Arithmetic Field Theory (for Elliptic Curves)

Minhyong Kim
International Centre for Mathematical Sciences, Edinburgh

ICTS Bengaluru
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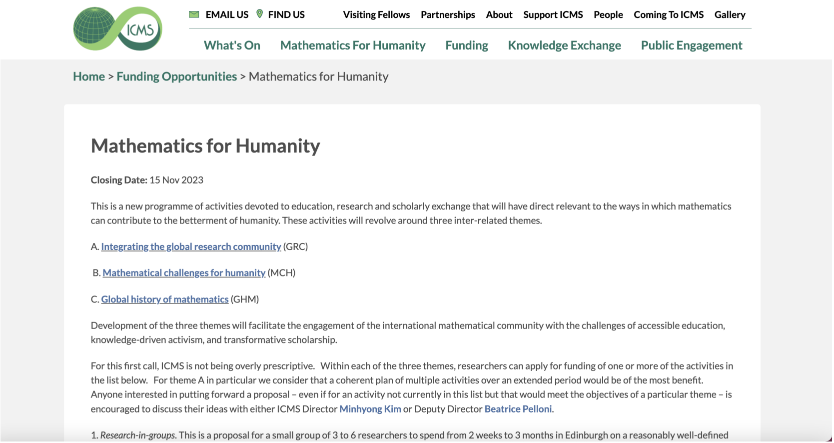
Mathematics for Humanity



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Mathematics for Humanity

Closing Date: 15 Nov 2023

This is a new programme of activities devoted to education, research and scholarly exchange that will have direct relevant to the ways in which mathematics can contribute to the betterment of humanity. These activities will revolve around three inter-related themes.

A. Integrating the global research community (GRC)

B. Mathematical challenges for humanity (MCH)

C. Global history of mathematics (GHM)

Development of the three themes will facilitate the engagement of the international mathematical community with the challenges of accessible education, knowledge-driven activism, and transformative scholarship.

For this first call, ICMS is not being overly prescriptive. Within each of the three themes, researchers can apply for funding of one or more of the activities in the list below. For theme A in particular we consider that a coherent plan of multiple activities over an extended period would be of the most benefit. Anyone interested in putting forward a proposal – even if for an activity not currently in this list but that would meet the objectives of a particular theme – is encouraged to discuss their ideas with either ICMS Director Minhyong Kim or Deputy Director Beatrice Pelloni.

1. *Research-in-groups*. This is a proposal for a small group of 3 to 6 researchers to spend from 2 weeks to 3 months in Edinburgh on a reasonably well-defined

Figure: Mathematics for Humanity, a project of the ICMS Edinburgh

Mathematics for Humanity

post-graduate students and early career researchers.

Submission Guidelines

Scientific Committee

The programme will be overseen by a specialised Scientific Committee (a sub-committee of the ICMS Programme Comı
the assessment of all submissions and will support the Director in the selection of proposals.

John Baez (UC Riverside)
Karine Chemla (Paris)
Sophie Dabo (Lille)
Nalini Joshi (Sydney)
Reviel Netz (Stanford)
Bao Chau Ngo (Chicago and VIASM)
Raman Parimala (Emory)
Fernando Rodriguez Villegas (ICTP, Trieste)
Emily Shuckburgh (Cambridge)
Terence Tao (UCLA)

Support Mathematics for Humanity

The Mathematics for Humanity Project of the International Centre for Mathematical Sciences supports the effort by m
betterment of humanity. It accepts a wide range of proposals with concentration on three themes.

Figure: Mathematics for Humanity, a project of the ICMS Edinburgh

Mathematics for Humanity: Programme 2024

‘Exploring scaling of mass extinction events for climate tipping point modelling’, Ivan Sudakov (Open University)

‘Mathematics of voting and representation’, Ismar Volic (Wellesley College, Boston)

‘Social Justice and Economic Recovery Mathematics’, Chris Budd (Bath)

‘Compositional game theory for governance design’, Jules Hedges (Strathclyde)

‘Mathematical modelling for 21st century decisions’, Erica Thompson (LSE)

‘Algebra and geometry from Africa’, G. Sankaran (Bath)

