Large sieve inequalities for families of automorphic forms

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Automorphic form notation

Let $\mathcal{F}$ be a family of automorphic forms. For $f \in \mathcal{F}$, write the associated L-function as

$$L(f, s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}.$$

Problems on families

Given a family $\mathcal{F}$, we would want to know:

- What is the size of $\mathcal{F}$ (Weyl's law).
- Low-lying zero statistics.
- Asymptotics or bounds on the moments of the L-functions $L(f, 1/2)$.
- Large sieve inequality.

Large sieve inequalities

A large sieve inequality for the family $\mathcal{F}$ takes the form

$$\sum_{f \in \mathcal{F}} \Big| \sum_{N/2 < n \leq N} a_n \lambda_f(n) \Big|^2 \leq \Delta(\mathcal{F}, N) \sum_{N/2 < n \leq N} |a_n|^2,$$

valid for *all* choices of coefficients $a_n \in \mathbb{R}$.

Ideally we want to prove this with Δ as small as possible.

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A good bound gives quantitatively strong orthogonality of the coefficients in the family.

Duality

The duality principle says that

$$\max_{|\mathbf{c}|=1} \sum_m \Big| \sum_n c_n a_{m,n} \Big|^2$$

equals

$$\max_{|\mathbf{b}|=1} \sum_n \Big| \sum_m b_m a_{m,n} \Big|^2.$$

Duality

In our context, this means

$$\sum_{f\in\mathcal{F}}\Big|\sum_{N/2<n\le N} a_n\lambda_f(n)\Big|^2 \le \Delta(\mathcal{F},N)\sum_{N/2<n\le N}|a_n|^2,$$

valid for all $a_n \in \mathbb{R}$ is equivalent to

$$\sum_{N/2<n\le N}\Big|\sum_{f\in\mathcal{F}} b_f\lambda_f(n)\Big|^2 \le \Delta(\mathcal{F},N)\sum_{f}|b_f|^2,$$

valid for all $b_f \in \mathbb{R}$.

GL_1 examples

Theorem

$$\sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi \pmod q} \Big|\sum_{n\le N} a_n\chi(n)\Big|^2 \le (Q^2+N)\sum_{n\le N}|a_n|^2.$$

GL_1 examples

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Applications include: Bombieri-Vinogradov, bounds on moments of twists of L-functions, etc.

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Some more advanced methods for studying moments of L-functions transform the original moment problem into one involving a completely different family, and finish by using a large sieve inequality to bound the 'dual' moment.

GL_1 examples, continued

Theorem (Heath-Brown)

$$\sum_{q \leq Q}^{*} \left| \sum_{n \leq N}^{*} a_n \left(\frac{n}{q} \right) \right|^2 \ll (Q + N)(QN)^{\varepsilon} \sum_{n \leq N} |a_n|^2.$$

GL_1 examples, continued

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Applications include: Improved bounds on nonvanishing of quadratic twists of L-functions, ...

GL_2 examples

Theorem (Deshouillers-Iwaniec)

$$\sum_{\substack{f \text{ level } q \\ t_f \leq T}} w_f^{-1} \Big| \sum_{n \leq N} a_n \lambda_f(n) \Big|^2 \ll (qT^2 + N)(qTN)^{\varepsilon} \sum_{n \leq N} |a_n|^2,$$

where: f is a Maass form with spectral parameter t_f, and

$$w_f = Res_{s=1} L(f \otimes \overline{f}, s).$$

Optimality

The previous GL_1 and GL_2 large sieve inequalities took the form

$$\Delta(\mathcal{F}, N) \ll (|\mathcal{F}| + N)(|\mathcal{F}|N)^{\varepsilon}.$$

This cannot be improved.

A cautionary tale

One might be tempted to conjecture that $\Delta(\mathcal{F}, N) \approx |\mathcal{F}| + N$ holds for any reasonable family. However, H. Iwaniec and Xiaoqing Li (Compositio, 2007) showed that the family of Hecke cusp forms on $\Gamma_1(q)$ (with q prime, and with fixed weight) does NOT satisfy this optimistic conjecture.

More reasons for caution

Making life exciting, there are a variety of different reasons that a family may not satisfy the optimistic bound:

- A family $\mathcal{F}$ may have cusp forms and Eisenstein series together, and the Eisenstein series may be problematic. Blomer and Buttcane showed this for the family $SL_n(\mathbb{Z})$, $n \geq 3$.

