Math Circle India
School of Mathematics TIFR Mumbai

The Dance of Shapes and Symmetry

September 6 2024

Today, we're about to embark on a journey through the world of shapes, patterns, and magical moves! You'll discover the wonders of symmetry, composition, and some intriguing mysteries of mathematical groups. Ready for some fun? Let's dive into the adventure!

The Legend of Symmetria and the Dancing Shapes

In the vibrant land of Symmetria, everything is about balance and harmony. The people of Symmetria believe that every shape has a hidden dance that brings out its beauty. Each dance move either keeps the shape looking the same or changes it in a way that can always be reversed. Let's explore the dances of some familiar shapes.

1 Symmetry in Action: The Shapes' Dance Moves

Meet the four famous dancers of Symmetria:

- **Trixie the Triangle**: She loves to spin around, and her favorite moves are rotating 120 degrees clockwise or counterclockwise. Sometimes she likes to flip over a line passing through her center too. See Figure 1 for some moves.
- **Sammy the Square**: Sammy enjoys flipping over lines and rotating by 90 degrees at a time.
- **Penny the Pentagon**: Penny's moves are a bit more complex, but she can rotate 72 degrees at a time or flip over lines passing through her center.
- **Hector the Hexagon**: Hector has six moves—rotating 60 degrees, 120 degrees, and so on. He also enjoys flipping over the lines connecting opposite corners.

ACTIVITY: Find the Hidden Dance Moves

For each shape below, draw the possible moves (rotations or flips) that keep the shape looking the same. How many different ways can each shape "dance"?

Shapes:

1. Equilateral Triangle
2. Square
3. Regular Pentagon
4. Regular Hexagon

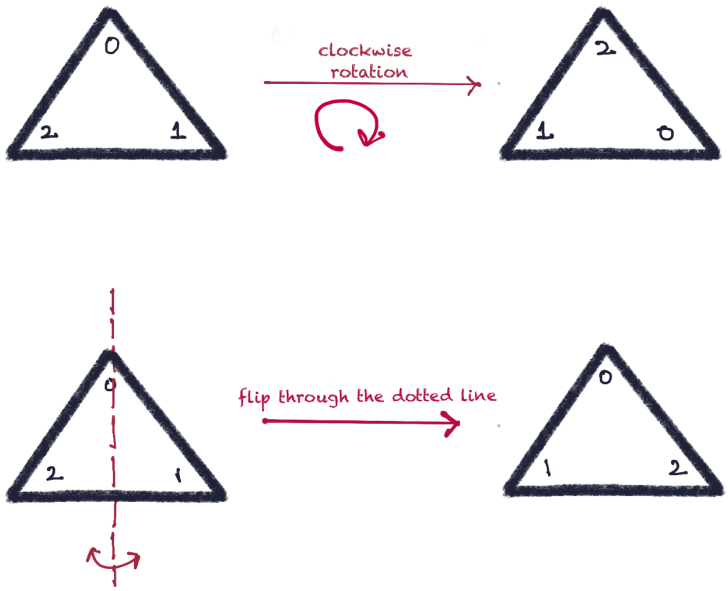

Figure 1: Some dance moves of Trixie the triangle

2 The Magic of Reversing Moves

In Symmetria, every dance move has an "undo" move. For example, if Trixie the Triangle spins 120 degrees clockwise, spinning 120 degrees counterclockwise brings her back to her starting position. The citizens of Symmetria call this an "undo" move.

ACTIVITY: Discover the Undo Moves

For each shape and its moves, find the move that "undoes" it. Try it out:

- What happens if you rotate Sammy the Square 90 degrees clockwise and then 90 degrees counterclockwise?
- Can you find all possible pair of moves for Hector the Hexagon that always brings him back to his starting position?

3 The Mystery of Composition

In Symmetria, sometimes two dance moves can be combined to create a new dance! This is called **composition**. For example, if Trixie rotates 120 degrees clockwise and then another 120 degrees clockwise, she ends up rotating a total of 240 degrees. See Figure 2.

ACTIVITY: Combine the Moves

Try combining different moves for each shape. What do you notice?

1. Choose any two moves for Sammy the Square. What happens when you do them one after the other? Does the order of the moves matter?
2. Pick any two flips for Hector the Hexagon. Try flipping along one line and then another. Does the order of flips change the outcome?

4 Basic dance moves

The elders of Symmetria believe that there are some basic dance moves that need to be mastered to become a great dancer.

