Noise-adapted Quantum Error Correction for Non-Markovian Noise

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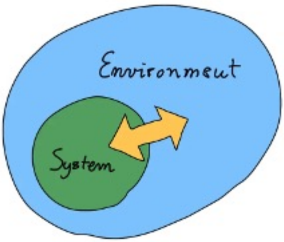
Decoherence



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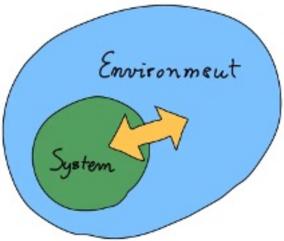

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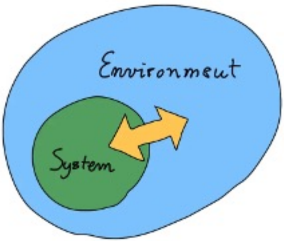

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- System dynamics are modelled by completely positive, trace-preserving (CPTP) maps.

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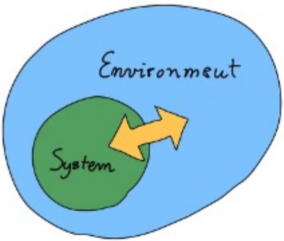

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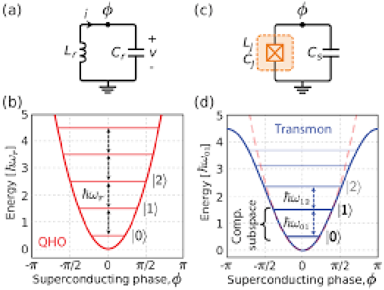
- Quantum channel: $\mathcal{E}$ is described by a set of noise (Kraus) operators $\{E_i\}$ acting on any given state ρ as,

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^{\dagger}; \; \sum_i E_i^{\dagger} E_i \leq I.$$

Quantum Channels

- Example: Transmon qubits



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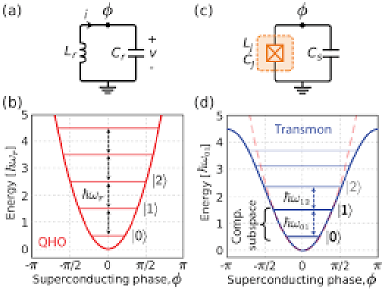
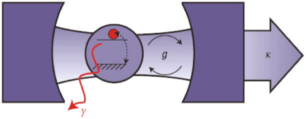
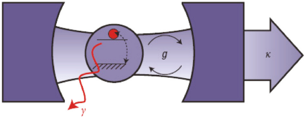


- Decoherence mechanism mimics energy dissipation in a 2-level system via the Jaynes-Cummings interaction



Amplitude-damping Channel



- Amplitude-damping channel (T_1 process), with Kraus operators,

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \; E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

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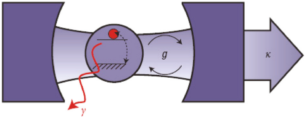

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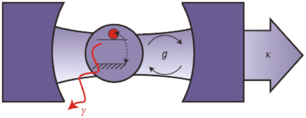

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- T_1 is the *coherence* time.

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- If (i) and (iii) hold, but (ii) is relaxed,
  $\Rightarrow$ a time-dependent GKSL master equation[1]

$$\frac{d\rho(t)}{dt} = \mathcal{L}(t)[\rho(t)] = \sum_j \Gamma_j(t)\left(L_j(t)\rho(t)L_j^\dagger(t) - \frac{1}{2}\{L_j^\dagger(t)L_j(t), \rho(t)\}\right)$$

$\{L_j(t)\}$ are the jump operators, $\{\Gamma_j(t)\}$ are the canonical decay rates.

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Non-Markovianity of Quantum Dynamical Maps

- When Γ_j are time-independent, $\mathcal{L}(t)$ is time-independent.
  $\Rightarrow$ One-parameter dynamical map $\mathcal{E}(t) = \exp(t\mathcal{L})$ (quantum channel!), the system state evolves as $\rho(t) = \mathcal{E}(t)[\rho(0)]$.

