Minimal surfaces & Labourie's Conjecture

Lecture 1: Equivariant harmonic maps
Lecture 2: Minimal surfaces + Labourie Conjecture
Lecture 3: Counterexamples to Labourie Conjecture

Equivariant harmonic maps

(M,μ), (N,ν) Riemannian manifolds.
$f : M \xrightarrow{C^2} N$, $df : TM \to TN$ interpreted as a section $df \in \Gamma(T^*M \otimes f^*TN)$.
μ, ν give rise to norm $|\cdot|_{\mu,\nu}$ and connection $\nabla = \nabla^{\mu,\nu}$ on $T^*M \otimes f^*TN$.
$|df|^2_{\mu,\nu} = tr_\mu f^*\nu$.

Defn: f is harmonic if $tr_\mu \nabla df = 0$.

M, N compact, f harmonic iff it's a critical point for the energy $\mathcal{E} = \frac{1}{2}\int_M |df|^2_{\mu,\nu}\, dV_\mu$
Basic examples:

- harmonic functions $\Omega^{\subset \mathbb{R}^n} \to \mathbb{R}^m$

- geodesics $[0,T] \to (N,\nu)$, more generally, totally geodesic maps
- holomorphic maps between Kähler manifolds
- Hopf fibrations $S^3 \to S^2$, $S^7 \to S^4$, etc

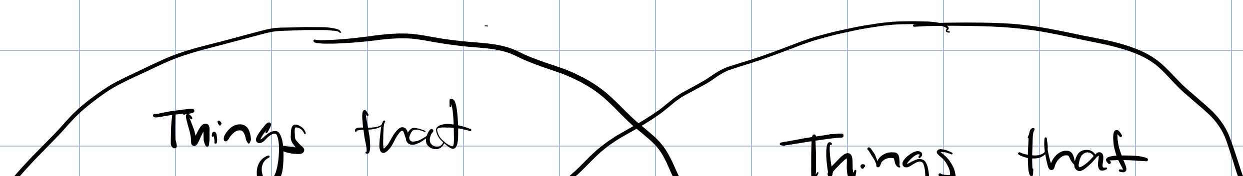
Thm. (Eells–Sampson, 1964) M, N compact, $\underbrace{K_N}_{\text{sectional curvature}} \le 0$. In every homotopy class of maps $M \to N$ $\exists$ a harmonic map $f : (M,\mu) \to (N,\nu)$

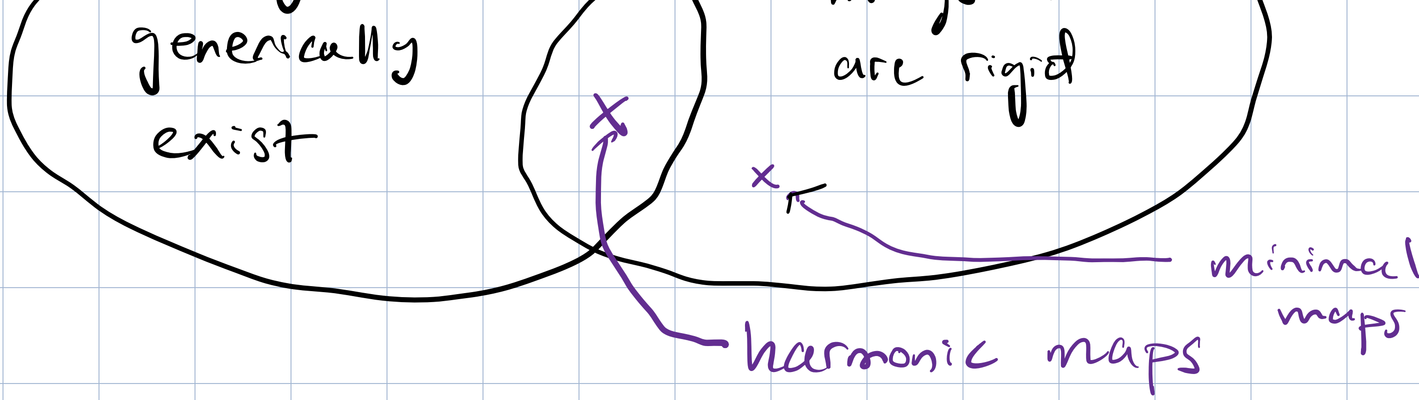
Proof by heat flow $F = F(x,t)$

$$\frac{\partial F}{\partial t} = \operatorname{tr}_\mu \nabla dF.$$

Thm. (Hartman, Sampson 1967, 1968) Above, if $K_N < 0$, then f is unique unless $f(M)$ is contained in a geodesic. Equivalently, $f_*(\pi_1 M)$ is abelian

$K_N \le 0$





Thm. (Siu, 1980) (M,μ), (N,ν) closed Kähler manifolds, $\dim_{\mathbb{C}} \geq 2$. Assume N complex hyperbolic. Then any degree 1 harmonic map $(M,\mu) \to (N,\nu)$ is a biholomorphism.