Mathematics for Humanity: Programme 2024

‘Coupling Mathematical Modelling and Computer Modelling Approaches to Support Flood Inundation Prediction: A Case Study of Dong Hoi City, Quang Binh Province’, Nguyen Hu Du (VIASM)

‘Current research on the history of mathematics in the ancient world: new questions and new approaches’, Karine Chemla (Univ. Paris SPHERE)

‘A global history of eclipse reckoning’, D. Kent (St. Andrews)

‘Supporting the Development of Mathematical Resilience Globally’, S. Johnston-Wilder (Warwick)

‘UK-Middle East Winter School on Mathematical Physics’, Ofer Aharony (Weizmann Institute)

‘Rewilding Mathematics’, M. Singer (UCL)

I. Fields

Fields

Roughly speaking, to a physicist, everything is a field.

However, more precisely, there is a stack

$$\mathscr{S} \longrightarrow M$$

over the spacetime manifold and fields are its sections:

$$\mathscr{F} := \Gamma(M, \mathscr{S})$$

These comprise the so-called kinematics of a theory.

The ‘dynamics’ are usually formulated in terms of an *action*

$$L : \mathscr{F} \longrightarrow \mathbb{C}$$

The solutions of the Euler-Lagrange equation

$$\mathcal{S}(\mathscr{F}) := \{\phi \in \mathscr{F} \mid dL(\phi) = 0\}$$

make up the *classical state space*.

Fields

Example:

$$\mathscr{S} = T^*M \longrightarrow M$$

$$\mathscr{F} = \Omega^1(M)$$

$$L(A) = Max(A) := \int_M \|dA\|^2 dvol_M$$

In this case, the E-L equation amounts to the equation

$$*dA = 0.$$

Together with $d(dA) = 0$, we get Maxwell's equations (for the six components of dA).

Can generalise this to $\mathscr{F} = Conn(\mathscr{L})$, the space of connections on a line bundle $\mathscr{L}$.

Fields

Also of interest is

$$\mathscr{S} = (TM)^{\otimes n} \otimes (T^*M)^{\otimes m}$$

or natural subquotients.

For example, when $\mathscr{S} = Met(M) \subset (T^*M)^{\otimes 2}$,

$$EH(g) = \int_M R(g) dvol_g,$$

where $R(g)$ is the scalar curvature of g, is the *Einstein-Hilbert action*.

The E-L equation

$$dEH(g) = 0$$

is the vacuum Einstein equation.

Fields

Another important example is

$$\mathscr{F} = M \times \Sigma,$$

where Σ is another manifold.

This kind of theory is called a *sigma model*. In that case, fields are identified with maps

$$\phi : M \longrightarrow \Sigma.$$

If Σ is equipped with a metric, then

$$L(\phi) = \int_M \|d\phi\|^2 dvol_M$$

defines an action, whose critical points are called the harmonic maps from M to Σ. It's often the case that Σ is equipped with other fields that are used to define the action.

Fields

The action is typically a global integral of local functions of the fields:

$$\int_M \langle D\phi, D\phi \rangle + \text{h.o.t.}$$

But there are other important functions in field theory that are supported on subspaces $N \subset M$.

For example, if $\mathscr{A}(P)$ is the space of connections on a principal G-bundle P, then a map $K : S^1 \longrightarrow M$ together with a representation V of G determines a function

$$Wil(K, V) : \mathscr{A} \longrightarrow \mathbb{C},$$

$$Wil(K, V)(\nabla) = Tr(Hol_K(f^*(\nabla))|V)$$

called the Wilson loop function.

Fields

In quantum field theory, we are interested in integrals like

$$\int_{\mathscr{F}} e^{-\frac{i}{\hbar}L(\phi)} d\phi$$

called the *partition function*.

Also, various correlation functions like

$$\int_{\mathscr{F}} f_1(\phi) f_2(\phi) \cdots f_k(\phi) e^{-\frac{i}{\hbar}L(\phi)} d\phi.$$

These are typically ill-defined (modern Zeno's paradox), but tremendously useful guides for plausible computations and formulation of conjectures, e.g., the definition of a conformal field theory or a topological quantum field theory.

