Math Circle India Workshop

The Game of SET

School of Mathematics, TIFR Mumbai

October 14, 2024

1 What is SET?

The game of SET was invented by Marsha Jean Falco in 1974. She was a population geneticist in England. She was studying epilepsy in German Shepherd dogs and began to record data as symbols on flash cards. While looking for patterns in the data, she came up with the idea of the game SET. After 17 years, Marsha's son and daughter convinced her to publish the game and make it available in the stores!

The game of SET consists of a deck of cards where each card is defined by four characteristics:

- **Color:** Red, Green, or Purple.
- **Shape:** Oval, Squiggle, or Diamond.
- **Shading:** Solid, Striped, or Empty.
- **Number:** One, Two, or Three shapes on the card.

The goal is to find a *Set* , where a *Set* is a collection of three cards such that for each of the four characteristics, the values are either all the same or all different.

Example 1. *Let's say we have the following three cards:*

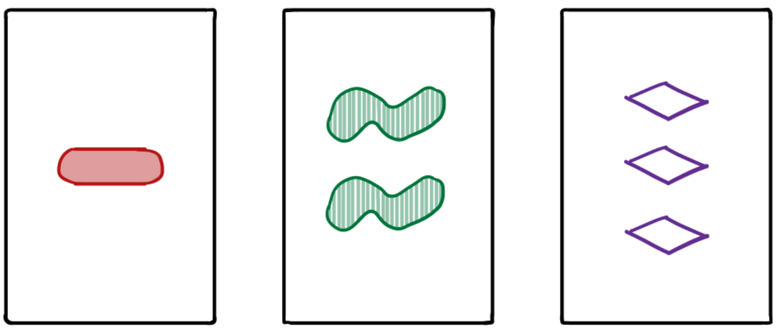

Figure 1: Example of a *Set*

These form a valid Set, because:

- *The* ***color*** *is all different: Red, Green, Purple.*
- *The* ***shape*** *is all different: Oval, Squiggle, Diamond.*
- *The* ***shading*** *is all different: Solid, Striped, Empty.*
- *The* ***number*** *is all different: 1, 2, 3.*

Example 2. *Do the following three cards form a Set ?*

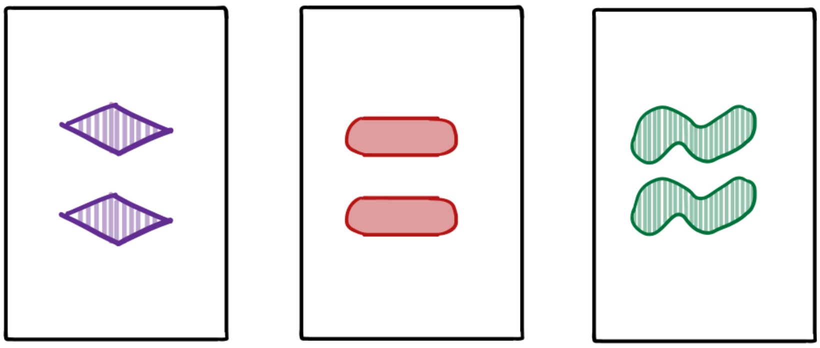

Figure 2: (a)

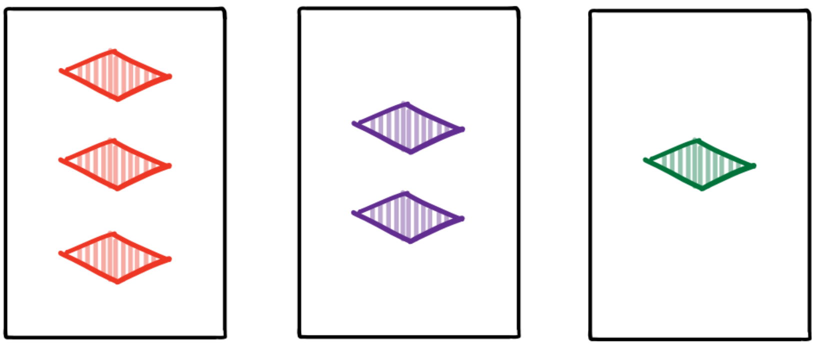

Figure 3: (b)

Lay out 12 cards on the table. Each time you find a *Set* , you remove those three cards from the table and replace them with new ones. Your goal is to find as many *Sets* as possible before the cards run out.