Corollary (Siu, 1980) (M,μ), (N,ν) as above. If $\pi_1 M$ is isomorphic to $\pi_1 N$ then M and N are biholomorphic or anti-biholomorphic.

Equivariant harmonic maps

$\rho: \pi_1 M \to \mathrm{Isom}(N,\nu)$, $\tilde{M}$ = universal cover of M, $\pi_1 M \curvearrowright \tilde{M}$ by Deck transformations, $\tilde{M}/\pi_1 M = M$.

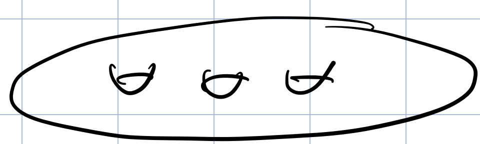
Defn: $f: \tilde{M} \to (N,\nu)$ is ρ-equivariant if $\forall \gamma \in \pi_1 M$, $f \circ \gamma = \rho(\gamma) \circ f$.

<u>Ex.</u> N_0 manifold with universal cover N

$$\begin{tikzcd} \tilde{f} : \tilde{M} \arrow[r] \arrow[d] & (\tilde{N}, v) \arrow[d] \\ f : M \arrow[r] & (N_0, v) \end{tikzcd}$$

$\rho = f_* : \pi_1 M \to \pi_1 N_0 < \mathrm{Isom}(N, v)$

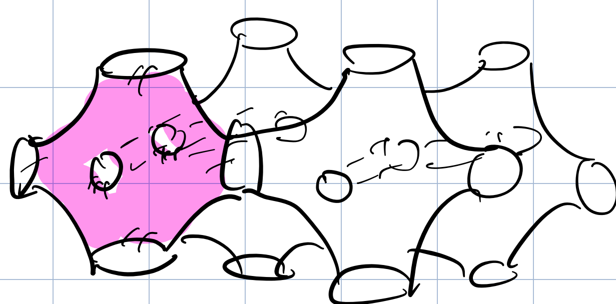
Σ_3



$$\begin{tikzcd} \Sigma_3 \arrow[r] & (S^1)^3 \cong \mathbb{R}^3/\mathbb{Z}^3 \\ \mathbb{H}^2 \arrow[u] \arrow[r] & \mathbb{R}^3 \arrow[u] \end{tikzcd}$$

$= \{ z \in \mathbb{C} : \mathrm{Re}\, z > 0 \}$

Schwarz P-surface 1880s



<u>Exercise</u>: f ρ-equivariant, $|df|^2_{\mu,v}$ is $\pi_1 M$-invariant and hence descends to M.

Thus, can define energy $\mathcal{E}(f) = \int_M |df|^2_{\mu,v} \, dV_\mu$

Now, assume $K_N \leq 0$, (N, v) complete and simply connected (C.H. thm $\Rightarrow N \simeq_{top} \mathbb{R}^n$)

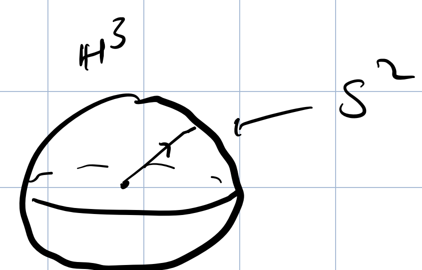
<u>Ex.</u> $(N, v) = (\mathbb{H}^n, x_n^{-2} \sum_i dx_i^2)$, $K_N = -1$.

$\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$

<u>Def'n</u>: Fix $O \in N$. Geodesic rays $\gamma_1, \gamma_2 : [0, \infty) \to (N, v)$, $\gamma_i(0) = O$, are equivalent if $\forall t$, $d(\gamma_1(t), \gamma_2(t)) \leq K$ (some $K > 0$). An equivalence class is called an endpoint.

The Gromov boundary is the set of endpoints of geodesics rays.