II. Arithmetic Topology

Arithmetic Topology

Let $\mathscr{O}_F$ be the ring of algebraic integers in a number field F and let

$$X := \mathrm{Spec}(\mathscr{O}_F).$$

It has many properties of a compact closed three-manifold.

If v is a maximal ideal in $\mathscr{O}_F$, then $k_v = \mathscr{O}_F/v$ is a finite field and the inclusion

$$\mathrm{Spec}(k_v) \hookrightarrow X$$

is analogous to the inclusion of a knot.

The completion $\mathrm{Spec}(\mathscr{O}_{F,v})$ is like the tubular neighbourhood of the knot.

Arithmetic Topology

The completion F_v of F is the fraction field of $\mathcal{O}_{F,v}$, so that

$$\mathrm{Spec}(F_v) = \mathrm{Spec}(\mathcal{O}_{F,v}) \setminus v$$

is like the tubular neighbourhood with the knot deleted, which should be homotopic to a torus.

If B is a finite set of primes and $\mathcal{O}_{F,B}$ is the set of B-integers, then

$$X_B := \mathrm{Spec}(\mathcal{O}_{F,B}) = \mathrm{Spec}(\mathcal{O}_F) \setminus B$$

is like a three-manifold with boundary, the boundary having one torus component $\mathrm{Spec}(F_v)$ for each prime in B.

$$\partial X_B = \coprod_{v \in B} \mathrm{Spec}(F_v) \longrightarrow X_B \hookrightarrow X.$$

Arithmetic topology: Dual Interpretation

Instead of the spaces themselves, can focus on moduli spaces

$$\mathscr{M}(X_B, R) := \{\rho : \pi_1(X_B) \longrightarrow R\}//R$$

for a p-adic Lie group R.

Then a pair (x, V), where $x \in X_B$ and V is a finite-dimensional representation of R, defines a function

$$\rho \mapsto Tr(\rho(Fr_x)|V)$$

on $\mathscr{M}(X_B, R)$, an *arithmetic Wilson loop*.

Other functions, e.g., actions?

Path integrals?

Other moduli spaces?

Arithmetic Topology and TQFT?

A 3d *arithmetic TQFT* will naturally assign a number

$$H(X)$$

to X: the value of the partition function.

A vector space

$$H(F_v)$$

to F_v: functions on the space of boundary conditions.

and a vector

$$H(X_B) \in H(B) = \otimes_{v \in B} H(F_v)$$

to X_B: function that assigns to a boundary condition the integral over fields that satisfy that condition.

III. Modular Curves

Modular curves

Can consider any scheme or stack as the target of a field theory:

$$Z \longrightarrow S$$

gives rise to

$$\mathscr{F} = Z(T)$$

$$= \left\{ \begin{array}{ccc} & & Z \\ & \nearrow & \downarrow \\ T & \longrightarrow & S \end{array} \right\}$$

So what are functions on $Z(T)$?

Modular curves

$S = \mathrm{Spec}(\mathbb{Z})$
$T = \mathrm{Spec}(\mathscr{O}_F)$ where F is an algebraic number field.
$\mathfrak{X}(1)$ compactified moduli stack of elliptic curves.
$\mathscr{F}$ = generalised elliptic curves over T.

Example of action might be the Faltings height:

$$12h_F(\mathscr{E}) := 12 \deg \omega_{\mathscr{E}/T}$$

Lemma

The sum

$$\sum_{\mathscr{E} \in \mathfrak{X}(1)(\mathbb{Z})} e^{-12h_F(\mathscr{E})}$$

converges.

Follows immediately from a theorem of Ruthi:

$$|\{E \mid 12h_F(E) < B\}| \sim Ce^{5B/6}$$

Modular curves

What about local functions?

For a prime ℓ, can consider

$$\mathscr{E} \mapsto a_\ell(\mathscr{E})$$

leading one to

$$\int_{\mathfrak{X}(1)(\mathbb{Z})} a_{\ell_1}(\mathscr{E}) a_{\ell_2}(\mathscr{E}) \cdots a_{\ell_k}(\mathscr{E}) e^{-12h(\mathscr{E})} dE$$

Lemma

This sum is absolutely convergent.

