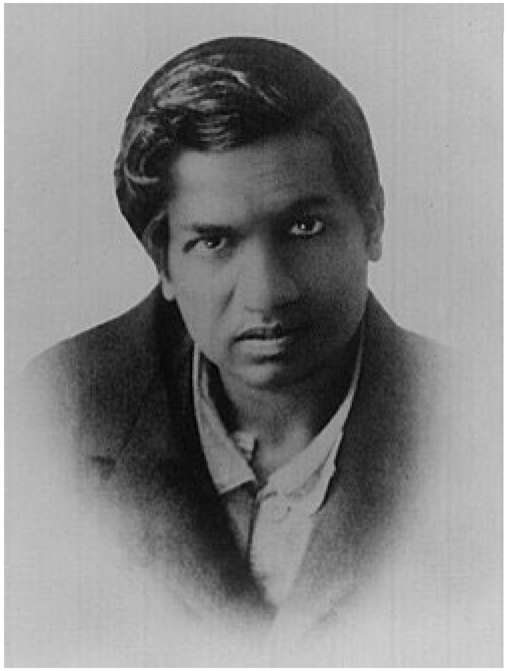
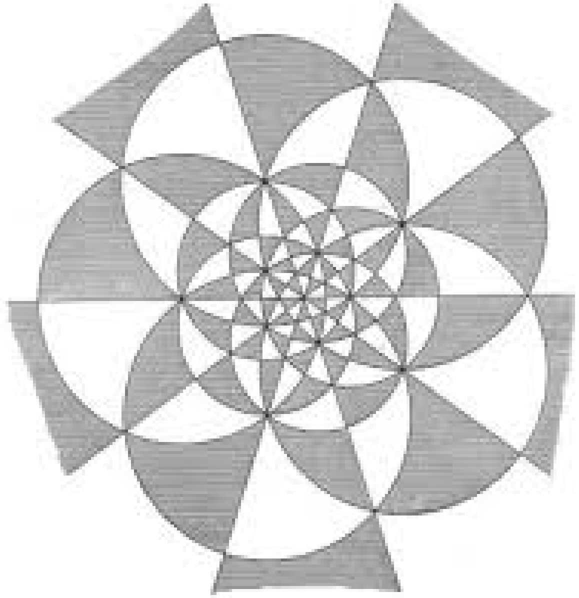
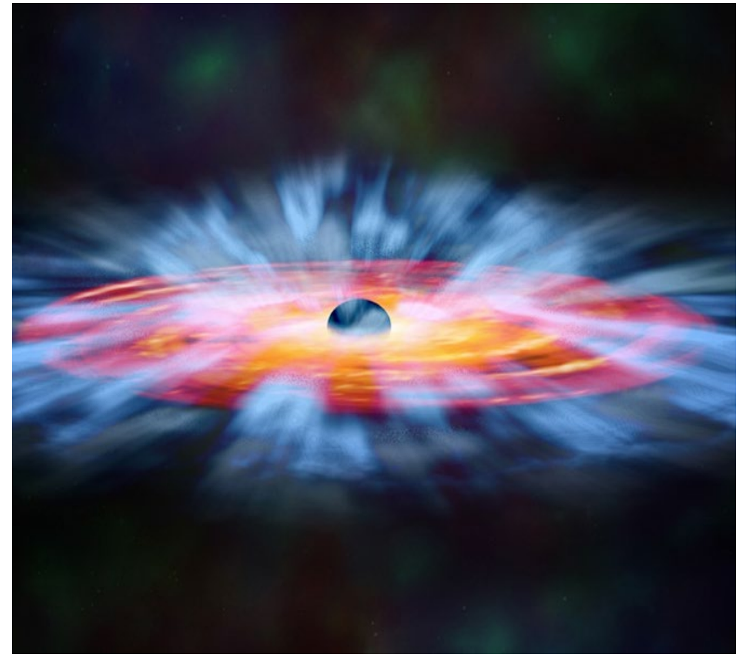