<u>Ex.</u> $\partial_\infty \mathbb{H}^n = S^{n-1}$



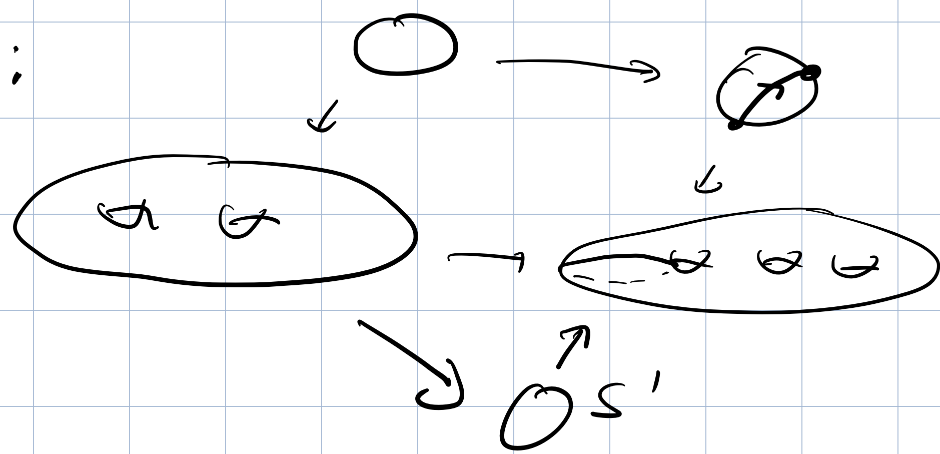
Any isometry of (N, v) extends to a bijection of $\partial_\infty N$.

Def'n: $\rho : \pi_1 M \to \mathrm{Isom}(N, \nu)$ is irreducible if $\forall \xi \in \partial_\infty N$ $\exists \gamma \in \pi_1 M$ s.t. $\rho(\gamma)\xi \neq \xi$.

Ex. $K_{N_0} < 0$

$M \to N_0$
closed manifolds

Non-example:



M compact

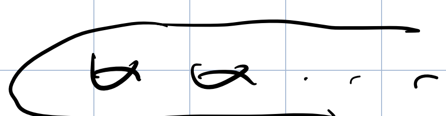
Thm. (Donaldson, Corlette, Labourie 1986, 1988, 1991)
(N, ν) as above, ρ irreducible. Then
$\exists!$ ρ-equiv. harmonic map $(\tilde{M}, m) \to (N, \nu)$.

Proof by heat flow.

Can generalize to non-compact situations.

Non-compact surfaces: Wolf, Simpson, Jost, Gupta, S. - 2019, Gupta - ?

Totally open: equivariant harmonic maps for infinite type surfaces



See Schoen Conjecture, Markovic, Benoist – Hulin,

Associated bundles: out of ρ we have

$$N_\rho = \tilde{M} \times_\rho N = \{(p,x) \in \tilde{M} \times N\} \big/ {\scriptstyle (p,x) \sim (\gamma \cdot p,\ \rho(\gamma) x)}$$

$$\begin{array}{c} N_\rho \\ \downarrow \\ M \end{array}$$

Comes with a flat connection D by taking exterior derivative in each triv.

$$\begin{tikzcd} \pi^* N_\rho \arrow[r] \arrow[d] & N_\rho \arrow[d] \\ \tilde{M} \arrow[u, bend left, "\pi^* s"] \arrow[r, "\pi"'] & M \arrow[u, bend right, "s"'] \end{tikzcd}$$

$\tilde{M}$ contractible, $\pi^* N_\rho \simeq \tilde{M} \times N$

$$\pi^* s(p,x) = (p, f(x))$$

for some ρ-equiv. $\tilde{M} \to N$.

Harmonic maps from Riemann surfaces

Metrics μ, μ' on M are conformally equivalent if $\exists\ v : M \to \mathbb{R}$ s.t. $\mu' = e^v \mu$.

From now on, M is a closed surface, genus $g \geq 2$, Σ_g. A conformal class of metrics on Σ_g is equivalent to a Riemann surface structure S on Σ_g.

(By Beltrami eqn, can find cdt z s.t. $\mu = \mu_0(z)|dz|^2$)

Exercise: $f:(\Sigma_g,\mu)\to(N,\nu)$, $v:\Sigma_g\to\mathbb{R}$, $\mathcal{E}(f)$ is the same if we replace μ with $e^v\mu$.

$\Longrightarrow$ Harmonic maps depend only on the conformal class of μ, or equivalently the Riemann surface structure.

Henceforth, we just specify R.S. S.

