Problem sheet 3 and 4 (11.09.2026)

1. Ramanujan's Discriminant Modular Form continued

We continue our discussion with Ramanujan's modular form, and finally introduce the general theory of modular forms for $\mathrm{SL}_2(\mathbb{Z})$.

1.1. **Ramanujan's Tau Function.** Recall that for any $z \in \mathbb{H}$, the Ramanujan's Discriminant Modular Form is given by the formula

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q(z)^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q(z)^n,$$

where the coefficients $\{\tau(n)\}_{n \geq 1}$ are given by the famous Ramanujan Tau function

$$\begin{aligned} \tau :& \mathbb{N} \longrightarrow \mathbb{Z} \\ & n \mapsto \tau(n). \end{aligned}$$

denote the upper half plane. and let $q(z) = e^{2\pi i z}$. As we will see in the next section, the coefficients $\{\tau(n)\}_{n \in \mathbb{N}}$ are known as the Fourier coefficients of Δ.

In 1916, Ramanujan famously conjectured the following three properties:

(i) If $(m, n) = 1$ we have the following multiplicative property

$$\tau(mn) = \tau(m)\tau(n). \tag{1}$$

(ii) For p a pime, and any $r \geq 1$, we have

$$\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1}).$$

(iii) For any prime p, we have the estimate

$$|\tau(p)| \leq 2p^{11/2}. \tag{2}$$

The first two properties of τ were proved by Mordell in 1917. However, the third property led to great advancements in modern mathematics, and was eventually realized by Grothendieck and Delgine in 1974, as a consequence of thier proof of the Weil conjectures.

We prove the multiplicative property (1) of the Ramanujan Tau function in this problem sheet.

1.2. **Proof of Multiplicative property.** For $n \geq 1$, define

$$F(n) = \sum_{d \mid n} d^{11} \tau\left(\frac{n}{d}\right) \tau(d). \tag{3}$$

Problems.

(i) Compute $\frac{d \log \Delta(q)}{dq}$, and show that where $\sigma_1(N) = \sum_{d \mid N} d$. Hence

$$\sum_{n=1}^{\infty} n\tau(n) q^n = \Delta(q) \left(1 - 24 \sum_{N=1}^{\infty} \sigma_1(N) q^N\right), \tag{4}$$

where $\sigma_1(N) = \sum_{d \mid N} d$.

(ii) Define the operator $\theta = q\frac{d}{dq}$. Applying θ to equation (4), derive

$$\sum_{n=1}^{\infty} F(n) q^n = \left(\sum_{n=1}^{\infty} \tau(n) q^n\right) \left(\sum_{n=1}^{\infty} n^{11} \tau(n) q^n\right).$$

(iii) For any $m, n \in \mathbb{N}$, show that $F(mn) = F(m)F(n)$.

(iv) For any $m \geq 1$, and a prime $p \not| m$, using induction or otherwise, show that

$$\tau(mp^k) = \tau(m)\tau(p^k),$$

for all $k \geq 0$. Hence, for any $(m, n) = 1$, from fundamental theorem of arithmetic, we arrive at following inequality

$$\tau(mn) = \tau(m)\tau(n)$$

which completes the proof of multiplicative property of τ (equation (1))

(v) For any $q \in \mathbb{C}$ with $|q| < 1$, recalling Jacobi product formula, derive

$$\Delta(q) = q\left(\prod_{n=1}^{\infty}(1-q^n)^3\right)^8 = q\left(\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}\right)^8. \tag{5}$$

(vi) Compute $\tau(4)$, $\tau(8)$, and in general $\tau(2^r)$ $(r \geq 1)$, using identity (5).

(vi) Assuming estimate (2), prove that the series

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n)q(z)^n \tag{6}$$

is absolutely convergent, and derive an upper bound for $\Delta(z)$ for any $z \in \mathbb{H}$.