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$$\mathcal{E}(t, t_0) = \mathcal{T} \exp\left\{ \int_{t_0}^{t} \mathcal{L}(\tau) d\tau \right\}, \;\; \mathcal{T} : \text{ time} - \text{ordering operator.}$$

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- The dynamics may be non-Markovian, when at least one of the decay rates $\Gamma(t)$ becomes negative for a certain interval of time[2].

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- A quantum channel $\mathcal{E}(t_2, t_0)$ is CP-divisible if: $\mathcal{E}(t_2, t_0) = \mathcal{E}(t_2, t_1)\mathcal{E}(t_1, t_0)$.

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- The corresponding Choi-Jamiolkowski matrix $\chi(t_2, t_1) = (\mathcal{E}(t_2, t_1) \otimes \mathcal{I})\left[|\Psi\rangle\langle\Psi|\right]$ is not positive[3]!

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- For such CP-indivisible maps, the non-CP intermediate map $\mathcal{E}(t_2, t_1)$ is still a linear, Hermiticity-preserving and trace-preserving (HPTP) map with an *Operator-sum-difference* representation:

$$\mathcal{E}^{HPTP}(t, \tau)[\rho] = \sum_i \mathrm{sign}(i) E_i(t, \tau) \rho E_i^\dagger(t, \tau)$$

$\mathrm{sign}(i) = -1$ if the map is non-CP, else it is $+1$.

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Example: Amplitude-Damping Noise

- Jaynes-Cummings model of a two-level system interacting with a dissipative bosonic reservoir at zero temperature.

$$H_{\text{tot}} = \frac{\omega_0 \sigma_z}{2} + \sum_j \omega_j a_j^\dagger a_j + \sum_j (g_j \sigma_+ a_j + g_j^* \sigma_- a_j^\dagger)$$

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- ω_0 represents the energy gap between ground state $|0\rangle$ and exited state $|1\rangle$, $\Delta = \omega - \omega_0$ is the detuning parameter.
  Γ_0 quantifies the strength of the system-environment coupling and b the spectral bandwidth.

Non-markovian Amplitude-Damping Noise[5]

- The reduced dynamics of the qubit corresponds to an amplitude-damping (AD) channel with Kraus operators

$$E_1(t) = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma(t)} \end{pmatrix} \quad ; \quad E_2(t) = \begin{pmatrix} 0 & \sqrt{\gamma(t)} \\ 0 & 0 \end{pmatrix}.$$

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- Damping paramter $\gamma(t) = 1 - |G(t)|^2$,

$$G(t) = e^{-bt/2}\left(\frac{b}{\sqrt{b^2 - 2\Gamma_0 b}}\sinh\left(0.5t\sqrt{b^2 - 2\Gamma_0 b}\right) + \cosh\left(0.5t\sqrt{b^2 - 2\Gamma_0 b}\right)\right).$$

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- The system undergoes non-Markovian evolution when $b \ll 2\Gamma_0$.
  When $b \gg 2\Gamma_0$, the dynamics is time-homogeneous Markovian – it has a Lindblad form with a constant decay rate ($\Gamma(t) = \Gamma_0$).

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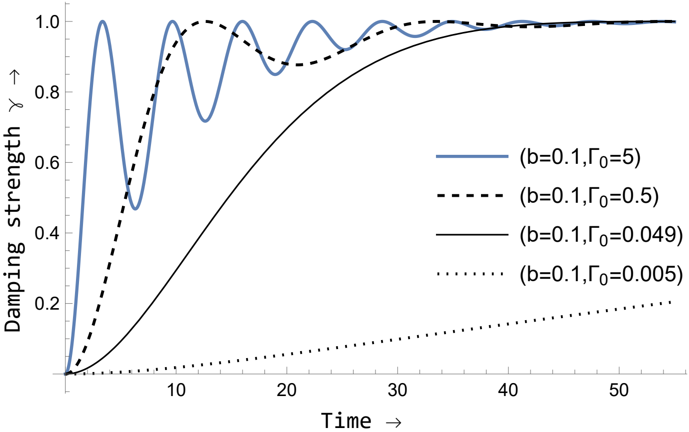
Markovian vs Non-Markovian regimes