In a multiplayer game, the dealer lays out 12 cards on the table and all the players (including the dealer) see the same collection of 12 cards. If you see a *Set* , say "Set" and then you get a chance to collect your three cards. There are no turns, the first player to call "Set" gets control of the board. The dealer then replaces the cards picked up. If everyone agrees that there no *Set* can be formed, then the dealer lays three more cards. The game ends when the dealer lays all the cards and no more *Sets* can be created from the cards on the table. Whoever makes the most number of *Sets* , wins the game. But beware—finding a *Set* isn't as easy as it sounds!

ACTIVITY:

Let's play a game of SET first before we dive into analysing the game! We can play online here. Up to 5 people can play without logging in.

`https://buddyboardgames.com/set?room=bW9uc3RlciByb29t&name=YWJj`

2 Appetizers

Let's start with some quick bites and create an appetite for the main course.

1. How many total cards are there in the deck?
2. If you draw two cards from the deck, how many different *Sets* can you form using these two cards?
3. How many different *Sets* can you form using all the cards in the deck?

Now let's play some probability games with these cards.

1. If you draw 3 cards at random from the deck, what is the probability that they form a *Set* ?
2. By now you have created and seen many *Sets* . Can you categorize the different kinds of *Sets* you found?
3. Once you categorize the different kinds of *Sets* , compute the percentage of a particular kind of *Set* among all the *Sets* that can be formed from the deck.

3 While we wait for the main course...

Exercise 1. *Can you find a set in the collection of cards shown below? If not, can you prove that there is no Set?*

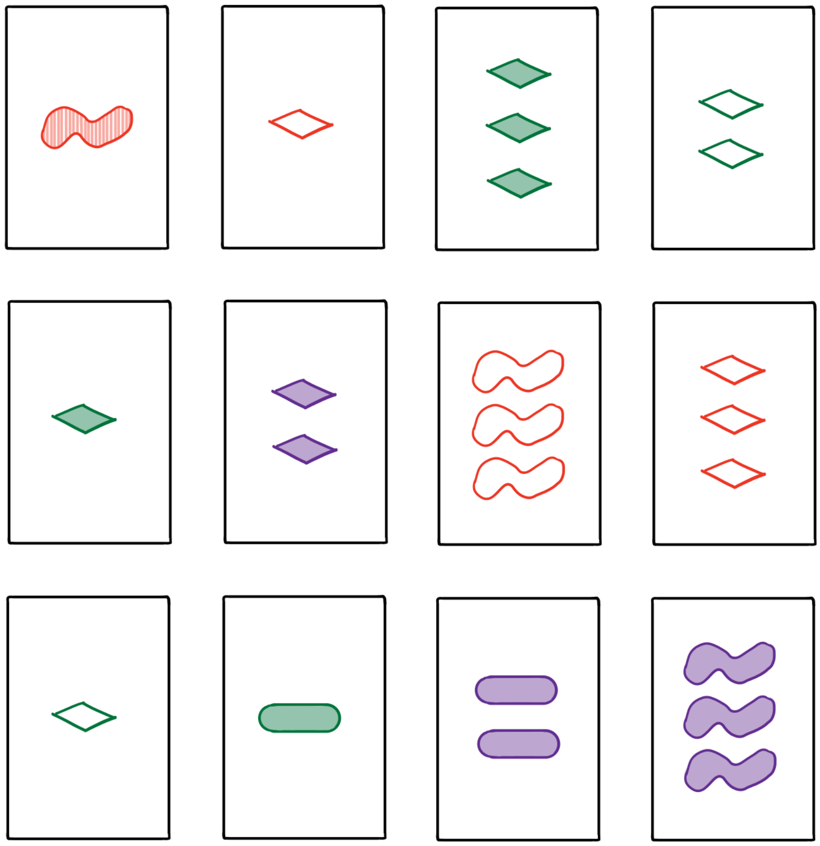

Figure 4: A collection of 12 cards

Exercise 2. *Can you add three more cards to this collection to guarantee that there is still no Set? (Your answers may vary.)*

How about adding another 3 cards?