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Making life exciting, there are a variety of different reasons that a family may not satisfy the optimistic bound:

- A family $\mathcal{F}$ may have cusp forms and Eisenstein series together, and the Eisenstein series may be problematic. Blomer and Buttcane showed this for the family $SL_n(\mathbb{Z})$, $n \geq 3$.
- A family may have a large *biased set*, that is, a set $\mathcal{N} \subset [N, 2N]$ so that

$$\sum_{f \in \mathcal{F}} \Big| \sum_{n \in \mathcal{N}} \lambda_f(n) \Big| \approx |\mathcal{F}| \cdot |\mathcal{N}|.$$

  Example: If $\mathcal{F}$ is the set of primitive quadratic characters, and $\mathcal{N}$ is the set of squares, then $|\mathcal{N}| \approx \sqrt{N}$ is a large biased set.

More reasons for caution

There is recent work of Dunn-Radziwiłł exhibiting subtle bias for families of cubic characters.

An Eisenstein series family

The newform Eisenstein series of level Q and trivial central character occur when $Q = q^2$, and are induced by a primitive Dirichlet character $\chi \pmod{q}$.

The n-th Fourier coefficient (or Hecke eigenvalue) of such an Eisenstein series is of the form

$$\lambda_{\chi,it}(n) = \sum_{ab=n} \chi(a)\overline{\chi}(b)(a/b)^{it}. \tag{1}$$

Large sieve

Define

$$\Delta(Q,N) = \max_{|\alpha|=1} \sum_{Q/2<q\leq Q} \sum_{\chi \pmod q}^{*} \Big| \sum_{N/2<n<N} \alpha_n \lambda_{\chi,0}(n) \Big|^2.$$

The classical large sieve inequality implies

$$\Delta(Q,N) \ll Q^2\sqrt{N} + N \log N.$$

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To see this, use duality and apply Dirichlet's hyperbola method.

Biased set

This is best possible: $\lambda_{\chi,0}(p^2) = 1 + \chi^2(p) + \overline{\chi}^2(p)$, which shows the squares of primes form a biased set.

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Recall that this means

$$|\sum_{p^2 \sim P} \lambda_{\chi,0}(p^2)| \approx \frac{\sqrt{P}}{\log P},$$

on average over χ.

Fixed definition

Trying again, define:

$$\Delta(Q,N) = \max_{|\alpha|=1} \sum_{Q/2<q\leq Q} \sideset{}{^*}\sum_{\chi \pmod q} \Bigg| \sum_{\substack{N/2<ab<N \\ (a,b)=1}} \alpha_{a,b}\chi(a)\overline{\chi}(b)\Bigg|^2.$$

Theorem (Y., 2022)

$$\Delta(Q,N) \ll (Q^2+N)^{1+\varepsilon}.$$

Reinterpretation: Rational large sieve

- Let $\mathbb{Q}_q = \{x \in \mathbb{Q} : v_p(x) \geq 0 \text{ for all } p|q\}$. (A ring)
- Let $\mathrm{red}_q : \mathbb{Q}_q \to \mathbb{Z}/q\mathbb{Z}$. (A ring hom.)
- For $x \in \mathbb{Q}_q$, define $\chi(x) = \chi(\mathrm{red}_q(x))$.
- For $a/b \in \mathbb{Q}_q$, define $\mathrm{ht}(a/b) = |ab|$.

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Then

$$\sum_{q \leq Q} \sideset{}{^*}\sum_{\chi \pmod{q}} \Bigg| \sum_{\substack{n \in \mathbb{Q}_q \\ \mathrm{ht}(n) \leq N}} \alpha_n \chi(n) \Bigg|^2 \ll (Q^2 + N)^{1+\varepsilon} |\alpha|^2.$$

Sieving application

- Let $\mathcal{N} = \{n \in \mathbb{Q}_{>0} : \mathrm{ht}(n) \leq N\}$.
- Let $\mathcal{P}$ be a finite set of primes.
- For each $p \in \mathcal{P}$, let $\Omega_p \subset \mathbb{Z}/p\mathbb{Z}$.
- Let $\omega(p) = |\Omega_p|$, and $h(p) = \frac{\omega(p)}{p-\omega(p)}$, extended multiplicatively to squarefrees.
- Let

$$\mathcal{S}(\mathcal{N}, \mathcal{P}, \Omega) = \{n \in \mathcal{N} : \text{for all } p,\ \mathrm{red}_p(n) \notin \Omega_p\}$$

Then

$$|\mathcal{S}(\mathcal{N}, \mathcal{P}, \Omega)| \ll \frac{(N + Q^2)^{1+\varepsilon}}{H}, \quad H = \sum_{q \leq Q} h(q).$$

Example

Take $|\Omega_p| = \frac{p-1}{2}$ for all $p \leq Q$, with $Q = \sqrt{N}$. Then

$$|\mathcal{S}| \ll N^{1/2+\varepsilon}.$$

Overview of proof

The proof relies on three results, each with a very different proof.