Ramanujan, Modular Forms and Beyond

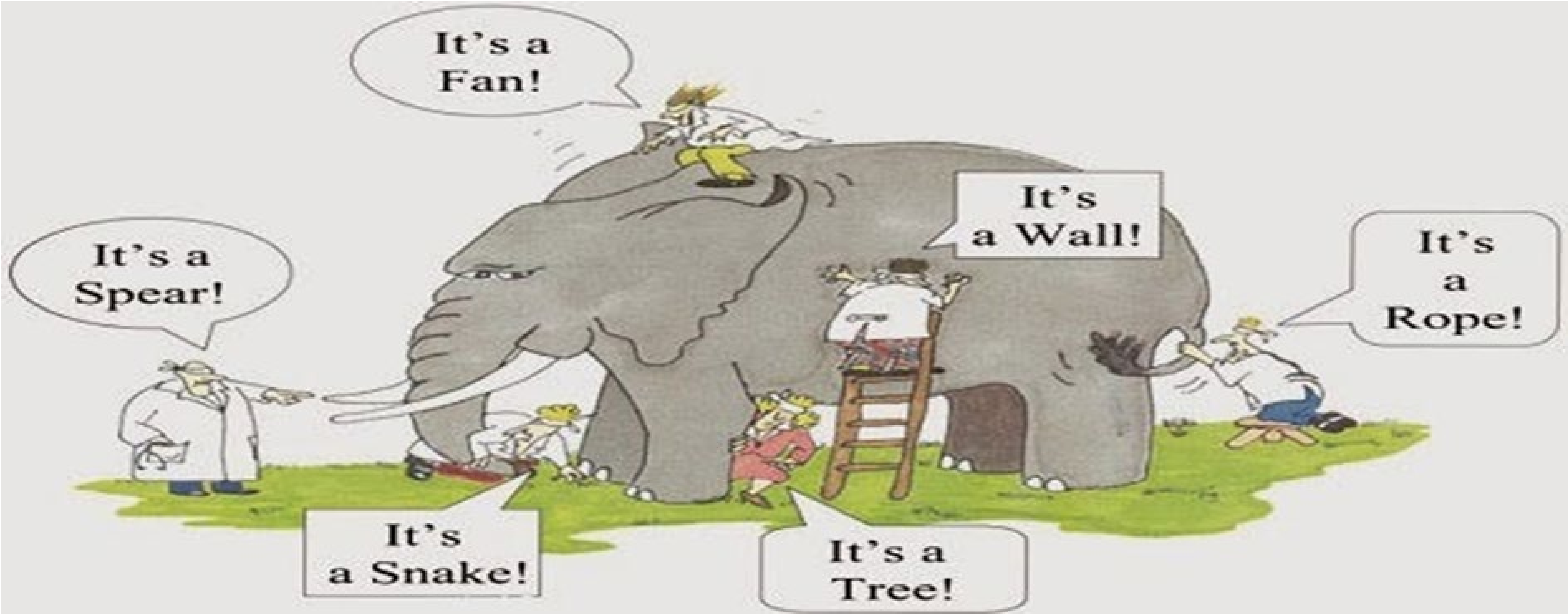
M. Ram Murty, FRSC,

Queen's Research Chair,

Queen's University, Kingston, Ontario, Canada

The Elephant in the Room

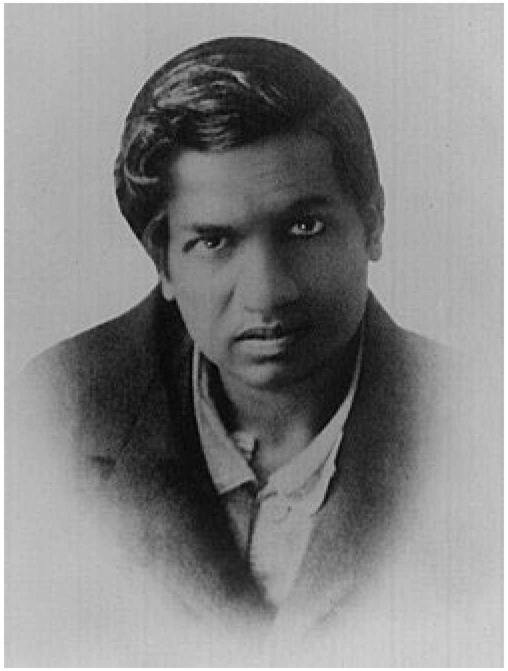
The fabled story from the Panchatantra of the blind men trying to figure out what an elephant is, applies to many situations in human history and particularly to the process of scientific discovery, especially as it pertains to our topic.



A Tale of Two Letters

We give an informal survey of the theories of modular forms, quasi-modular forms, and mock modular forms. These topics owe much to Ramanujan whose mystic vision of mathematics in many ways shaped 20^{th} century number theory and now continues to shape the 21st century going beyond number theory and into combinatorics, representation theory and even mathematical physics!

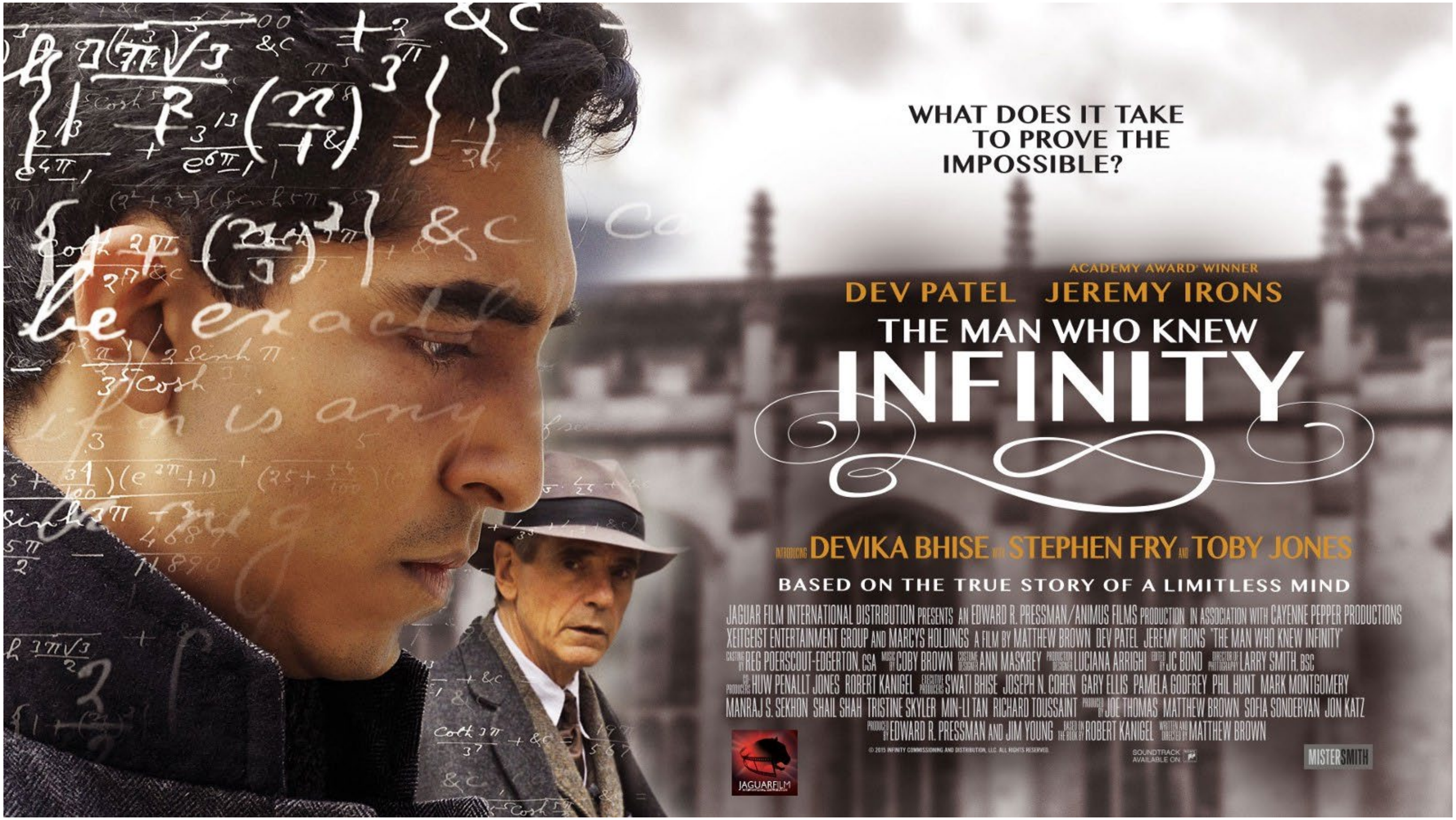
His short life of 32 years did not give him enough time to develop many of these ideas. So he formulated them only in outline but with profound intuition and insight.