Exercise 3. *What is the maximum number cards that you can lay out and still not have a Set ? In other words, what is the minimum number of cards on the table that guarantee a Set . These are hard questions! Let's try a simpler case.*

Disclaimer: The simpler cases might still be hard!

(Baby) Case I

Let's redefine a *Set* to understand a baby example. Say a collection three of cards forms a *Set* if either all three cards have the same color or they all have different colors.

Exercise 4. *What is the minimum number of cards needed on the table to guarantee a Set as per the new definition?*

Case II

Let's assume all the cards have the same color and the same shape. In other words, we now have only 2 characteristics that vary - number and shading.

Exercise 5. *How many such cards are there?*

Suppose there are 6 cards of the same color and shape on the table. Is there a Set among these cards?

What is the minimum number of cards of the same color and same shape you should lay on the table to guarantee a Set ?

Case III

Let's assume all the cards have the same color. In other words, we now have only 3 characteristics that vary - number, shape and shading.

Exercise 6. *How many such cards are there?*

What is the minimum number of cards needed on the table to guarantee a Set ?

You should have been able to figure out Case I and maybe you figured out Case II. Probably Case III got too difficult already! In the next section we will see that by introducing some mathematical language to the game, some of these questions seem more approachable.

4 Main course (coordinates)

We will introduce some coordinates for the cards so it is easy to describe them. But first let's recall some modular arithmetic.

mod 3

The set of remainders of natural numbers when divided by 3 is the collection $\{0, 1, 2\}$. We denote this collection by $\mathbb{Z}/3\mathbb{Z}$, called "$\mathbb{Z}$ mod 3 $\mathbb{Z}$". For a natural number A we write

$$A \equiv 0 \mod 3$$

if the remainder of A when divided by 3 is 0. Similarly, we write $A \equiv 1 \mod 3$ and $A \equiv 2 \mod 3$.

Suppose

$$A \equiv 1 \mod 3$$

$$B \equiv 2 \mod 3,$$

then A+B is divisible by 3, in other words

$$A + B \equiv 1 + 2 = 3 \equiv 0 \mod 3.$$

Since $1 + 2 \equiv 0 \mod 3$, we say 1 and 2 are *inverses* of each other. Sometimes we write $2 \equiv -1 \mod 3$.

Back to SET

Now let's define some coordinates for the cards in the deck so it is easy to call the cards.

Color:	Red 0	Green 1	Purple 2
Shape:	Oval 0	Squiggle 1	Diamond 2
Shading:	Solid 0	Striped 1	Empty 2
Number:	One 0	Two 1	Three 2

For each card we record the coordinates as a tuple in the order (color, shape, shading, number).

Example 3. *The card in Figure 5 has color green (=1), shape Squiggle (=1), shading striped (=1) and number two (=1). Therefore, it's coordinates are (1, 1, 1, 1).*

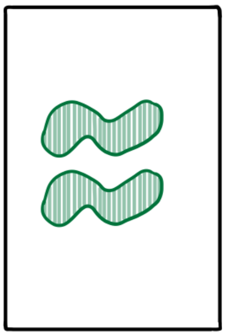

Figure 5: (1,1,1,1)

Exercise 7. *Find the coordinates for each card in Figure 6, Figure 7 and Figure 8.*

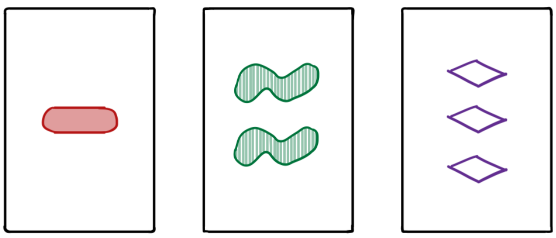

Figure 6: (a)

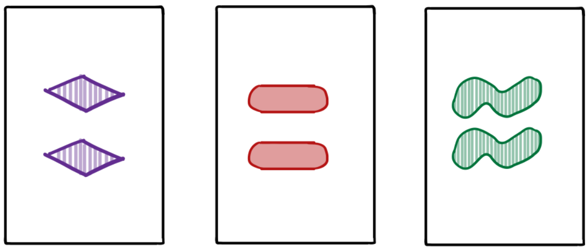

Figure 7: (b)