Damping strength as a function of time, for a fixed bandwidth b

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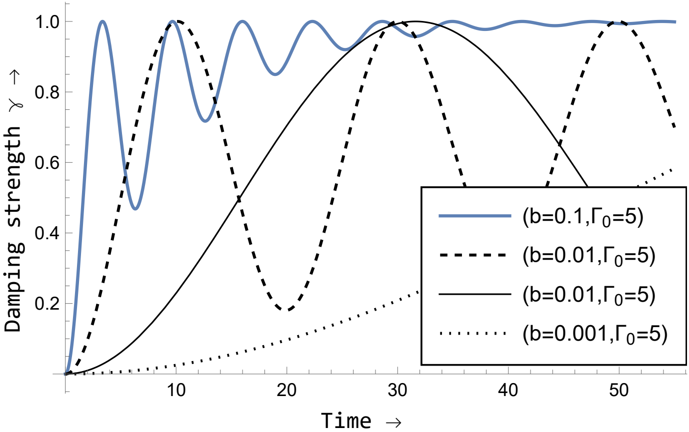

Damping strength as a function of time, for a fixed coupling Γ_0

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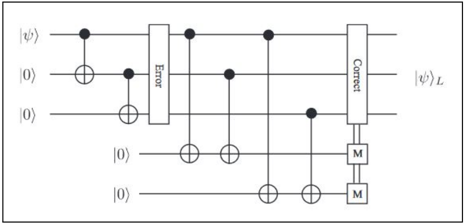

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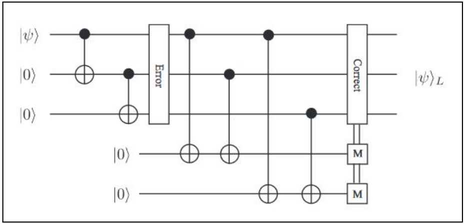

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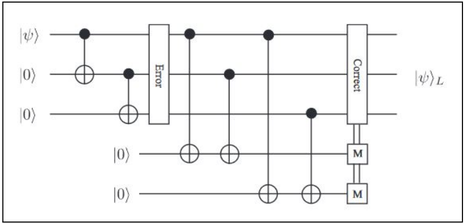

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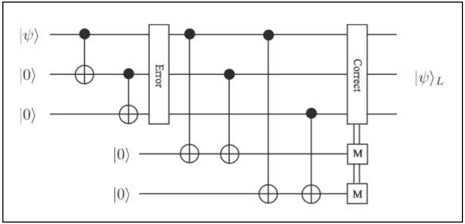

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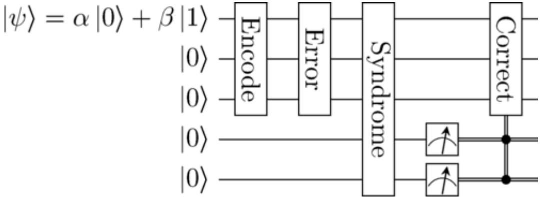
  The 3-qubit code corrects for single-qubit bit-flip noise, provided $p < \frac{1}{2}$.

QEC Schematic



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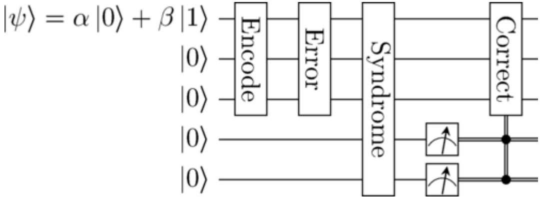

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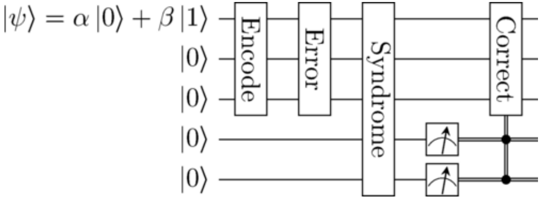

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- A QEC protocol is characterized by a code $\mathcal{C}$ (encoding) and the corresponding recovery map $\mathcal{R}$.
- For Pauli errors – X, Y, Z errors – the recovery operation is simply the inverse of the noise operator.