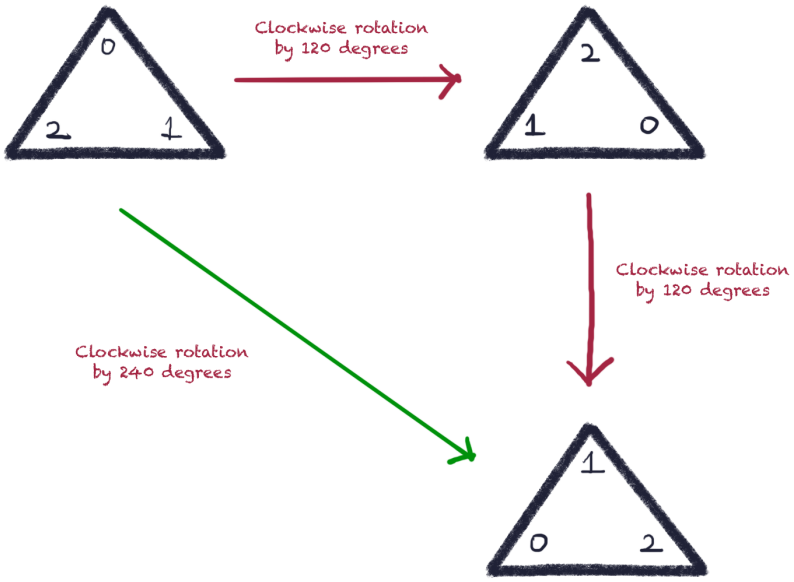

Figure 2: Composing two dance moves

For instance, the composition of two clockwise 120 degree rotations of an equilateral triangle is the same as a counter clockwise rotation by 120 degrees, which is the same as the undo move of clockwise rotation by 120 degrees. See Figure 2 carefully. So we can say that clockwise rotation by 120 degrees is a basic dance move of an equilateral triangle.

ACTIVITY:

Can you help Trixie the Triangle identify her basic dance moves so she can focus on those and strengthen their basics? Give the basic moves some names and then use them to write down all the other moves. Now can you also help Sammy the Square?

5 Passing on the Tradition (challenging)

The people of Symmetria are very proud of their dance heritage and have requested the wisest of all, **Rafiki the circle**, to record all the dance moves of the various citizens so they are never forgotten. See Figure 3 for an example of a diagram for the equilateral triangle.

ACTIVITY:

Can you help Rafiki draw a table/diagram of all the moves for each of the shapes, so that they can be easily explained to the future generations? You can use the basic moves to organize your table/diagram.

1. Equilateral Triangle
2. Square
3. Regular Pentagon
4. Regular Hexagon

6 Discovering Permutation Groups

In the kingdom of Symmetria, some citizens discovered that the dance moves of the shapes are just like rearranging objects in different orders. This idea is called a **permutation**.