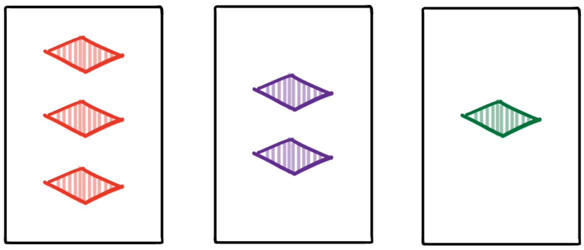

Figure 8: (c)

We can **add the coordinates of two cards** coordinate wise and then reduce mod 3. Here is an example:

$$(0,1,2,0) + (1,2,2,0) = (0+1, 1+2, 2+2, 0+1) = (1,3,4,1)$$

After taking remainders of each coordinate by 3, we get the final result $(1,0,1,1)$. Similarly, we can add the coordinates of three cards as well.

Exercise 8. *Draw the cards corresponding to the coordinates* $(0,1,2,0),(1,2,2,0)$ *and the inverse of* $(1,0,1,1)$*. Do you notice anything about this collection of three cards?*

Exercise 9. *In Figure 6, add the coordinates of any two cards. How is the result related to the coordinates of the third card? Repeat the exercise for Figure 7 and Figure 8. Do you see any connection between the sum of the coordinates of all three cards in each figure and whether the collection is a Set ?*

Exercise 10. *Let* A, B, C *be the coordinates of three distinct cards. Prove or disprove that the three cards form a Set if and only if* $A + B + C = (0,0,0,0)$ *mod 3.*

Exercise 11. *We saw earlier that given two cards from the deck, there is a unique card that forms a Set with the two cards. Can you use the language of coordinates to prove this again?*

The space of all cards

Using the coordinates defined above, we can visualize the collection of all cards in the deck as a 4-dimensional space!

The idea is as follows: Divide the deck into three piles, each pile corresponding to a color. Pick the Red pile. Sort the Red pile into three more piles according to shape.

Place the three piles in a nice line. Now for each stack, sort it according to shading into three more stacks. Now now you have 9 stacks, arranged in a grid. Now for each of these 9 stacks, arrange them vertically by first placing all the cards with One shape, then all the cards with Two shapes and last with all cards with Three shapes. Do the same sorting for the Green and the Purple pile as well. So basically, we have three $3 \times 3 \times 3$ grids of different colors and we can think of color as the 4^{th} dimension.

Now using the coordinates, we can say the following: Recall $\mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$. The product $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ is the set of all pairs of elements

$$\begin{matrix} (0,0) & (0,1) & (0,2) \\ (1,0) & (1,1) & (1,2) \\ (2,0) & (2,1) & (2,2) \end{matrix}$$

As you can see, it is naturally a 2-dimensional space.

We take four copies of $\mathbb{Z}/3\mathbb{Z}$ corresponding to color, shape, shading and number. Then $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ is the set of all coordinates of the cards in the SET deck. And it is a 4-dimensional space. We call it the **SET geometry**.

Back to Case II

Let's assume all the cards have the same color and the same shape. In other words, we now have only 2 characteristics that vary - number and shading. By fixing a color and a shape, and only remembering the number and shading coordinates, our space of cards is $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3Z$.

Also recall that three cards A, B, C form a *Set* if and only if in the coordinates $A + B + C = (0, 0, 0, 0)$ mod 3. Since color and shape are fixed, we need only check the other two coordinates.

Exercise 12. *Can you identify all collections of 3 points in $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3Z$ whose coordinates add up to (0,0) mod 3?*

You will notice that most of these collections of 3 points as above look like **lines**. We call the other ones lines too. So we have noticed that in this special Case II, a collection of three cards is a *Set* if an only if the corresponding points in SET geometry lie on a line. In fact, this is true in the full 4-dimensional SET geometry too, though the lines can look skewed.