Beyond Pauli noise

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  $E_0 = (1 + \sqrt{1-\gamma})I/2 + (1 - \sqrt{1-\gamma})Z/2$, $E_1 = \sqrt{\gamma}(X + iY)/2$.
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- Quantum Hamming Bound: shortest perfect QEC code requires 5 qubits to protect one.
- For amplitude-damping noise, a 4-qubit code was constructed[6], that provides a degree of protection comparable to that of 5-qubit code.

[6] D.W.Leung et al, Phys.Rev.A **56**, 2567 (1997).

Beyond Pauli noise

- A *continuum* of different errors can occur on a single qubit : erasure, phase-flip ($T2$-process), amplitude-damping ($T1$-process), depolarizing noise.

- Discretize the errors in terms of a finite, unitary set of errors
  Pauli error basis: $\{X, Y, Z, I\}$
- For example, amplitude-damping in the Pauli basis:
  $E_0 = (1 + \sqrt{1-\gamma})I/2 + (1 - \sqrt{1-\gamma})Z/2$, $E_1 = \sqrt{\gamma}(X + iY)/2$.
- Quantum Hamming Bound: shortest perfect QEC code requires 5 qubits to protect one.
- For amplitude-damping noise, a 4-qubit code was constructed[6], that provides a degree of protection comparable to that of 5-qubit code.
- **Noise-adapted QEC:** Develop efficient QEC protocols for specific noise models and qubit architectures.

[6] D.W.Leung et al, Phys.Rev.A **56**, 2567 (1997).

Beyond *perfectly* correctable codes

- A 4-qubit code that corrects for single qubit amplitude damping errors [7]:

$$|0\rangle_L = \frac{1}{\sqrt{2}}\left(|0000\rangle + |1111\rangle\right)$$
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It is stabilized by the 4-qubit Pauli subgroup $\langle XXXX, ZZII, IIZZ\rangle$.

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  - The 4-qubit code: $F^2_{\min} = 1 - 1.75\gamma^2 + \mathrm{O}(\gamma^3)$.

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  $\Rightarrow$ The $[4, 1]$ code is a shorter code of comparable fidelity!

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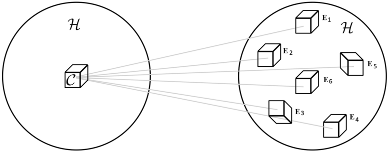
Perfect vs Approximate QEC



- Knill-Lafllame condition: $PE_i^\dagger E_j P = \lambda_{ij} P$
- At least five qubits are necessary to correct arbitrary single qubit noise.
- There exists a recovery $\mathcal{R} \sim \{U_i^\dagger P_i\}$[a]. P_i is the projector onto the $i-th$ syndrome subspace.

[a] M. A. Nielsen, I. L. Chuang Quantum Computation and Quantum Information, Cambridge University Press

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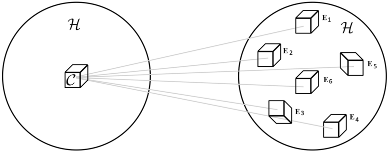

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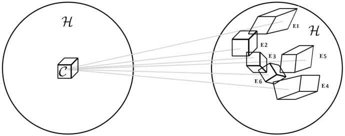

- AQEC condition [a]: $PE_i^\dagger E_j P = \lambda_{ij} P + PB_{ij}P$.
- The error subspaces are not orthogonal to each other. The unitarity (or deformability) condition gets violated.
- Noise-adapted Recovery: could be a CPTP map!

[a] C. Bény and O. Oreshkov, PRL, 104(12):120501, 2010

Noise-adapted QEC

- A triple optimization problem:

$$\max_{\mathcal{C}} \; \max_{\mathcal{R}} \min_{|\psi\rangle \in \mathcal{C}} F^2 \left[|\psi\rangle, (\mathcal{R} \circ \mathcal{E}) \left(|\psi\rangle\langle\psi| \right) \right].$$

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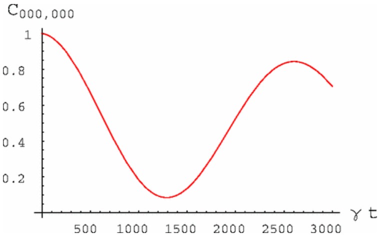
QEC for non-Markovian noise?