Srinivasa Ramanujan (1887-1920)

This talk is literally about a tale of two letters, written by Ramanujan. The first in 1913 and the second in 1920, on his deathbed.

The man who knew infinity



The notion of a modular form

A quotation attributed to Eichler is the following "joke": there are only **five operations of mathematics**: addition, subtraction, multiplication, division and "modular forms".

If one had to point to a single mathematical idea that transformed 20th century mathematics, it would have to be the concept of a modular form.

Ramanujan did not discover this idea because it already existed in the 19th century.

But he brought it to the foreground. He connected it to number theory and made it the center piece of his attention.

His 1916 paper modestly titled "On certain arithmetical functions" had a tremendous impact on number theory of the 20th century.

It led to the development of the theory of modular forms and the now emerging theory of mock modular forms which was hastily described by Ramanujan at the end of his life.

Modular forms and the 20th century

Ramanujan's 1916 paper sowed the seeds of the theory of automorphic forms and gave rise to the **Langlands program**.

It led to an expansion of our idea of a zeta function and brought about the marriage between **number theory and algebraic geometry.**

Ramanujan's conjectures were solved using this connection.

The theory also led to the **classification of finite simple groups** using the Monster group constructed via modular forms.

It was central to the proof of **Fermat's Last Theorem** by Wiles and Taylor.

Now, **mock modular forms** are emerging to describe the birth of the cosmos in **theoretical physics**.

But let me begin the story with G.H. Hardy and his discovery of Ramanujan.

The 1936 lectures of G.H. Hardy

In 1936, G.H. Hardy was invited by Harvard University to deliver some lectures on Ramanujan and his work.

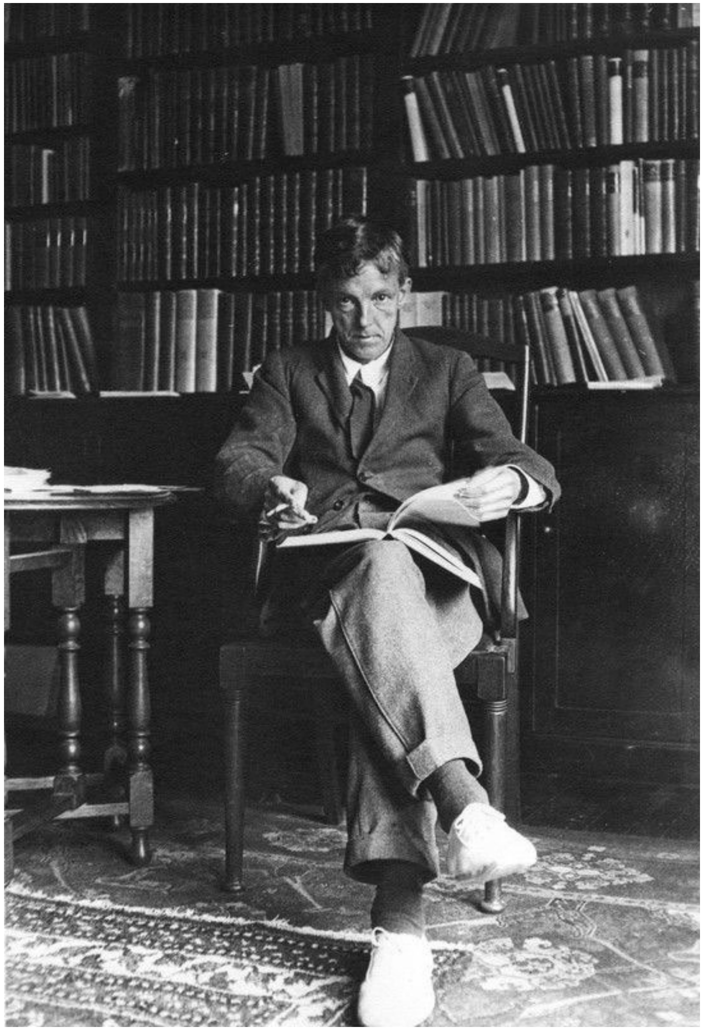
These lectures were later published as a book.

They comprise a faithful snapshot of Ramanujan along with his main contributions to mathematics.

Hardy confesses that Ramanujan was "the most romantic figure in the recent history of mathematics" and "a very great mathematician."

Ramanujan's story has all the fascination and enchantment of a fairy tale.

But as truth is sometimes stranger than fiction, it was a real-life story with enormous consequences for mathematics and more generally, for science.



Ramanujan's letter to Hardy

Hardy knew full well that he was the first mathematician to have taken Ramanujan seriously.

Earlier, Ramanujan had written to three other reputable mathematicians in England. These were Baker, Hobson and Hill.

But Hardy was the first to respond to his now famous letter of 16th January 1913.

The letter opens with the humble lines "Dear Sir. I beg to introduce myself to you as a clerk ... I have had no university education ... but I am striking out a new path for myself. I have made a special investigation of divergent series in general and the results I get are termed by the local mathematicians as 'startling'."

What was in this letter?

Two entries of this first may be of interest to a general audience. They are startling indeed!

$$1 + 2 + 3 + 4 + \cdots = -\frac{1}{12},$$
$$1^3 + 2^3 + 3^3 + 4^3 + \cdots = \frac{1}{120}.$$

Both of these results have to do with summability of divergent series. One method (but not the only one) has to do with the method of analytic continuation, which is a topic in complex analysis. The two relations above become evident if we consider the Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which converges for $\mathrm{Re}(s) > 1$. In 1859, Riemann derived an extension of $\zeta(s)$ to the entire complex plane, apart from a simple pole at $s = 1$. The two startling formulas above are the values $\zeta(-1)$ and $\zeta(-3)$. But as Hardy explains, Ramanujan knew little or no complex analysis, nor was he aware of Riemann's work. This means that he had other methods of summing divergent series which were not based on complex analysis.

Some sample pages

This is the first page of Ramanujan's letter to Hardy. After a brief introduction, it goes on for 10 pages giving theorem after theorem and formula after formula!

(9) $$\frac{1}{1^7 \cosh \frac{1}{2}\pi\sqrt{3}} - \frac{1}{3^7 \cosh \frac{3}{2}\pi\sqrt{3}} + \dots = \frac{\pi^7}{23040}.$$

(10) $$\left\{1+\left(\frac{n}{1}\right)^3\right\}\left\{1+\left(\frac{n}{2}\right)^3\right\}\left\{1+\left(\frac{n}{3}\right)^3\right\}\dots$$

can always be exactly found if n is any integer positive or negative.