Modular curves

Functions on $\mathfrak{X}_1(p)$:

For a prime $\ell \equiv 1 \mod p$ fix

$$\mu_p(\mathbb{F}_\ell) \simeq \mu_p(\mathbb{C}).$$

Get a function of $x \in \mathscr{E}[p](\mathbb{F}_\ell)$ via

$$t_\ell(x) := \langle \delta(x)(Fr_\ell), x \rangle \in \mu_p(\mathbb{C})$$

Here, $\delta(x) \in H^1(\mathbb{F}_\ell, \mathscr{E}[p])$, so that $\delta(x)(Fr_\ell) \in \mathscr{E}[p](\bar{\mathbb{F}}_\ell)$. Depends on choice of cocyle representative, but the Weil pairing $\langle \delta(x)(Fr_\ell), x \rangle$ does not.

Modular curves

Thus, get a local function on $\mathfrak{X}_1(p)$ and can try to compute

$$\int_{\mathfrak{X}(p)_1(\mathbb{Z})} t_{\ell_1}(\mathscr{E}) t_{\ell_2}(\mathscr{E}) \cdots t_{\ell_k}(\mathscr{E}) e^{-12h(\mathscr{E})} d\mathscr{E}$$

Modular curves

Also interesting to consider the moduli stack of curves of genus 1,

$$\mathscr{M}_1,$$

admitting maps

$$\mathfrak{X}(1) \longrightarrow \mathscr{M}_1 \longrightarrow \mathfrak{X}(1).$$

Consider

$$\mathscr{F} := (\mathscr{M}_1 \times_{\mathfrak{X}(1)} \mathscr{M}_{1,2})(\mathbb{Z}).$$

Given $\phi = (C, \mathscr{E}, x)$, where C is an $\mathscr{E}$-torsor and $x \in \mathscr{E}(\mathbb{Z})$, for any ℓ of good reduction, we then have a local Tate pairing

$$u_\ell(\phi) := \langle C, x \rangle \in \mathbb{Q}/\mathbb{Z}.$$

Modular curves

Compute

$$\int_{\mathscr{F}} e^{2\pi i u_{\ell_1}(\phi)} e^{2\pi i u_{\ell_2}(\phi)} \cdots e^{2\pi i u_{\ell_k}(\phi)} e^{-12h(\phi)} d\phi \qquad ?$$

IV. Some more examples of arithmetic actions

Arithmetic Actions

For technical reasons, we will assume throughout that F is totally complex.

Would like to define

$$S : \mathscr{M}(X_B, R) = H^1(\pi_1(X_B), R) \longrightarrow K$$

as well as path integrals

$$\int_{\rho \in \mathscr{M}(X_B, R)} \exp(-S(\rho)) d\rho$$

possibly also on more general fields and/or related moduli spaces.

Arithmetic Duality

Let μ_n be the n-th roots of 1. Then

$$H^3(X, \mu_n) = H^3(\mathrm{Spec}(\mathscr{O}_F), \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

This follows from

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{(\cdot)^n} \mathbb{G}_m \longrightarrow 1,$$

leading to

$$H^3(X, \mu_n) \simeq H^3(X, \mathbb{G}_m)[n].$$

Meanwhile

$$H^3(X, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}.$$

Arithmetic Duality

Local class field theory:

$$H^2(F_v, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}$$

Global class field theory:

$$0 \longrightarrow H^2(F, \mathbb{G}_m) \xrightarrow{\text{loc}} \oplus_v H^2(F_v, \mathbb{G}_m) \xrightarrow{\sum} \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

$$0 \longrightarrow H^2(X_B, \mathbb{G}_m) \xrightarrow{\text{loc}_B} \oplus_{v \in B} H^2(F_v, \mathbb{G}_m) \xrightarrow{\sum} \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

But

$$\oplus_{v \in B} H^2(F_v, \mathbb{G}_m) = H^2(\partial X_B, \mathbb{G}_m),$$

so that

$$coker(\text{loc}_B) \simeq H^3_c(X_B, \mathbb{G}_m) \simeq H^3(X, \mathbb{G}_m).$$

Finite Arithmetic Chern-Simons Functionals

Assume $\mu_n \subset F$. Then

$$H^3(X, \mathbb{Z}/n) \simeq H^3(X, \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

so we get a map

$$\text{inv} : H^3(\pi_1(X), \mathbb{Z}/n) \longrightarrow H^3(X, \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

Let R have trivial $\pi_1(X)$-action. On the moduli space

$$\mathscr{M}(X, R) = \text{Hom}(\pi_1(X), R)//R,$$

of continuous representations of $\pi_1(X)$, a Chern-Simons functional is defined as follows.