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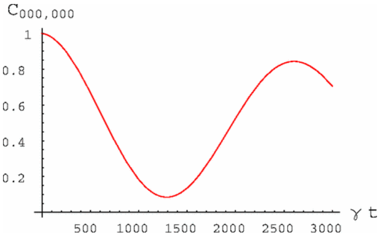

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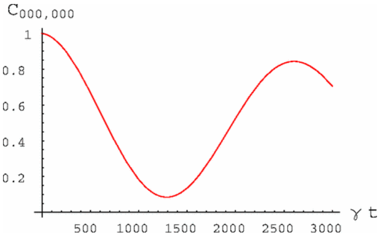

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- A codespace with projector P that satisfies the perfect QEC conditions for $\{E_i\}$ may no longer be in the domain of $\mathcal{E}$.

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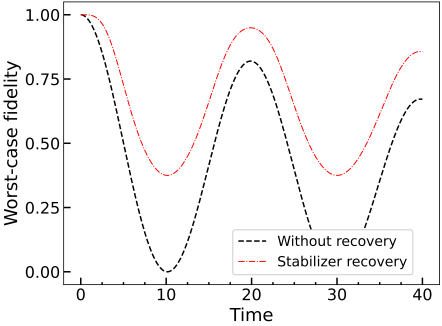
Perfect QEC for non-Markovian AD noise



- Code : $[[5,1,3]]$ code
- Recovery: $\mathcal{R} \sim \{U_i^\dagger P_i\}$. $\{U_i\}$s are the Pauli matrices.
- Noise : Non-Markovian AD $(b = 0.01, \Gamma_0 = 5)$.

Correcting non-Markovian noise using the Petz map[11]

- Defining a Petz recovery map for the noise process $\mathcal{E}[.] = \sum_i \mathrm{sign}(i) E_i[.] E_i^\dagger$

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- For a HPTP map $\mathcal{E}(t) \sim \{\text{sign}(i), E_i(t)\}$, code $\mathcal{C}$ with projector P. Let $\Delta_{ij}(t) \in \mathcal{B}(\mathcal{C})$ be traceless operators such that

$$PE_i^\dagger(t)\mathcal{E}[P]^{-1/2}E_j(t)P = \beta_{ij}(t)P + \Delta_{ij}(t),$$

  where $\beta_{ij}(t) \in \mathbb{C}$. Then,
  - The fidelity loss achieved using $\mathcal{R}_{\text{P}}^{NM}$ is,

$$1 - F^2(t) = \eta(t) = \sum_{i,j} \text{sign}(i)\text{sign}(j)(\langle\psi|\Delta_{ij}^\dagger(t)\Delta_{ij}(t)|\psi\rangle - |\langle\psi|\Delta_{ij}(t)|\psi\rangle|^2).$$

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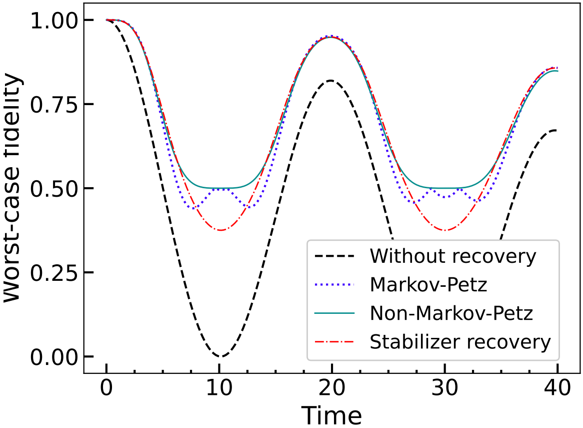
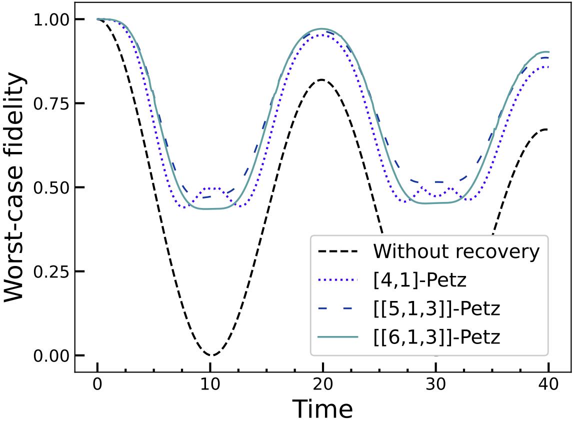
Petz recovery *vs.* Stabilizer recovery