(11) $$\frac{2}{3}\int_0^1 \frac{\tan^{-1} x}{x}\,dx - \int_0^{2-\sqrt{3}} \frac{\tan^{-1} x}{x}\,dx = \frac{\pi}{12}\log(2+\sqrt{3}).$$

VI. ......

(2) $$\frac{\log 1}{\sqrt{1}} - \frac{\log 3}{\sqrt{3}} + \frac{\log 5}{\sqrt{5}} - \dots = \left(\tfrac{1}{4}\pi - \tfrac{1}{2}\gamma - \tfrac{1}{2}\log 2\pi\right)\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} - \dots\right),$$

where $\gamma = \cdot 5772\dots$, the Eulerian constant.

......

(4) If $$\int_0^\infty \phi(p, x)\cos nx\,dx = \psi(p, n),$$

then $$\frac{\pi}{2}\int_0^\infty \phi(p, x)\,\phi(q, nx)\,dx = \int_0^\infty \psi(q, x)\,\psi(p, nx)\,dx.$$

(5) If $\alpha\beta = \pi$ then $$\sqrt{\alpha}\int_0^\infty \frac{e^{-x^2}}{\cosh \alpha x}\,dx = \sqrt{\beta}\int_0^\infty \frac{e^{-x^2}}{\cosh \beta x}\,dx.$$

......

VII. (1) $$1^2\log 1 + 2^2\log 2 + 3^2\log 3 + \dots + x^2\log x$$
$$= \tfrac{1}{6}x(x+1)(2x+1)\log x - \tfrac{1}{9}x^3 + \frac{1}{4\pi^2}\left(\frac{1}{1^3} + \frac{1}{2^3} + \dots\right) + \frac{x}{12} - \frac{1}{360x} + \dots.$$

......[The missing page included the formulæ headed VIII.]

IX. ......

(2) If

$$P = \frac{\Gamma\{\frac{1}{4}(x+m+n+1)\}\,\Gamma\{\frac{1}{4}(x+m-n+1)\}\,\Gamma\{\frac{1}{4}(x-m+n+3)\}\,\Gamma\{\frac{1}{4}(x-m-n+3)\}}{\Gamma\{\frac{1}{4}(x-m+n+1)\}\,\Gamma\{\frac{1}{4}(x-m-n+1)\}\,\Gamma\{\frac{1}{4}(x+m+n+3)\}\,\Gamma\{\frac{1}{4}(x+m-n+3)\}},$$

then $$\frac{1-P}{1+P} = \frac{m}{x+}\,\frac{1^2-n^2}{x+}\,\frac{2^2-m^2}{x+}\,\frac{3^2-n^2}{x+}\,\frac{4^2-m^2}{x+\dots}.$$

(3) If $$z = 1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2 x^2 + \dots,$$

and $$y = \frac{\pi}{2}\,\frac{1 + \left(\frac{1}{2}\right)^2(1-x) + \left(\frac{1.3}{2.4}\right)^2(1-x)^2 + \dots}{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2 x^2 + \dots},$$

then $$\frac{1}{(1+\alpha^2)\cosh y} + \frac{1}{(1+9\alpha^2)\cosh 3y} + \frac{1}{(1+25\alpha^2)\cosh 5y} + \dots$$
$$= \frac{1}{2}\left\{\frac{z\sqrt{x}}{1+}\,\frac{(\alpha z)^2}{1+}\,\frac{(2\alpha z)^2 x}{1+}\,\frac{(3\alpha z)^2}{1+}\,\frac{(4\alpha z)^2 x}{1+\dots}\right\},$$

α being any quantity.

......

I have got theorems on divergent series, theorems to calculate the convergent values corresponding to the divergent series, viz.

The letter contained 120 theorems!

Hardy was confused as to what to make of the letter. Was Ramanujan a crank or a genius, he wondered.

He invited his colleague J.E. Littlewood and together, they sat with the letter well past midnight.

Finally, after three hours of careful study, they concluded: "a single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true, because **if they were not true, no one would have the imagination to invent them**."

So Hardy invited Ramanujan to come to England in 1914 against the background of the First World War.

Hardy and Littlewood (circa 1920)

The circle method

One formula of this first letter is worth highlighting for this exposition. It is formula (7) in section 7:

The coefficient of x^n in $(1-2x+2x^4-2x^9+2x^{16}-\cdots)^{-1}$

$$= \text{the nearest integer to } \frac{1}{4n}\left\{\cosh(\pi\sqrt{n}) - \frac{\sinh(\pi\sqrt{n})}{\pi\sqrt{n}}\right\}.$$

Though this is not exactly correct, it is approximately true. The reciprocal of the power series appearing in the middle of the formula is a modular form of weight $-1/2$ and the last expression can be written as the derivative

$$\frac{1}{2\pi}\frac{d}{dn}\left(\frac{\sinh \pi\sqrt{n}}{\pi\sqrt{n}}\right).$$

This cannot be derived without the circle method and shows that Ramanujan had the idea before he got to England.

The Hardy-Ramanujan collaboration

There are two big discoveries that can be credited to the Hardy-Ramanujan collaboration.

The first is the normal order method which led to the development of probabilistic number theory.

This work says that the number of prime factors of a random integer n is "usually" log log n.

Their second work is the asymptotic formula for the number of partitions of n which led to the development of the circle method.

The τ-function and beyond

But it is his single-authored 1916 paper on the τ-function that is the breakthrough paper of the 20th century. This function is defined by:

$$\sum_{n\geq 1} \tau(n)q^n = q\prod_{n\geq 1}(1-q^n)^{24} = \eta(z)^{24} = \Delta(z)$$

Though Ramanujan did not invent the idea of a modular form, he can be credited with the discovery of "mock" modular forms. His last letter to G.H. Hardy, written literally on his death bed in 1920, envisaged a grand theory which only now seems to be entering center stage, not only in number theory, but also in other branches of mathematics and surprisingly theoretical physics.

Ramanujan conjectures

In 1916, Ramanujan was studying the coefficients of a certain infinite product and he observed some remarkable properties.