Finite Arithmetic Chern-Simons Functionals

The functional will depend on the choice of a cohomology class (a level)

$$c \in H^3(R, \mathbb{Z}/n).$$

Then

$$CS_c : \mathscr{M}(X, R) \longrightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

is defined by

$$\rho \mapsto \rho^*(c) \in H^3(\pi_1(X), \mathbb{Z}/n) \mapsto \mathrm{inv}(\rho^*(c)).$$

Finite Arithmetic Chern-Simons Functionals

Example:

Let $R = \mathbb{Z}/n$. Then

$$\mathscr{M}_X = Hom(\pi_1(X), \mathbb{Z}/n) = H^1_{et}(X, \mathbb{Z}/n).$$

Take $c \in H^3(R, \mathbb{Z}/n)$ to be given as

$$a \cup \delta a,$$

where $a \in H^1(R, \mathbb{Z}/n) = \mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}/n)$ is the class coming from the identity map, while

$$\delta : H^1(R, \mathbb{Z}/n) \longrightarrow H^2(R, \mathbb{Z}/n)$$

is the Bockstein map coming from the extension

$$0 \longrightarrow \mathbb{Z}/n \longrightarrow \mathbb{Z}/n^2 \longrightarrow \mathbb{Z}/n \longrightarrow 0.$$

Then

$$CS_{a \cup \delta a}(\rho) = \mathrm{inv}(\rho^*(a) \cup \rho^*(\delta a)).$$

BF-theory

Have a function

$$H^1(X, V) \times H^1(X, D(V)) \xrightarrow{BF} \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

defined by

$$(a, b) \mapsto \mathrm{inv}(da \cup b)$$

For this, V is a finite n-torsion group scheme that admits a suitable Bockstein map

$$d : H^1(X, V) \longrightarrow H^2(X, V)$$

and $D(V)$ is the Cartier dual.

Variant:

$$H^1(X_B, V) \times H^1_c(X_B, D(V)) \xrightarrow{BF} \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

Remark on arithmetic differentials

The Bockstein map

$$d : H^1(X, \mathbb{Z}/n) \longrightarrow H^2(X, \mathbb{Z}/n)$$

is very much like a differential. In crystalline cohomology of varieties over perfect fields of positive characteristic, Bockstein maps on crystalline cohomology sheaves are used to construct the De Rham-Witt complex.

In general, whenever you have an extension

$$0 \longrightarrow V \longrightarrow E \longrightarrow V \longrightarrow 0,$$

there is a differential

$$H^1(X, V) \longrightarrow H^2(X, V)$$

that can be used to construct arithmetic functionals.

More general differentials arise from deformation theory.

V. Arithmetic Path Integrals

Arithmetic Chern-Simons

[Joint work with H. Chung, D. Kim, G. Pappas, J. Park, H. Yoo]

Let $n = p$, a prime and assume the Bockstein map

$$d : H^1(X, \mathbb{Z}/p) \longrightarrow H^2(X, \mathbb{Z}/p)$$

is an isomorphism.