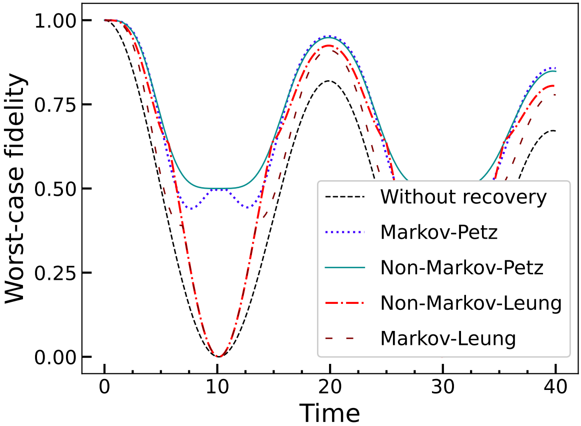


Performance of the Petz recoveries for the $[4,1]$ code subject to non-Markovian AD noise with the noise parameters $b = 0.01$ and $\Gamma_0 = 5$. The Markovian Petz is adapted to the noise regime with $b = 0.1$ and $\Gamma_0 = 0.005$. The stabilizer recovery is for the $[[5,1,3]]$ code.

Performance of different codes



Leung Recovery



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Thank You!

A counter-example study for correcting NCP error

- Given that any NCP map can be represented as

$$\mathcal{E}^{NCP}(\rho) = \sum_i \mathsf{sign}(i) E_i \rho E_i^\dagger = \mathcal{E}_1(\rho) - \mathcal{E}_2(\rho), \tag{1}$$

  where $\mathcal{E}_i$ are CP maps[13]. When $\mathcal{E}_2(P\rho P) \neq 0 \implies$ violation of $PE_i^\dagger E_j P = d_i \delta_{ij} P$.
- In other words, the code space is no more in the domain of the error map.
- consider the 3-qubit bit flip code $\mathcal{C} \overset{\text{span}}{=} \{|000\rangle, |111\rangle\}$.
- Consider $\Phi_{\text{bit-flip}}(\rho \in \mathcal{C}) = c_0 \rho + c_1 \sum X_n \rho X_n$.
- After the noise, if we measure the syndrome, the probability of detection the i_{th} single qubit bit-flip $\text{Tr}(P_i \Phi(\rho))$ can be negative.
- Makes the perfect QEC approach impractical.

[13] Shabani and Lidar (2009), PRA, **80**, 012309; A. Gonzales, D. Dilley, Mark Byrd (2020) PRA **102** 062415

Implementing noise-adapted recovery protocols[15]

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- These maps are trace increasing in general, but the overall map obtained by composing these three maps is indeed trace-preserving.
- Past work: An algorithm based on Quantum Singular Value Transform [14] Approximate, probabilistic implementation requiring $\mathcal{O}(4^{4n} + n^2 4^n)$ gates for an n-qubit code subject to arbitrary noise.

[14] A. Gilyén et al, Phys Rev Lett **128**, 220502 (2022)
[15] D. Biswas, G. Vaidya and P. Mandayam, Phys. Rev. Res. **6** (4), 043034 (2024).

Implementing noise-adapted recovery protocols[15]

- The map $\mathcal{R}_{\mathrm{P}}$ can be decomposed as a composition of,
  - (a) the adjoint map $\mathcal{E}^{\dagger}(.)$ with Kraus operators $\{E_i^{\dagger}\}$,
  - (b) the normalization or amplification map $\mathcal{E}(P)^{-1/2}(.)\mathcal{E}(P)^{-1/2}$ and
  - (c) the projection map $\mathcal{P}(.) = P(.)P$.
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- We demonstrate three different circuit constructions for implementing the Petz map and estimate the resource requirements in each case.