- This product is given by: $$q\prod_{n\geq 1}(1-q^n)^{24} = \sum_{n\geq 1}\tau(n)q^n$$

$$q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + 84480q^8 - 113643q^9 - 115920q^{10} + 534612q^{11} - 370944q^{12} - 577738q^{13} + 401856q^{14} + 1217160q^{15} + 987136q^{16} - 6905934q^{17} + 2727432q^{18} + 10661420q^{19} - 7109760q^{20} - 4219488q^{21} - 12830688q^{22} + 18643272q^{23} + 21288960q^{24} - 25499225q^{25} + 13865712q^{26} - 73279080q^{27} + 24647168q^{28} + 128406630q^{29} - 29211840q^{30} + \dots$$

Ramanujan conjectured:

τ is multiplicative: $\tau(mn)=\tau(m)\tau(n)$ whenever m and n are coprime.

τ satisfies a second order recurrence relation for prime powers.

$|\tau(p)| < 2p^{11/2}$

These conjectures led to the development of the theory of modular forms.

Ramanujan's conjectures: precise form

$$\sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

In other words, if we expand the right hand side as a power series in q, the coefficient of q^n is designated as $\tau(n)$. He made three conjectures concerning this function. They are:

(a) $\tau(mn) = \tau(m)\tau(n)$ whenever m and n are relatively prime;

(b) for each prime p, $\tau(p^{m+1}) = \tau(p)\tau(p^m) - p^{11}\tau(p^{m-1})$ for $m \geq 1$;

(c) for each prime p, $|\tau(p)| \leq 2p^{11/2}$.

The first two conjectures were proved by Mordell in 1917.

They signalled a general theory developed later by Hecke in 1935.

The third conjecture was deeper and was proved by Deligne in 1974 as a consequence of his solution to the Weil conjectures. Deligne was awarded the Fields Medal for this.

Congruences for τ-function and Galois representations

Ramanujan also observed that the τ-function satisfies some special congruences such as:

$$\tau(n) \equiv \sum_{d|n} d^{11} \pmod{691}.$$

For a long time, these congruences seemed strange until the 1970's, when Serre, Swinnerton-Dyer and Deligne developed the theory of Galois representations to explain them. In particular, Deligne proved that for each prime ℓ, there is an ℓ-adic representation

$$\rho_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Z}_\ell)$$

such that $\mathrm{tr}\rho_\ell(\sigma_p) = \tau(p)$ for all primes $p \neq \ell$ and σ_p is the Frobenius automorphism attached to p [16]. Here, $\mathbb{Z}_\ell$ denotes the ring of ℓ-adic integers. It is this representation that explains Ramanujan's congruence. It is also this representation that led Deligne to his proof of Ramanujan's third conjecture (c) in 1974.

Ramanujan's first two conjectures motivated the rapid development of the theory of modular forms largely formulated by Erich Hecke in 1935.

This is called Hecke theory today.

What is a modular form?

To understand the significance of Ramanujan's conjectures, we need to understand modular forms. Now, a century after Ramanujan, we see that if there is a single word that captures the life and work of Ramanujan, it is the word "modular". So what is a modular form?

Mathematics can be said to be the study of symmetries of objects and it is amazing that the set of such symmetries forms a group. Thus, when we look at the cosmos and see symmetries and beauty, we literally see an all pervading presence of mathematics. Mathematics is omnipresent. It also seems to be omniscient!

The set of complex numbers z with imaginary part positive is called the upper half-plane and is denoted by $\mathfrak{H}$. The group of 2 by 2 matrices

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

is called the modular group and is denoted as $SL_2(\mathbb{Z})$. This group "operates" on $\mathfrak{H}$ via (what are called) modular transformations:

$$z \mapsto \frac{az+b}{cz+d},$$

as is easily checked. In other words, $SL_2(\mathbb{Z})$ is a group of symmetries of the upper half-plane.

This group action on the upper half-plane seems to be at the heart of how the cosmos works!

The space of modular forms

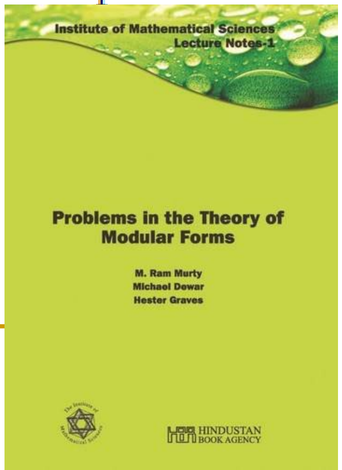
For the moment, let k be a positive integer. A holomorphic function f on $\mathfrak{H}$ is called a modular form of weight k if it satisfies

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \qquad \text{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

and it is also holomorphic at "infinity." The last condition means that f has a Fourier expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_f(n) e^{2\pi i n z},$$

and we speak of the sequence of numbers $a_f(n)$ as the Fourier coefficients of f. If f vanishes at infinity (that is, $a_f(0) = 0$), then f is called a cusp form. If we relax the condition of holomorphy at infinity to meromorphy at infinity, we obtain what is called a "weakly" holomorphic modular form of weight k. A gentle introduction to modular forms can be found in my book with Dewar and Graves and published by the Hindustan Book Agency and IMSC in their Lecture Notes Series.



The vector spaces of modular forms

The vector spaces of cusp forms, modular forms and weakly holomorphic modular forms are denoted by the symbols S_k, M_k, and $M_k^!$ respectively. We have the containments $S_k \subset M_k \subset M_k^!$. Ramanujan observed that the growth rate of Fourier coefficients in each of these spaces is different. Indeed, if we put $q = e^{2\pi i z}$ in (2), the function becomes a cusp form of weight 12 on the upper half-plane. Ramanujan's third conjecture (Deligne's theorem) predicts the growth rate of these coefficients. By contrast, the growth of Fourier coefficients of elements of $M_k^!$ which are not elements of M_k is exponential, as in the case of the j-function (to be discussed below).

Some examples may help the reader. If k is an even integer ≥ 4, we can define

$$G_k(z) = \sum_{(m,n)\neq(0,0)} (mz+n)^{-k}.$$

If we set $E_k(z) := G_k(z)/2\zeta(k)$,

then $E_k(z)$ has the Fourier expansion

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \qquad q = e^{2\pi i z},$$

Here B_k is the k-th Bernoulli number

where

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1},$$

The j-function and the Monster group

Another important function that arises from the Eisenstein series is the modular invariant called the j-function:

$$j(z) := \frac{E_4^3(z)}{\Delta(z)} = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots$$

In 1984, John McKay observed that the coefficients of the j-function were related to degrees of irreducible representations of the Monster group.