Now that we know a *Set* corresponds to a line. Can you use this to answer the question from before:

Exercise 13. *What is the minimum number of cards of the same color and same shape you should lay on the table to guarantee a Set ?*

Lines, planes and hyperplanes in SET geometry

We saw above that three points A, B, C in the SET geometry form a line if and only if $A + B + C = (0, 0, 0, 0)$. We also saw that any two points determine a line, in other words, for any two cards A, B, there is a unique card C that forms a *Set* with the two cards.

Now consider three points A, B, D that are not on a line, that is, they don't form a set. We say such points are **non-colinear**. Place the corresponding three cards in a 'corner' as here

A	B
D	

There is a line passing through each pair of cards. Including those in, we get

A	B	C
D		
E		F

Finally, you can find 3 more cards by filling in the lines joining any two points in the collection $\{A, B, C, D, E, F\}$ to get a **plane**.

A	B	C
D	G	H
E	I	F

We just saw that any 3 non-colinear points lie on a plane.

Exercise 14. *Now add another card to a collection of cards in a plane. Repeat the above process to 'complete' the collection. How many cards are there in total in the 'complete' collection? What does it look like?*

We observe that by taking 4 **non-coplanar points**, we get a **hyperplane**.

Exercise 15. *Add another card to the hyperplane obtained above and 'complete' the collection as above. What do you get now?*

Let's do some counting exercises. It might help to think about *Sets* to answer some of these problems.

Exercise 16. *How many lines pass through a point in SET geometry?*

Exercise 17. *How many lines are there in a plane?*

Exercise 18. *How many planes pass through a point?*

Exercise 19. *How many planes pass through a given line?*

Exercise 20. *How many hyperplanes pass through a given point?*

We say two lines ℓ_1 and ℓ_2 in the SET geometry are **parallel** if the points on ℓ_1 can be obtained from the points on ℓ_2 by adding a constant $C = (c_1, c_2, c_3, c_4)$. In particular, two parallel lines don't intersect.

Exercise 21. *Can you find two lines in the SET geometry that are not parallel and don't intersect?*

Such lines are called **skew** lines.

Exercise 22. *How many lines are parallel to a given line?*

Exercise 23. *How many lines intersect a given line?*

Exercise 24. *Translate the sentence 'Two lines ℓ_1 and ℓ_2 intersect in a point." into SET language. Can you write an equation to express two intersecting lines? Also see* super-set *defined in the last section.*

Planes can also be parallel, skew or intersecting in the SET geometry.

Exercise 25. *Give an example, using SET cards, of two planes that are parallel, two planes that intersect in a line and two planes that are skew.*

Exercise 26 (Open ended)**.** *Every line in the SET geometry corresponds to a Set . Does a plane in the SET geometry have a meaning in the game SET?*

5 What's missing?

Imagine you're in the middle of a game. There are 8 cards on the table, but one card is missing (maybe a mischievous ghost stole it!). The remaining 72 cards are in the hands of the players, neatly arranged in *Sets* . Can you find out which card is missing?

6 More food for thought

We didn't completely answer the question about finding the minimum number of cards needed on a table to find a *Set* . You can now the use the coordinates to see if you get closer to the answer.

We used SET geometry to explain the game of SET. But you can observe phenomenon in the SET geometry and ask what they mean in the game of SET. For example, when two lines intersect in a point, then that means there is a unique card that makes a *Set* with two pairs of card. We call the two pairs (that is, 4 cards) a **super-*Set*** . Now you can ask all kinds of questions about super-*Sets* !

Why stop at 4 characteristics? How do things work in a deck with 5 characteristics? What would be a nice example of a 5^{th}-characteristic.

Make your own deck of SET and play!

Invent your own games using the SET deck! Here are some variantions
`http://magliery.com/Set/SetVariants.html`

7 References

1. 'SET' by Brian Conrey and Diana Donaldson

   `https://mathcircles.org/wp-content/uploads/2021/02/Set-Activity-Guide.pdf`

2. 'SETs and Anti-SETs: The math behind the game of SET' by Charlotte Chan

   `https://web.math.princeton.edu/~charchan/SET.pdf`

3. 'The card game SET' by Benjamin Davis Lent and Diane Maclagan, The Mathematical Intelligencer 25, 33–40 (2003).

https://doi.org/10.1007/BF02984846