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- We demonstrate three different circuit constructions for implementing the Petz map and estimate the resource requirements in each case.
  - Isometric Extension ($\mathcal{O}(n^2 4^{2n})$)
  - A sequence of general quantum measurements (POVMs) ($\mathcal{O}(4^{2n}(5n^2 + O(n)))$)
  - Algorithmic approach using block-encoding ($\mathcal{O}(n^2 4^n + 4^{4n})$)

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Calculation of $F^2_{\min} = 1 - \eta$

1. The action of a channel Φ can be represented by a matrix M acting on the vectors in a Hilbert-Schmidt space , with the matrix elements
$$M_{\alpha,\beta} = \mathrm{Tr}[\mathcal{O}_\alpha \Phi(\mathcal{O}_\beta)] \tag{2}$$
For a qubit channel $\mathcal{O}_\alpha$ are the Pauli operators.
2. For a $[n, 1]$- qubit code , the cardinality of the set $\{\mathcal{O}_\alpha\}$ is four, and the operators are
$$\mathcal{O}_0 = P(\text{projector onto the code space}) \tag{3}$$
$$\mathcal{O}_1 = |0_L\rangle\langle 1_L| + |1_L\rangle\langle 0_L| \tag{4}$$
$$\mathcal{O}_2 = i(|0_L\rangle\langle 1_L| - |1_L\rangle\langle 0_L|) \tag{5}$$
$$\mathcal{O}_3 = |0_L\rangle\langle 0_L| - |1_L\rangle\langle 1_L| \tag{6}$$
3. The matrix M is then a 4×4 matrix and has the structure as
$$\left(\begin{array}{c|c} 1 & 0 \\ \hline \vec{\tau} & T_{3\times 3} \end{array}\right). \tag{7}$$
4. For a $d-$ dimension code $\mathcal{C}$,
$$\left(\begin{array}{c|c} 1 & 0 \\ \hline \vec{\tau} & T_{d-1\times d-1} \end{array}\right) \tag{8}$$

Calculation

- The fidelity between a state $|\psi\rangle \in \mathcal{C}$ and the state $\Phi(|\psi\rangle\langle\psi|)$ is

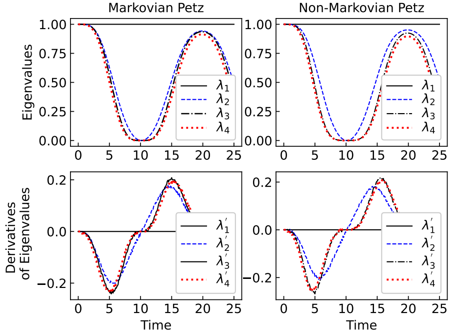
$$F^2 = \frac{1}{2}(r^{\mathrm{T}}.T.r + \vec{\tau}.\vec{r}). \quad (9)$$

  $\vec{r}$ is the Bloch vector of the encoded Bloch sphere.
- If the channel Φ is unital onto the code space $\implies \Phi(P) = P$, then $F^2 = \frac{1}{2}(r^{\mathrm{T}}.T.r)$.
- The min of the $F^2_{min} = \frac{1}{2}(1 - t_{min})$, t_{min} is the minimum eigenvalue of the T.

Non-Markovinity of a Map

A Hermitian dynamical map is said to be P-divisible iff [16]

$$\frac{d}{dt}\lambda_k(t) = \lambda_k' \leq 0 \quad \leftarrow k^{th} - \text{eigenvalue of the matrix } M. \tag{10}$$



- Code : [4,1]-Leung code.
- Noise: Non-Markovian amplitude damping $(b = 0.01, \Gamma_0 = 5)$.
- Petz Recovery:
  1. Non-Markov Petz: Adapted to the noise $(b = 0.01, \Gamma_0 = 5)$.
  2. Markov Petz: Adapted to the Markovian regime $(b = 0.1, \Gamma_0 = 0.005)$ of the noise.

[16] D.Chruściński, C. Macchiavello, and S. Maniscalco, PRL: 118(8):080404, 2017