2. Modular forms for $\mathrm{SL}_2(\mathbb{Z})$

We now discuss general theory of modular forms of weight-k, with respect to $\mathrm{SL}_2(\mathbb{Z})$.

2.1. **Basic Definitions.** For any $k \in \mathbb{N}$, a function $f : \mathbb{H} \mapsto \mathbb{C}$ is called a modular form of weight-k with respect to $\mathrm{SL}_2(\mathbb{Z})$, if it satisfies:

(i) f is holomorphic on $\mathbb{H}$.

(ii) For any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and $z \in \mathbb{H}$, f satisfies the following transformation formula

$$f(\gamma z) = f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \text{ where } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}).$$

(iii) A modular form f is called a cusp form, if it satisfies the following extra condition: $f(i\infty) = 0$.

For a given $k \in \mathbb{N}$, let $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ ($\mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z}))$) denote the set of modular forms (cusp forms), of weight-k, with respect to $\mathrm{SL}_2(\mathbb{Z})$. Let f_1, $f_2 \in \mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ (or $\mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z}))$), we have the following:

(i) Clearly $\mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z})) \subset \mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$.

(ii) Constant function $0 \in \mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z}))$.

(iii) The function $f := \lambda_1 f_1 + \lambda_2 f_2 \in \mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ (or $\mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z}))$), and for any $\lambda_1, \lambda_2 \in \mathbb{C}$.

The set of functions $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ and $\mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z}))$ are known as complex vector spaces. Furthermore, $\mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z}))$ is a complex vector subspace of $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$.

Problems.

(i) Show that $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z})) = \emptyset$, for $k = 2n+1$, for $n \geq 0$.

(iii) For any $k \geq 2$, and $f \in \mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ with $\tau \in \mathbb{H}$, show that

$$f(\tau+1) = f(\tau).$$

(iii) Show that the q-function

$$q : \mathbb{H} \longrightarrow \mathbb{D}^* := \big\{z \in \mathbb{C}\backslash\{0\} \big| \, |z| < 1\big\}$$

is a surjective map. Using which

$$\begin{aligned} g :& \mathbb{D}^* \longrightarrow \mathbb{C} \\ & g(q) \mapsto f(\log(q)/2\pi i). \end{aligned}$$

Show that g is analytic.

(iv) Given that any analytic function $g : \mathbb{D}^* \longrightarrow \mathbb{C}$ can be expressed as

$$g(q) = \sum_{n=0}^{\infty} a_n q^n,$$

using which show that

$$f(z) = \sum_{n=0}^{\infty} a_n q^n. \tag{7}$$

(iv) For any $n \geq 1$, show that

$$a_n = e^{2\pi} \int_0^1 f(x + i/n) e^{-2\pi i n x} dx.$$

Using which show that

$$|a_n| \leq c n^{k/2},$$

for some constant $c > 0$, independent of n and k.

Remark 2.1. We have just proved that, given any $f \in \mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$, it admits a q-series expansion of the form described in equation (7), which is called the Fourier expansion of f. Furthermore, the constants a_n's are called the Fourier coefficients of f. Fourier coefficients of modular forms carry a lot of arithmetic information, and play a fundamental role in modern number theory.

3. Fourier expansions of Modular Forms

We continue our discussion on Fourier expansion of modular forms, via a example, namely, the Eisenstein series.

3.1. **Eisenstein series $\mathcal{G}_k$.** For any $k > 2$, and $z \in \mathbb{H}$, define

$$\mathcal{G}_k(z) = \sum_{(c,d) \in \mathbb{Z}^2 \backslash \{(0,0)\}} \frac{1}{(cz+d)^k}.$$

Problems.

(i) For any $k > 2$, show that the series $\mathcal{G}_k(z)$ converges absolutely on compact sets (bounded subsests) of $\mathbb{H}$. Furthermore, show that $\mathcal{G}_k(z)$ is an analytic function.