This led to the theory of "monstrous moonshine" connecting group theory and modular forms.

This study finally led to the classification of finite simple groups, a remarkable achievement of 20th century mathematics.

Ramanujan's P, Q and R

Ramanujan seems to have discovered the Eisenstein series independently and he singled out three which he called P,Q and R.

He denoted E_4 and E_6 as Q and R respectively

Of special interest is E_2 which he called P and in his 1916 paper, he noticed but did not prove that all "quasi-modular" forms are generated by P,Q and R. This was later proved formally by Kaneko and Zagier in 1995.

$$\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728}.$$

The reader will observe that the denominator is one less than the famous "taxicab number"

The taxi cab story

There is an interesting story about this number connected to Ramanujan's life. When he was ill in a sanatorium, Hardy came to visit him in a taxi.

On entering Ramanujan' s room, Hardy remarked, "I just came in a Taxi numbered 1729, which looks like a dull number. I hope it is not a bad omen."

Ramanujan replied, "On the contrary, it is very interesting. It is the smallest number that can be written as the sum of two cubes in two different ways."

$$1729 = 1^3 + 12^3 = 10^3 + 9^3.$$

Hardy then asked him if he knew the answer to the corresponding problem for fourth powers. After a moment's thought, Ramanujan replied that he "could see no obvious example and thought the first such number must be very large." Indeed, the smallest such number is 635,318,657:

$$635318657 = 134^4 + 133^4 = 158^4 + 59^4,$$

Differential equations for P,Q and R

let D be the differential operator $q\frac{d}{dq}$.

E_2 was first studied by Hurwitz in his PhD thesis.

$$\begin{aligned} DP &= \frac{1}{12}(P^2 - Q) \\ DQ &= \frac{1}{3}(PQ - R) \\ DR &= \frac{1}{2}(PR - Q^2). \end{aligned} \tag{5}$$

The significance of (5) is that it enables us to endow the set of all modular forms with an algebraic structure. This can be described as follows. It was already mentioned that $E_2(z)$ is not a modular form. However, writing $z = x + iy$ with $x, y \in \mathbb{R}$ and $y > 0$, we set

$$E_2^*(z) := E_2(z) - \frac{3}{\pi y}$$

and observe that $E_2^*(z)$ transforms like a modular form of weight two. In other words, $E_2(z)$ is holomorphic but not modular and $E_2^*(z)$ is modular but not holomorphic. The difference between the two is a "real-analytic" function and it is this sort of behaviour that is characteristic of mock modular forms that we shall see later. In fact, E_2 is the simplest example of a "mock" modular form (to be defined below).

Δ and E_2

The relationship between E_2 and Δ is given by

$$\frac{\Delta'(z)}{\Delta(z)} = 2\pi i E_2(z).$$

Derivatives of modular forms are rarely modular forms. However, Ramanujan's D operator allows us to define (what is now called the Ramanujan-Serre derivative):

$$f \mapsto Df - \frac{k}{12} E_2 f$$

which maps the space of modular forms of weight k into the space of modular forms of weight $k+2$.

Another related application of the D operator is to the construction of new modular forms. This is based on the Rankin-Cohen bracket which is defined as follows. Given two modular forms f and g of weight k and ℓ respectively, we set

$$[f,g]_n := \sum_{r+s=n} (-1)^r \binom{k+n-1}{r} \binom{\ell+n-1}{s} (D^s f)(D^r g).$$

Then, $[f,g]_n$ is a modular form of weight $k+\ell+2n$. If $n=0$, the Rankin-Cohen bracket $[f,g]_0$ is simply the product fg. If $n=1$, the bracket endows the space of modular forms with a Lie algebra structure.

Maass forms and differential operators

It is now clear that differential operators are an essential part of any general theory of modular forms (or more appropriately Maass forms) and that the condition of holomorphy is too restrictive. The condition of holomorphy can be understood in terms of differential operators. Indeed, to say that f is holomorphic implies (but is not equivalent to) the assertion that f is a solution to the non-euclidean Laplace equation:

$$-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)f=0.$$

In 1948, Maass was the first to study real-analytic modular forms satisfying the above equation. He realized we need to define something more general than the Laplacian.

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+iky\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$$

so that Maass forms satisfy $\Delta_0 f = 0$. In this way, we are led to consider "real analytic" or smooth functions $f : \mathfrak{H} \to \mathbb{C}$ such that

(a) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$, for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$;

(b) $\Delta_k(f) = 0$;

(c) f is meromorphic at infinity.

For Maass forms, we have the analog of the Ramanujan conjecture which is still open, though some progress has been made. It is conjectured that **generically**, Fourier coefficients are transcendental.

Harmonic Maass forms of manageable growth

These observations lead to the discovery of new spaces of generalized real-analytic forms.

Technically, we should consider subgroups of $SL_2(\mathbb{Z})$ but we will leave these details out of our discussion here. The space of such forms is denoted H_k and elements of H_k are called harmonic Maass forms of weight k. If we relax condition (c) to be

$$f(z) = O(e^{cy}) \qquad \text{as} \quad y \to \infty,$$

we get a larger space of (harmonic Maass forms of manageable growth) which is denoted $H_k^!$. We thus have the inclusions $M_k \subset H_k$ and $M_k^! \subset H_k^!$.

So far, we have restricted our attention to positive integral k. As will be seen below, we must enlarge further to allow for half-integral weights and even real numbers.

The partition function

The partition function, $p(n)$ is the number of ways of writing n as a sum of positive numbers. For instance, $p(1) = 1$, and $p(2) = 2$ since we can write 2 as simply 2 or 1+1. The number 3 can be written as $1+1+1$, or $1+2$, or simply 3, so, $p(3) = 3$. Looking at 4, we see that

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1$$

are the partitions of 4, so that $p(4) = 5$. Looking at 5, we see that

$$5, \quad 4+1, \quad 3+2, \quad 3+1+1, \quad 2+2+1, \quad 2+1+1+1, \quad 1+1+1+1+1,$$

so that $p(5) = 7$. Formally, $p(n)$ is the number of ways of writing n as

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k, \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1,$$

with the λ_i being whole numbers. The partition function $p(n)$ grows very rapidly.

Ramanujan noticed that p(n) is the Fourier coefficient of a modular form of weight -1/2.

By contrast, the τ-function is the Fourier coefficient of a modular form of weight 12.