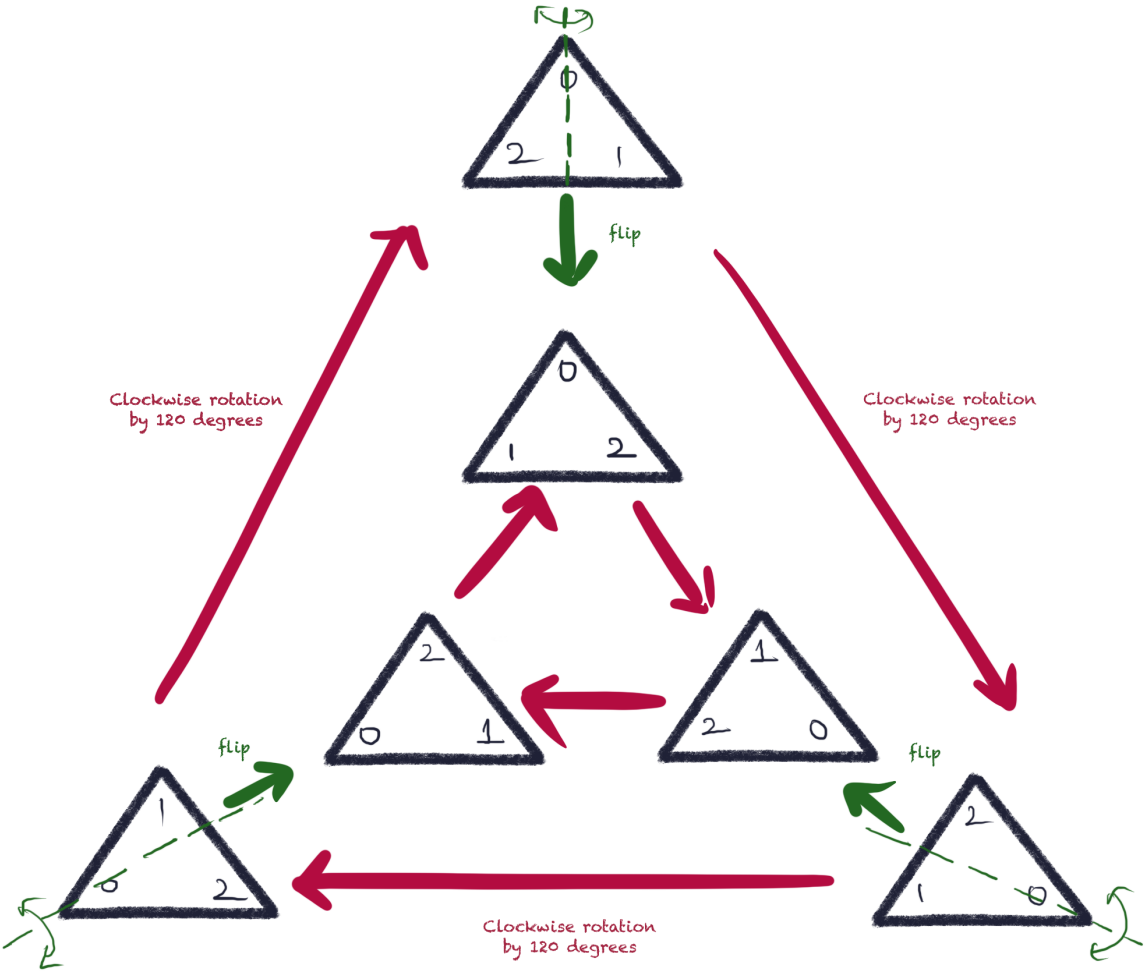

Figure 3: Recording all the dance moves of Trixie the Triangle

ACTIVITY: Arrange and Rearrange

Imagine you have three colored balls: red, blue, and green. How many different ways can you arrange them in a line? List out all the possible arrangements.

Challenge:

What if you add a fourth ball (yellow)? How many different ways can you arrange them now?

7 Every Group is Part of a Bigger Dance

Here's a fascinating discovery from the people of Symmetria: Every group of dance moves is like a smaller part of a bigger group of rearrangements (permutations). This means every group is a **subgroup** of a larger **permutation group**.

ACTIVITY: Find the Bigger Dance Group

Take the dance moves of Sammy the Square. Try to think of how these moves could be seen as rearrangements of numbers or objects. Can you list all possible rearrangements (permutations) that include Sammy's moves?

- Can you find a connection between Sammy's dance moves and rearranging four objects?
- Similarly, try to find a connection between Hector's dance moves and rearranging six objects.

8 Abstraction (challenging)

Consider the collection of dance moves of a square under the operation of composition. Let's call this collection D_4 and the composition operation $*$. Observe

1. When we compose two dance moves, we get a dance move again.
2. Let's say standing still is also a dance move. It can be done before or after your favorite dance move without changing your dance move.
3. For every dance move, there is an undo dance move. Doing one of the other amounts to standing still move.

We were using the word 'group' of moves in the previous sections to mean a collection of moves. In mathematics, the word 'group' has a very special meaning. Any set with an operation which satisfies the properties above is called a **group**. So we will say $(D_4, *)$ is a (mathematical) group.

Challenge:

Is the set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ with the operation of addition a group? That is,

- If you add two natural numbers do you get a natural number?
- Is there a number in $\mathbb{N}$ such that adding it does not change any number in $\mathbb{N}$?
- For every number in $\mathbb{N}$, there is another number in $\mathbb{N}$ to 'cancel' it?

What about the set of integers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ with the operation of addition? with the operation of multiplication?

Abstraction in mathematics refers to the process of generalizing and simplifying concepts by focusing on their essential properties rather than specific details. This allows mathematicians to create more general theories and apply them to a wide range of problems.

Conclusion

Congratulations! You have uncovered the secrets of symmetry, composition, permutation groups, and special dance groups in the magical land of Symmetria. Keep looking for these patterns in the world around you (next time we will look at *Frieze patterns*), and remember: the dance of math is everywhere!