(ii) Show that $\mathcal{G}_k(z)$ is a modular form of weight k with respect to $\mathrm{SL}_2(\mathbb{Z})$.

(iii) For any $z \in \mathbb{H}$, prove the identity

$$\frac{1}{z} + \sum_{d=1}^{\infty} \left(\frac{1}{z-d} + \frac{1}{z+d} \right) = \pi \cot(\pi z) = \pi i - 2\pi i \sum_{m=0}^{\infty} q(z)^m. \tag{8}$$

(iv) Differentiating (8), derive the following identity

$$\sum_{d \in \mathbb{Z}} \frac{1}{(z+d)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} q(z)^m.$$

(iv) Finally, recalling the functions

$$\zeta(k) := \sum_{d=1}^{\infty} \frac{1}{d^k}; \ \sigma_{k-1}(n) := \sum_{\substack{m|n \\ m>0}} m^{k-1},$$

show that

$$\mathcal{G}_k(z) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q(z)^n.$$

Here $\zeta(k)$ is known as the Riemann zeta function. Furthermore, we have shown that $\mathcal{G}_k(i\infty) = 2\zeta(k) \neq 0$, which implies that $\mathcal{G}_k(z)$ is a modular form, but not a cusp form.

3.2. **Number of ways or representing a nonnegative integer as a sum of 4 squares.** Lagrnage's famous four square theorem tells us that every integer can be represented as a sum of 4 squares. Using theory of modular forms, we now find a formula compute $r_4(n)$, the number of ways of representing a non negative integer n as a sum of 4 squares.

We assume that the theory of modular forms extends to

$$\Gamma_0(4) := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \, 4|c \right\}.$$

Problems.

(i) For any $z \in \mathbb{H}$, consider the function

$$\Theta(z) := \sum_{n \in \mathbb{Z}} r_4(n) q(nz) = \sum_{(x_1,x_2,x_3,x_4) \in \mathbb{Z}^4} e^{2\pi i (x_1^2 + x_2^2 + x_3^2 + x_4^2)}$$

Show that

$$\Theta(z) = \left(\sum_{n \in \mathbb{Z}} q(n^2 z) \right)^4 = \vartheta^4(z),$$

where ϑ is the standard Jacobi Theta function defined in the previous problem sheets.

(ii) Consider the functions

$$\mathcal{G}_{2,2}(z) := \mathcal{G}_2(z) - 2\mathcal{G}_2(2z);$$
$$\mathcal{G}_{2,4}(z) := \mathcal{G}_2(z) - 4\mathcal{G}_2(4z).$$

Show that both these functions admits the following q-expansions (Fourier expansions)

$$\mathcal{G}_{2,2}(z) = -\frac{\pi^2}{3}\left(1 + 24\sum_{n=1}^{\infty}\Big(\sum_{\substack{0<d|n \\ d\,\mathrm{odd}}} d\Big)q(nz)\right); \tag{9}$$

$$\mathcal{G}_{2,4}(z) = -\pi^2\left(1 + 8\sum_{n=1}^{\infty}\Big(\sum_{\substack{0<d|n \\ 4\nmid d}} d\Big)q(nz)\right). \tag{10}$$

(ii) Finally, one can show that $\mathcal{G}_{2,2}, \mathcal{G}_{2,2}, \Theta \in \mathcal{M}_2(\Gamma_0(4))$. From certain advanced theorems from arithmetic geometry (namely, Riemann Roch theorem from modular curves), one can show that $\mathcal{M}_2(\Gamma_0(4))$ is a two dimensional complex vector space, i.e., there exists two constants λ_1 and λ_2 such that

$$\Theta(z) = \lambda_1\mathcal{G}_{2,2}(z) + \lambda_2\mathcal{G}_{2,4}(z),$$

for all $z \in \mathbb{C}$. Determine λ_1 and λ_2 using equations (9), (10), and compute $r_4(8)$.