The growth rates of these functions are quite different.

Young diagrams and partitions

Each partition of $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ can be visualised as a "Young diagram" (sometimes called a "Ferrers diagram") as an arrangement of dots with λ_1 dots in the first row, λ_2 dots in the second row and so on:

$$\begin{array}{ccccccl}
\bullet & \bullet & & & \cdots & \bullet & \lambda_1 \text{ nodes} \\
\bullet & \bullet & \cdots & \bullet & & & \lambda_2 \text{ nodes} \\
\vdots & \vdots & \vdots & & & & \\
\bullet & \cdots & \bullet & & & & \lambda_k \text{ nodes}
\end{array}$$

The Durfee square of such a diagram is the largest square of nodes in the upper left hand corner of the diagram. This divides the partition into a perfect square and two "smaller" partitions whose number of parts does not exceed the side length of the Durfee square. If we denote by $a_m(n)$ the number of partitions of n that do not exceed m, then the generating function for $a_m(n)$ is clearly

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)} = \sum_{n=0}^{\infty} a_m(n) q^n.$$

The study of these formal series point naturally into a theory of mock modular forms which apparently Ramanujan saw at the end of his life.

Durfee squares and combinatorial identities

If $b_m(n)$ is the number of partitions of n with Durfee square of size m, then by similar reasoning,

$$\frac{q^{m^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^m)^2} = \sum_{n=0}^{\infty} b_m(n)q^n.$$

Summing over m gives the combinatorial identity

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^{-1} = 1 + \sum_{m=1}^{\infty} \frac{q^{m^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^m)^2}.$$

Before his arrival in England, Ramanujan had discovered variations of these identities:

$$\prod_{n=1}^{\infty}(1-q^{5n+1})^{-1}(1-q^{5n+4})^{-1} = 1 + \sum_{m=1}^{\infty} \frac{q^{m^2}}{(1-q)(1-q^2)\cdots(1-q^m)};$$

$$\prod_{n=1}^{\infty}(1-q^{5n+2})^{-1}(1-q^{5n+3})^{-1} = 1 + \sum_{m=1}^{\infty} \frac{q^{m(m+1)}}{(1-q)(1-q^2)\cdots(1-q^m)};$$

Ramanujan's correspondence with Rogers

Ramanujan had no proofs for either of these but in 1917, while in England and sitting in the library of Trinity College, he stumbled upon some old volumes of the Proceedings of the London Mathematical Society where he found an 1894 paper of Rogers in which these identities were proved. Hardy later recalled, "I can remember very well his surprise and admiration which he expressed for Rogers' work. A correspondence [between them] followed in the course of which Rogers was led to a considerable simplification of his original proof." These identities are now called the Rogers-Ramanujan identities.

In their 1918 paper, Hardy and Ramanujan derived the asymptotic formula,

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}},$$

as n tends to infinity. Twenty years later, Rademacher discovered an explicit formula:

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k} A_k(n) \frac{d}{dn} \left(\frac{\sinh(\frac{\pi}{k}\sqrt{\frac{2}{3}}(n - \frac{1}{24}))}{\sqrt{n - \frac{1}{24}}} \right), \qquad A_k(n) = \sum_{0 \le m < k, (m,k)=1} e^{\pi i [s(m,k) - 2nm/k]},$$

$$s(m,k) = \frac{1}{4k} \sum_{j=1}^{k-1} \cot\frac{\pi j}{k} \cot\frac{\pi j m}{k}.$$

Selberg writes that if Hardy had looked at Ramanujan's first letter, they could have derived Rademacher's formula!!

Ramanujan's last letter to Hardy

In his last letter, it is clear that Ramanujan discovered an extension of the theory of modular forms and was frantically trying to write it down by giving examples of what he called "mock" theta functions.

$$f(q) = 1 + \sum_{m=1}^{\infty} \frac{q^{m^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^m)^2}$$

$$\phi(q) = 1 + 1 + \sum_{m=1}^{\infty} \frac{(-q)^{m^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2m})}$$

$$\psi(q) = 1 + 1 + \sum_{m=1}^{\infty} \frac{(-q)^{m^2}}{(1+q)(1+q^3)\cdots(1+q^{2m+1})}$$

These have a striking similarity with the series appearing in the Rogers-Ramanujan identities with "some sign changes".

Fourier expansions of mock modular forms

recall that the Γ-function $\Gamma(z)$ is first defined by the integral

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx,$$

for $\mathrm{Re}(z) > 0$. It is easy to show that in this region, $\Gamma(z)$ satisfies the functional relation $\Gamma(z+1) = z\Gamma(z)$. This functional relation is then used to extend $\Gamma(z)$ to the entire complex plane with only simple poles at $z = 0, -1, -2, \ldots$ For $u \in \mathbb{R}$, the incomplete Γ-function is defined as

$$\Gamma(z, u) = \int_u^\infty e^{-x} x^{z-1} dx.$$

Given a modular form g of weight $2-k$ with Fourier expansion

$$\sum_{n=0}^{\infty} b_n q^n, \qquad q = e^{2\pi i z},$$

define $g^*(z)$ by the series

$$\overline{b}_0 \frac{(4\pi y)^{1-k}}{k-1} + \sum_{n=1}^{\infty} n^{k-1} \overline{b}_n \Gamma(1-k, 4\pi n y) q^{-n}.$$

Coefficients of mock modular forms are still a mystery. Some have arithmetic meaning. Others are expected to be transcendental numbers.

A mock modular form h of weight k is a holomorphic function on H with at most exponential growth at infinity to which corresponds a "shadow" modular form g of weight 2-k, such that h(z) + g *(z) transforms like a modular form of weight k.

Theorem:
(RM & Ekata Saha)

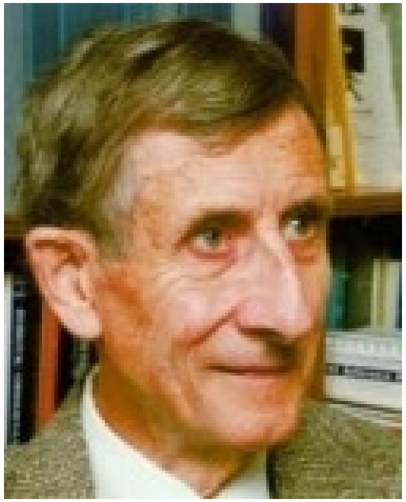
for all $z = x + iy$ such that $\pi y \in \overline{\mathbb{Q}}$, $\Gamma(k-1, 4\pi|n|y)$ is transcendental.