Then

$$\sum_{\rho \in H^1(X, \mathbb{Z}/p)} \exp[2\pi i CS(\rho)]$$

$$= \sqrt{|Cl_X[p]|} \left(\frac{\det(d)}{p} \right) i^{\left[\frac{(p-1)^2 \dim(Cl_X[p])}{4}\right]}.$$

Arithmetic BF-theory: [Joint work with Magnus Carlson]

$$BF : H^1(X, \mu_n) \times H^1(X, \mathbb{Z}/n) \longrightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

$$(a, b) \mapsto \mathrm{inv}(da \cup b).$$

Proposition

For $n >> 0$,

$$\sum_{(a,b)\in H^1(X,\mu_n)\times H^1(X,\mathbb{Z}/n)} \exp(2\pi i BF(a, b))$$

$$= |Cl_X[n]||\mathcal{O}_X^\times/(\mathcal{O}_X^\times)^n|.$$

Compare with

$$\frac{L^{(r)}(Triv, 0)}{r!} = -|Cl_X|\|\det(\mathcal{O}_F^\times)\|$$

Arithmetic BF-theory

Similarly, if E is an elliptic curve with Neron model $\mathscr{E}$, then we have

$$0 \longrightarrow \mathscr{E}[n] \longrightarrow \mathscr{E}[n^2] \longrightarrow \mathscr{E}[n] \longrightarrow 0$$

for n coprime to the conductor and the orders of component groups of $\mathscr{E}$.

This gives us a map

$$BF : H^1(X, \mathscr{E}[n]) \times H^1(X, \mathscr{E}[n]) \longrightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

as

$$(a, b) \longrightarrow \mathrm{inv}(da \cup b).$$

Arithmetic BF-theory

Proposition

Assume Sha(E) is finite. For $n >> 0$ as above,

$$\sum_{(a,b)\in H^1(X,\mathscr{E}[n])\times H^1(X,\mathscr{E}[n])} \exp(2\pi i BF(a,b))$$

$$= |Ш(A)[n]||E(F)/n|^2.$$

Compare

$$\frac{L^{(r)}(T_pE,0)}{r!} = (\prod_v c_v)|Ш_E|\|\det(E(F))\|^2$$

Chern-Simons Theory for Elliptic Curves

For $a \in H^1(X, \mathscr{E}[p])$, define

$$CS(a) := BF(a, a).$$

This is a mod p version of the p-adic height.

Local operators: Let $\ell \equiv 1 \mod p$ a prime of good reduction and $y \in \mathscr{E}(\mathbb{F}_\ell)$, define

$$O_{\ell,y} : H^1(X, \mathscr{E}[p]) \longrightarrow \mu_p$$

as

$$O_{\ell,y}(a) := \langle a \mod \ell, y \rangle$$
$$= (a(Fr_\ell), y),$$

where the last bracket is the Weil pairing.

Chern-Simons Theory for Elliptic Curves

$$\sum_{a \in H^1(X, \mathscr{E}[p])} O_{\ell_1, y_1}(a) O_{\ell_2, y_2}(a) \cdots O_{\ell_k, y_k}(a) \exp(2\pi i CS(a)) = ?$$

VI. Chern-Simons with Boundaries

Finite Arithmetic Chern-Simons Functionals with Boundaries

$X_B = \mathrm{Spec}(\mathscr{O}_F[1/B])$ for a finite set B of primes;

$\partial X_B = \coprod_{v \in B} \mathrm{Spec}(F_v)$.

$$\pi_1(X_B) := \mathrm{Gal}(F_B^{un}/F), \quad \pi_v := \mathrm{Gal}(\bar{F}_v/F_v),$$

and fix a tuple of homomorphisms

$$i_S = (i_v : \pi_v \longrightarrow \pi_1(X_B))_{v \in B}$$

corresponding to embeddings $\bar{F} \hookrightarrow \bar{F}_v$.

Assume B contains all places dividing n.

Finite Arithmetic Chern-Simons Functionals with Boundaries

In addition to the global moduli space

$$\mathscr{M}(X_B, R) = \mathrm{Hom}(\pi_1(X_B), R)//R$$

we have the local moduli space

$$\mathscr{M}(\partial X_B, R) := \{\phi_B = (\phi_v)_{v \in B} \mid \phi_v : \pi_v \longrightarrow R\}//R$$

Thus, we get a localisation map

$$\mathrm{loc}_B = i_B^* : \mathscr{M}(X_B, R) \longrightarrow \mathscr{M}(\partial X_B, R)$$

Finite Arithmetic Chern-Simons Functionals with Boundaries

Key cohomological facts:

$$H^2(\pi_v, \mathbb{Z}/n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

$H^i(\pi_v, \mathbb{Z}/n) = 0$ for $i > 2$.