The simplest of these three is monotonicity:

Proposition

If $Q' \gg Q \log QN$ *then*

$$\Delta(Q,N) \ll \Delta(Q',N).$$

If $N' \gg Q \log QN$ *then*

$$\Delta(Q,N) \ll \Delta(Q,N').$$

Monotonicity

Idea for N-monotonicity:

$$\sum_{q,\chi}\Big|\sum_{a,b}\alpha_{a,b}\chi(a)\overline{\chi}(b)\Big|^2 = \frac{1}{P^*}\sum_{p\sim P}\sum_{q,\chi}\Big|\sum_{a,b}\alpha_{a,b}\chi(a)\overline{\chi}(b)\Big|^2$$
$$\approx \frac{1}{P^*}\sum_{p\sim P}\sum_{q,\chi}\Big|\sum_{a,b}\alpha_{a,b}\chi(ap)\overline{\chi}(b)\Big|^2.$$

Now ap can be glued into a new variable, making the inner sum have length PN.

This kind of idea was apparently first used in the context of the large sieve in a paper of Forti and Viola (1973).

Functional equation

Trying the functional equation:

$$\sum_{(a,b)=1} w\left(\frac{ab}{N}\right)\Big|\sum_{q,\chi}\beta_\chi\chi(a)\overline{\chi}(b)\Big|^2$$

$$= \sum_{\substack{q_1,\chi_1\\ q_2,\chi_2}} \beta_{\chi_1}\beta_{\chi_2} \sum_{(a,b)=1} w\left(\frac{ab}{N}\right)\chi_1\overline{\chi_2}(a)\overline{\chi_1}\chi_2(b)$$

$$= \frac{1}{2\pi i}\int_{(2)} \sum_{\substack{q_1,\chi_1\\ q_2,\chi_2}} \beta_{\chi_1}\beta_{\chi_2}\widetilde{w}(s)N^s\frac{L(s,\chi_1\overline{\chi_2})L(s,\chi_2\overline{\chi_1})}{\zeta(2s)}ds.$$

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The condition $(a,b)=1$ is necessary to avoid the biased set, but it leads to the highly problematic $\zeta(2s)$ in the denominator.

Functional equation

A compromise:

$$S := \sum_{(a,b)=1} w\left(\frac{ab}{N}\right)\Big|\sum_{q,\chi}\beta_\chi\chi(a)\overline{\chi}(b)\Big|^2$$

$$\leq \sum_{\frac{ab}{(a,b)^2}>Y} w\left(\frac{ab}{N}\right)\Big|\sum_{q,\chi}\beta_\chi\chi(a)\overline{\chi}(b)\Big|^2 =: S_{>Y}.$$

Here $Y < N/100$ is a parameter to be chosen later.

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Here $Y < N/100$ is a parameter to be chosen later.

This type of upper bound sieve was used by Heath-Brown in his proof of the quadratic large sieve.

Splitting

Inclusion-exclusion: $S_{>Y} = S_\infty - S_{\leq Y}$.

$$S_{\leq Y} = \sum_{\frac{ab}{(a,b)^2} \leq Y} w\left(\frac{ab}{N}\right) \Big| \sum_{q,\chi} \beta_\chi \chi(a) \overline{\chi}(b) \Big|^2$$

$$\approx \int_{(2)} \widetilde{w}(s) \zeta(2s) \sum_{\substack{ab \leq Y \\ (a,b)=1}} \frac{N^s}{(ab)^s} \Big| \sum_{q,\chi} \beta_\chi \chi(a) \overline{\chi}(b) \Big|^2 \frac{ds}{2\pi i}.$$

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Now shift contours to the line $\mathrm{Re}(s) = 0$.

Good news/bad news

Bad news: We get a pole at $s = 1/2$ of shape

$$S_{\leq Y}^{\text{pole}} = \tfrac{1}{2}\widetilde{w}(1/2) \sum_{\substack{ab \leq Y \\ (a,b)=1}} \frac{N^{1/2}}{(ab)^{1/2}} \Big| \sum_{q,\chi} \beta_\chi \chi(a) \overline{\chi}(b) \Big|^2,$$

which is not small on its own.