String theory and mock modular forms

The theory of relativity and the discovery of quantum mechanics are the two pinnacles of achievement of 20th century physics. The former is the physics of the macrocosm and the latter, the physics of the microcosm. Both theories work in the sense that they have been tested and the observed phenomena was explained using the appropriate theory. However, the theories are incompatible. A good part of Einstein's later life was spent in search of a unified field theory from which both quantum mechanics and relativity can be derived. String theory emerged in the 1960's, after Einstein's death. It proposes such a unified theory which is mathematically elegant though there is still not widespread consensus on the claim that it is the long sought unified field theory.

The traditional model holds that fundamental particles resemble mathematical points. String theory begins with the axiom that each type of elementary subatomic particle is a kind of vibrating geometric entity that mathematically resembles a string. Each of the fundamental forces of nature corresponds to a vibrational mode of these strings. For example, one of these corresponds to gravity. Another corresponds to the electromagnetic force and so on. Thus, the standard atomic theory is replaced by the theory of vibrational modes of these strings. These strings are infinitesimally small. To get a sense of their microscopic size, the ratio of the size of a string to the size of proton is about the ratio of the size of a proton to the size of the entire solar system. Though the mathematical theory of strings is elegant, it requires more dimensions than the customary four of space-time as in the theory of relativity or quantum mechanics. These strings seem to vibrate in ten dimensions.

Freeman Dyson on Ramanujan



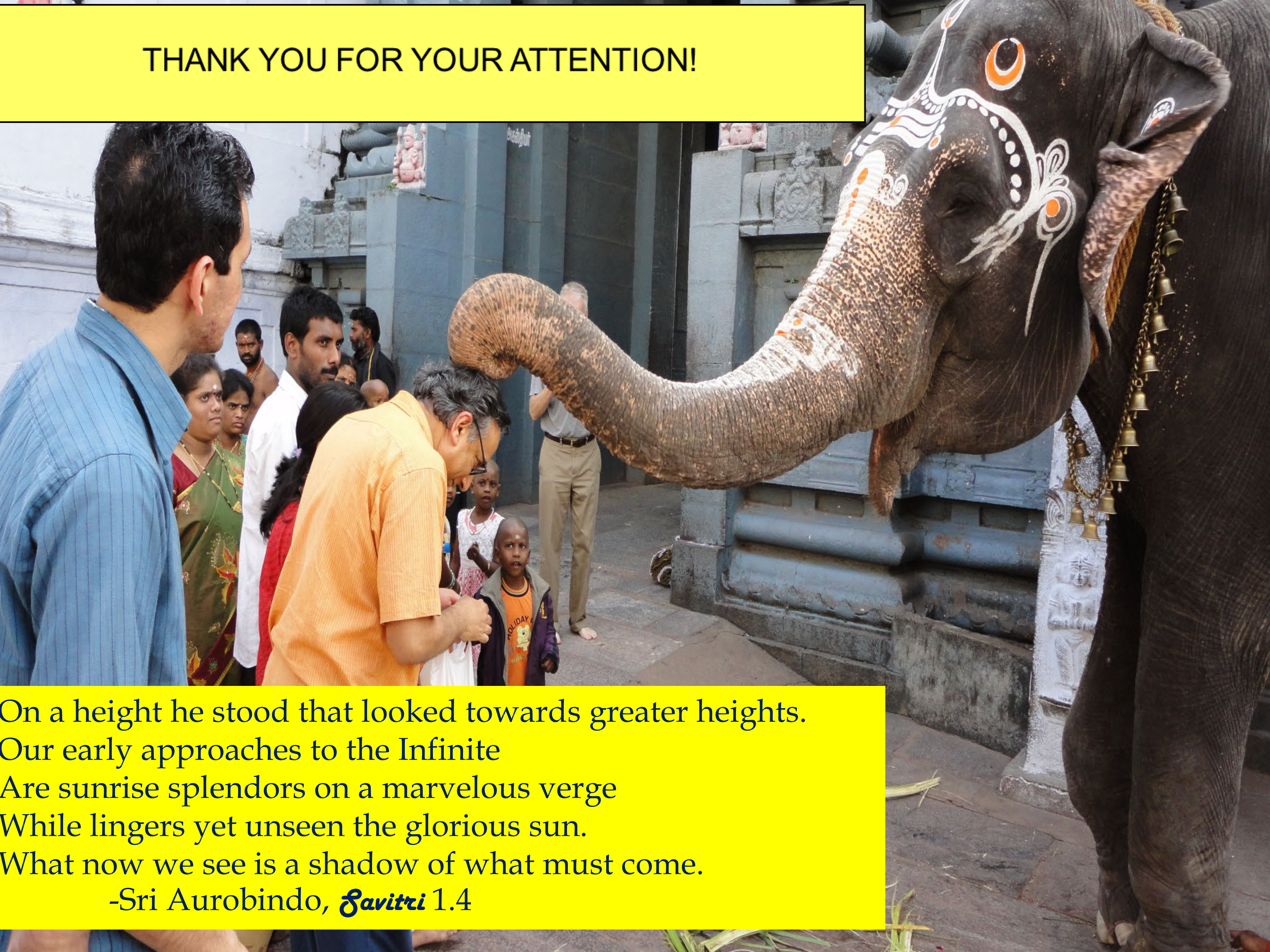
Freeman Dyson (1923-2020)

"The mock theta functions give us tantalizing hints of a grand synthesis still to be discovered … It should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta functions of Jacobi. This remains a challenge for the future."

It seems that we are on the cusp [no pun intended] of a new development and the 21st century will see the emergence of a grand theory of mock modular forms grander than the classical Hecke theory of the 20th century. Moreover, this theory may have a central role in unifying the diverse strands of fundamental particles, the holy grail of theoretical physics.

The quantum theory of black holes is a chapter in string theory.
In work by Dabholkar, Murthy and Zagier, certain generating functions that arise in this theory turn out to be mock modular forms!!

THANK YOU FOR YOUR ATTENTION!



On a height he stood that looked towards greater heights.
Our early approaches to the Infinite
Are sunrise splendors on a marvelous verge
While lingers yet unseen the glorious sun.
What now we see is a shadow of what must come.
-Sri Aurobindo, *Savitri* 1.4