There is a symplectic non-degenerate pairing

$$H^1(\pi_v, \mathbb{Z}/n) \times H^1(\pi_v, \mathbb{Z}/n) \longrightarrow H^2(\pi_v, \mathbb{Z}/n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

There is an exact sequence

$$0 \longrightarrow H^1(X_B, \mathbb{Z}/n) \longrightarrow \prod_{v \in B} H^1(\pi_v, \mathbb{Z}/n) \xrightarrow{\sum} \frac{1}{n}\mathbb{Z}/\mathbb{Z} \longrightarrow 0.$$

Finite Arithmetic Chern-Simons Functionals with Boundaries

Now $c \in Z^3(R, \mathbb{Z}/n)$ will denote a 3-cocycle.

For any $\phi_B = (\phi_v)$, each $\phi_v^*(c) \in Z^3(\pi_v, \mathbb{Z}/n)$ is trivial. Thus,

$$\mathscr{T}_v := d^{-1}(\phi_v^*(c)) \in C^2(\pi_v, \mathbb{Z}/n)/B^2(\pi_v, \mathbb{Z}/n)$$

is a torsor for $H^2(\pi_v, \mathbb{Z}/n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}$.

Hence,

$$\prod_{v \in B} \mathscr{T}_v$$

is a torsor for

$$\prod_{v \in B} H^2(\pi_v, \mathbb{Z}/n) \simeq \prod_{v \in B} \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

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Finite Arithmetic Chern-Simons Functionals with Boundaries

We push this out using the sum map

$$\Sigma : \prod_{v \in B} \frac{1}{n}\mathbb{Z}/\mathbb{Z} \longrightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

to get a $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsor

$$\mathscr{T}(\phi_B) := \Sigma_*(\prod_v d^{-1}(\phi_v)).$$

As ϕ_B varies, we get a $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsor

$$\mathscr{T} \longrightarrow \mathscr{M}(\partial X_B, R)$$

over the local moduli space.

Finite Arithmetic Chern-Simons Functionals with Boundaries

Can use the map

$$\exp 2\pi i : \frac{1}{n}\mathbb{Z}/\mathbb{Z} \longrightarrow S^1.$$

to push $\mathscr{T}$ out to a unitary line bundle $\mathscr{U}$ over $\mathscr{M}(\partial X_B, R)$ and define

$$H_{CS}(B) := \Gamma(\mathscr{M}(\partial X_B), R), \mathscr{U})$$

This is the Hilbert space associated by finite arithmetic CS theory to B.
Should define

$$H_{CS}(X_B) \in H_{CS}(B).$$

Finite Arithmetic Chern-Simons Functionals with Boundaries

If $\rho \in \mathscr{M}(X_B, R)$, because $H^3(\pi_1(X_B), \mathbb{Z}/n) = 0$, we can solve

$$d\beta = \rho^*(c) \in Z^3(\pi_1(X_B), \mathbb{Z}/n),$$

and put

$$\mathbb{CS}(\rho) = \Sigma_*(\mathrm{loc}_B(\beta)) \in \mathscr{T}_{\mathrm{loc}_{B(\rho)}}.$$

Lemma

$\mathbb{CS}(\rho)$ *is independent of the choice of* β.

This follows immediately from the reciprocity sequence

$$0 \longrightarrow H^2(\pi_1(X_B), \mathbb{Z}/n) \longrightarrow \prod_{v \in B} H^2(\pi_v, \mathbb{Z}/n) \xrightarrow{\Sigma} \frac{1}{n}\mathbb{Z}/\mathbb{Z} \longrightarrow 0,$$

Finite Arithmetic Chern-Simons Functionals with Boundaries

Exponentiating, we get

$$\exp(2\pi i \mathbb{CS}(\rho)) \in \mathscr{U}_{\mathrm{loc}_B(\rho)}$$

and

$$\int_{\{\rho \mid \mathrm{loc}_B(\rho)=\rho_B\}} \exp(2\pi i \mathbb{CS}(\rho)) \in \mathscr{U}_{\rho_B}.$$

As ρ_B varies get an element

$$H_{CS}(X_B) \in H_{CS}(B).$$