Good news: The contribution on the line 0 is at most $\Delta(Q, Y)|\beta|^2$, and Y is small compared to N.

Functional equation

$$S_\infty = \sum_{a,b\geq 1} w\left(\frac{ab}{N}\right)\Big|\sum_{q,\chi}\beta_\chi \chi(a)\overline{\chi}(b)\Big|^2$$

$$= \frac{1}{2\pi i}\int_{(2)} \sum_{\substack{q_1,\chi_1\\ q_2,\chi_2}} \beta_{\chi_1}\beta_{\chi_2}\widetilde{w}(s)N^s L(s,\chi_1\overline{\chi_2})L(s,\chi_2\overline{\chi_1})ds.$$

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Problem: χ_1 and χ_2 are primitive, but $\chi_1\overline{\chi_2}$ may not be, which affects the functional equation.

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Problem: χ_1 and χ_2 are primitive, but $\chi_1\overline{\chi_2}$ may not be, which affects the functional equation.

To make life easier, let's pretend either $\chi_1 = \chi_2$, or that $(q_1, q_2) = 1$. The diagonal $\chi_1 = \chi_2$ is easy to understand, and gives $O(N|\beta|^2)$.

Functional equation

Changing s to $1-s$, and then the functional equation gives:

$$\int_{(2)} \sum_{\substack{q_1,\chi_1 \\ q_2,\chi_2}} \beta_{\chi_1}\beta_{\chi_2}\widetilde{w}(1-s)N^{1-s}L(1-s,\chi_1\overline{\chi_2})L(1-s,\chi_2\overline{\chi_1})\frac{ds}{2\pi i}$$

$$\approx \int_{(2)} \sum_{\substack{q_1,\chi_1 \\ q_2,\chi_2}} \beta_{\chi_1}\beta_{\chi_2}\widetilde{w}(1-s)\frac{N^{1-s}}{(q_1^2q_2^2)^{\frac{1}{2}-s}}L(s,\chi_1\overline{\chi_2})L(s,\chi_2\overline{\chi_1})\frac{ds}{2\pi i}$$

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The new sum can be truncated at $ab \ll \frac{Q^4}{N}$.

Functional equation (cont.)

We need $(a, b) = 1$ again, so pull out the gcd here. This forms yet another $\zeta(2s)$, giving rise to

$$\int_{(2)} \sum_{\substack{(a,b)=1 \\ ab \ll Q^4/N}} \sum_{\substack{q_1,\chi_1 \\ q_2,\chi_2}} \beta_{\chi_1}\beta_{\chi_2}\widetilde{w}(1-s)\zeta(2s)\frac{N^{1-s}}{(q_1^2 q_2^2)^{\frac{1}{2}-s}}\frac{\chi_1(a\bar{b})\chi_2(b\bar{a})}{(ab)^s}\frac{ds}{2\pi i}.$$

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We move the contour back to $\mathrm{Re}(s) = 0$, crossing a pole at $s = 1/2$. If we take $Y = Q^4/N$, then this polar term cancels the polar term $S_{\leq Y}^{\text{pole}}$. The new integral part is bounded by

$$\frac{N}{Q^2}\Delta(Q, \frac{Q^4}{N}).$$

Summary

Theorem (Functional equation)

$$\Delta(Q, N) \ll N + \frac{N}{Q^2}\Delta(Q, \frac{Q^4}{N}).$$

This is good if $N \gg Q^2$.

The dual side

The third main tool works on the dual side, gives:

Theorem (Family sum)

$$\Delta(Q, N) \ll Q^2 + \frac{Q^2}{N}\Delta(N/Q, N).$$

This is good if $Q^2 \gg N$.

The dual side

Trying orthogonality:

$$\sum_{q} w(q/Q) \sum_{\substack{\chi \pmod{q} \\ \chi \text{ primitive}}} \left| \sum_{(a,b)=1} \alpha_{a,b}\chi(a)\overline{\chi}(b) \right|^2$$

$$= \sum_{\substack{(a_1,b_1)=1 \\ (a_2,b_2)=1}} \alpha_{a_1,b_1}\alpha_{a_2,b_2} \sum_{q} w(q/Q) \sum_{\substack{d|q \\ d|a_1b_2-a_2b_1}} \varphi(d)\mu(q/d)$$

$$= \frac{1}{2\pi i}\int Q^s \frac{\widetilde{w}(s)}{\zeta(s)} \sum_{\substack{(a_1,b_1)=1 \\ (a_2,b_2)=1}} \alpha_{a_1,b_1}\alpha_{a_2,b_2} \sum_{d|a_1b_2-a_2b_1} \frac{\varphi(d)}{d^s}\, ds.$$

