 - \frac{b_1^2}{4}\right) = (2Z - b_2)(Z^2 - b_0) - \frac{b_1^2}{4} = 0.$$

Since $b_1 \neq 0$, it implies that $2Z - b_2 \neq 0$, which implies that

$$Z^2 - b_0 = \frac{b_1^2}{4(2Z_2 - b_2)}. \tag{4}$$

(iii) Then, combining equations (2)–(4), we observe that

$$(Y^2 + Z)^2 = Y^4 + 2Y^2Z + Z^2 = (2Z - b_2)Y^2 - b_1Y + \frac{b_1^2}{4(2Z - b_2)} =$$

$$\left(\sqrt{2Z - b_2}Y - \frac{b_1}{2\sqrt{2Z - b_2}}\right)^2. \tag{5}$$

(iv) From equation (5), we arrive at the equation

$$(Y^2 + Z)^2 - \left(\sqrt{2Z - b_2}Y - \frac{b_1}{2\sqrt{2Z - b_2}}\right)^2 = 0$$

$$\implies \left(Y^2 + Z + \sqrt{2Z - b_2}Y - \frac{b_1}{2\sqrt{2Z - b_2}}\right)\left(Y^2 + Z - \sqrt{2Z - b_2}Y + \frac{b_1}{2\sqrt{2Z - b_2}}\right) = 0. \tag{6}$$

Finally, solving the quadratics in equation (6), we derive that the solutions of equation (2) are

$$Y_1 = \frac{1}{2}\left(\sqrt{2z - b_2} + \sqrt{-2z - b_2 - \frac{2b_1}{\sqrt{2z - b_2}}}\right) \quad Y_2 = \frac{1}{2}\left(\sqrt{2z - b_2} - \sqrt{-2z - b_2 - \frac{2b_1}{\sqrt{2z - b_2}}}\right);$$

(7)

$$Y_3 = \frac{1}{2}\left(\sqrt{2z - b_2} + \sqrt{-2z - b_2 + \frac{2b_1}{\sqrt{2z - b_2}}}\right) \quad Y_4 = \frac{1}{2}\left(\sqrt{2z - b_2} - \sqrt{-2zb_2 + \frac{2b_1}{\sqrt{2z - b_2}}}\right),$$

where z is any solution of cubic equation (3).

Problem 2

(i) Show that the solutions of the quartic $f(X) = 0$ (equation (2)), which are as described in equation (7), are independent of the choice of z (solutions of equation (3)).

(ii) Solve the equation

$$f(X) = X^4 + 6X^2 - 60X + 36 = 0.$$

Problem 3

Let S_4 denote the set

$$\big\{f : \{1, 2, 3, 4\} \longrightarrow \{1, 2, 3, 4\} \big| \, f \,\text{is a bijection}\big\}. \tag{8}$$

Any bijection $f \in S_4$ can be represented by a cycle, as in the case of S_3. For example the cycle $(1, 2)(3, 4)$ corresponds to the function

$$\gamma(1) = 2,\ \gamma(2) = 1,\ \gamma(3) = 4,\ \gamma(4) = 3.$$

Similarly the cycle (123) corresponds to the function

$$\gamma(1) = 2,\ \gamma(2) = 3,\ \gamma(3) = 3,\ \gamma(4) = 4.$$

(i) Let

$$A_4 := \big\{[\gamma_1, \gamma_2] \big| \, \gamma_1, \gamma_2 \in S_4\big\} \subset S_4.$$

Using a computer or otherwise, show that

$$A_4 = \big\{\mathrm{Id}, (1, 2, 3), (1, 3, 2), (1, 2, 4), (1, 4, 2), (1, 3, 4), (1, 4, 3),$$
$$(2, 3, 4), (2, 4, 3), (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\big\}.$$

(ii) Let

$$V_4 := \{[\gamma_1, \gamma_2] \,|\, \gamma_1, \gamma_2 \in A_4\} \subset A_4 \subset S_4.$$

Using a computer or otherwise, show that

$$V_4 = \{\text{Id}, (12)(34), (14)(23), (13)(24)\}.$$

(iii) Show that the set

$$\mathcal{S} := \{[\gamma_1, \gamma_2] \,|\, \gamma_1, \gamma_2 \in V_4\} = \{\text{Id}\}.$$

(iv) Let $f = a_0 + a_1X + a_2X^2 + a_3X^3 + X^4 \in \mathcal{F}_4$ be a fixed polynomial of degree 4 with complex coefficients, and distinct roots. Let Γ_1, Γ_2, be the paths in $\mathbb{C}^4$ corresponding to the permutations $(1, 2, 3)$, $(2, 3, 4)$, respectively. Consider $\gamma := [\gamma_1, \gamma_2] = \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$, the path corresponding to the permutation $(1, 4)(2, 3)$, acting on the polynomial f (by identifying f with the point $(a_0, a_1, a_2, a_3) \in \mathbb{C}^4$). Show that the functions

$$R_0(a_0(\gamma(t)), a_1(\gamma(t)), a_2(\gamma(t))a_3(\gamma(t))),\ R_1(a_0(\gamma(t)), a_1(\gamma(t)), a_2(\gamma(t))a_3(\gamma(t))),$$
$$R_2(a_0(\gamma(t)), a_1(\gamma(t)), a_2(\gamma(t))a_3(\gamma(t)))$$

follow a loop. Thus, eliminating a formula for the roots of f with two nested sequence of roots.

Observation Continuing with the hypothesis as above, we observe that γ_1 and γ_2 both belong to A_4, (commutator of S_4). Now $\gamma = [\gamma_1, \gamma_2] \in V_4$ (commutator of A_4). The process terminates with commutator of V_4 being the identity Id. Hence, we have the possibility of three nested sequence of radicals in the formula for the roots of a quartic, as demonstrated in equation (7).

1.3. **Quintics.** We now give a heuristic proof for the proof of Abel-Ruffini's theorem by Arnold.

Problem 4

(i) Let

$$A_5 := \{[\gamma_1, \gamma_2] \,|\, \gamma_1, \gamma_2 \in S_5\} \subset S_5.$$

Show that the permutation $(1, 2, 3, 4, 5) \in A_5$.

(ii) Using a computer or otherwise, show that

$$\{[\gamma_1, \gamma_2] \,|\, \gamma_1, \gamma_2 \in A_5\} = A_5$$

Observations

(i) Let $f \in \mathcal{F}_5$ be a fixed polynomial of degree 5, complex coefficients, and distinct roots. Since $\gamma_0 = (1, 2, 3, 4, 5)$ is a path which permutes the roots of the polynomial, there exists a loop in $\mathbb{C}^5$ at f, corresponding to γ_0.

(ii) Now for any path $\gamma_1 \in A_5$, since $[\gamma_0, \gamma_1] \in A_5$, we keep landing back in A_5 for every such operation with γ_0. Each such operation corresponds to nesting of radicals, one for each such path. Hence, we may never arrive at a formula for the roots of f, as the paths may lead to infinite nesting.