A compromise

$$S := \sum_q w(q/Q) \sum_{\substack{\chi \pmod q \\ \chi \text{ primitive}}} \Big| \sum_{(a,b)=1} \alpha_{a,b}\chi(a)\overline{\chi}(b)\Big|^2$$

$$\leq \sum_q w(q/Q) \sum_{\substack{\chi \pmod q \\ \mathrm{cond}(\chi) > Y}}^{*} \Big| \sum_{(a,b)=1} \alpha_{a,b}\chi(a)\overline{\chi}(b)\Big|^2 := S_{>Y}.$$

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Next write $S_{>Y} = S_\infty - S_{\leq Y}$.

Truncated part

For $S_{\leq Y}$, write $\chi \to \chi\chi_0$, $q \to qq_0$ where χ_0 is trivial modulo q_0, χ is primitive modulo q, and $q \leq Y$. The sum over q_0 forms a zeta, giving

$$S_{\leq Y} \approx \frac{1}{2\pi i} \int_{(2)} \widetilde{w}(s) \sum_{q \leq Y} \sideset{}{^*}\sum_{\chi \pmod{q}} \left(\frac{Q}{q}\right)^s \zeta(s) \Big| \sum_{(a,b)=1} \alpha_{a,b} \chi(a) \overline{\chi}(b) \Big|^2.$$

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Again, we want to shift contours to the line $\mathrm{Re}(s) = 0$, but there is a pole at $s = 1$, giving

$$S_{\leq Y}^{\text{pole}} = \widetilde{w}(1) \sum_{q \leq Y} \sideset{}{^*}\sum_{\chi \pmod q} \frac{Q}{q} \Big| \sum_{(a,b)=1} \alpha_{a,b} \chi(a) \overline{\chi}(b) \Big|^2.$$

Truncated part

The part on the new line of integration is bounded by

$$\Delta(Y, N).$$

Extended part

$$S_\infty = \sum_q w(q/Q) \sum_{\chi \pmod{q}} \Big| \sum_{(a,b)=1} \alpha_{a,b}\chi(a)\overline{\chi}(b)\Big|^2$$

$$\approx Q \sum_{\substack{(a_1,b_1)=1\\(a_2,b_2)=1}} \alpha_{a_1,b_1}\alpha_{a_2,b_2} \sum_{q|a_1b_2-a_2b_1} w_1(q/Q).$$

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Important point: The diagonal $a_1 b_2 = a_2 b_1$ forces $a_1 = a_2$ and $b_1 = b_2$, using $(a_1, b_1) = (a_2, b_2) = 1$.

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Now the idea is to use a divisor-switching trick and convert back to Dirichlet characters. This idea was inspired by Conrey-Iwaniec-Soundararajan's paper on the asymptotic large sieve.

Divisor switch

Write

$$a_1 b_2 - a_2 b_1 = qr,$$

where now $0 < |r| \ll \frac{N}{Q}$.

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S_∞ is roughly

$$\sum_{r \le N/Q} \sum_{\chi \pmod{r}} r^{-1} \sum_{\substack{(a_1,b_1)=1 \\ (a_2,b_2)=1}} \alpha_{a_1,b_1} \alpha_{a_2,b_2} \chi(a_1 b_2 \overline{a_2 b_1}) w_1\left(\frac{a_1 b_2 - a_2 b_1}{Qr}\right).$$

Final steps

To complete the circuit, we need to get back to primitive characters. Write $\chi \to \chi\chi_0$, $r \to rr_0$. The r_0-sum creates another zeta:

$$\int_{(2)} \widetilde{w_1}(-s)\zeta(1+s) \sum_{r \le N/Q} \sum_{\chi \pmod r}^{*} r^{-1} \sum_{\substack{(a_1,b_1)=1 \\ (a_2,b_2)=1}} \alpha_{a_1,b_1}\alpha_{a_2,b_2}\chi(a_1 b_2 \overline{a_2 b_1})\left(\frac{a_1 b_2 - a_2 b_1}{Qr}\right)^s ds.$$

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Shifting contours to $\mathrm{Re}(s) = -1$ passes a pole at $s = 0$. If $Y = N/Q$, it cancels with $S^{\text{pole}}_{\le Y}$. The new line gives $\frac{Q}{N}\Delta(N/Q, N)$.

Thank you for listening!