Complex geometry:

Warm-up: harmonic functions

$$\left\{\begin{array}{c}\text{harmonic} \\ f:\mathbb{C}\to\mathbb{R}\end{array}\right\}\Big/\text{trans.} \longleftrightarrow \left\{\begin{array}{c}\text{holomorphic} \\ \phi:\mathbb{C}\to\mathbb{C}\end{array}\right\}$$

$$\frac{\partial^2 f}{\partial\bar{z}\partial z}=0 \qquad f\longmapsto \partial f/\partial z \qquad \frac{\partial\phi}{\partial\bar{z}}=0$$

$$\phi\longmapsto f(z)=\int^z \operatorname{Re}\phi(\zeta)\,d\zeta$$

$\int_{z_0} \ldots$

Riemann surfaces : $S \to (N, v)$

$$T^*S^{\mathbb{C}} = \underset{dz}{(T^*S)^{1,0}} \oplus \underset{d\bar z}{(T^*S)^{0,1}}$$

Split $df = \underset{f_z dz}{\partial f} + \underset{f_{\bar z} d\bar z}{\bar\partial f}$, $\nabla = \nabla^{1,0} + \nabla^{0,1}$

<u>Exercise</u>: f is harmonic iff $\nabla^{0,1}\partial f = 0$

$$\partial f \in (T^*S)^{1,0} \otimes_{\mathbb{C}} f^*TN^{\mathbb{C}}$$

<u>Thm.</u> (Koszul–Malgrange) Given a complex v. bundle E over a complex manifold M, with an operator $\bar\partial_E : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ satisfying the $\bar\partial$-Liebniz rule, if $\bar\partial_E^2 = 0$, then $\exists$ holomorphic v. bundle structure on E s.t. $\bar\partial_E$ is the del-bar operator.

Del-bar operator: $F \to M$ hol. v. bundle,
$s_1, \ldots, s_n$ local frame of hol. sections,

$\partial_F\left(\sum_i f_i s_i\right) = \sum_i \partial f_i \otimes s_i)$

Upshot: $\nabla^{0,1}$ induces hol. structure on $f^*TN^{\mathbb{C}}$ in which ∂f is a hol-$f^*TN^{\mathbb{C}}$-valued 1-form.

<u>Harmonic maps from surfaces to symmetric spaces</u>

$$X_n^{\mathbb{C}} = SL(n,\mathbb{C})/SU(n) = \{A \in SL(n,\mathbb{C}) : A = \bar{A}^T,\ A > 0\}$$

$$= \{\text{Hermitian metrics on } \mathbb{C}^n \text{ inducing } 1 \text{ on } \Lambda^n\mathbb{C}^n\}$$

$$X_n \subset X_n^{\mathbb{C}},\quad X_n = SL(n,\mathbb{R})/SO(n,\mathbb{R})$$

$$= \{A \in SL(n,\mathbb{R}) : A = A^T,\ A > 0\}$$

$$= \{\text{Inner products on } \mathbb{R}^n \text{ inducing } 1 \text{ on } \Lambda^n\mathbb{R}^n\}$$

$$T_H X_n^{\mathbb{C}} = \{A \in M(n,\mathbb{C}) : A = \bar{A}^T,\ H^{-1}A \text{ traceless}\}$$

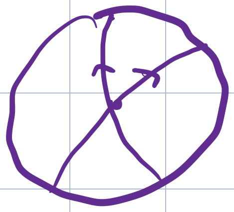
Metric v on $X_n^{\mathbb{C}}$: $v_I(A,B) = \frac{n}{2}\operatorname{tr}(AB)$

$SL(n,\mathbb{C})$-invariant: $v_H(A,B) = \frac{n}{2}\operatorname{tr}(H^{-1}AH^{-1}B)$.

For $n=2$, $X_n = \mathbb{H}^2$, $X_n^{\mathbb{C}} = \mathbb{H}^3$

- $\forall n$, $K_{X_n^{\mathbb{C}}} \leq 0$
- $X_n^{\mathbb{C}}$ complete, simply connected
- through each point in $X_n^{\mathbb{C}}$ $\exists$

$(n-1)$-dimensional flat subspaces



$\mathbb{H}^2$ Ex. At Id, can take real diagonal matrices.

Flatness: $R(X,Y)Z = -[[X,Y],Z]$.

$\rho : \pi_1 \Sigma_g \to SL(n,\mathbb{C}) \curvearrowright X_n^{\mathbb{C}}$ by isometries

irreducible iff composition of ρ w/ $ad : SL(n,\mathbb{C}) \to sl(n,\mathbb{C})$ totally reducible with finite centralizer.

$\tilde{S} \longrightarrow X_n^{\mathbb{C}}$ ρ-equiv., ρ irreducible

1) $E_\rho = \tilde{S} \times_\rho \mathbb{C}^n$ with flat connection D.

2) $(X_n^{\mathbb{C}})_\rho = \tilde{S} \times_\rho X_n^{\mathbb{C}} = \mathrm{Met}_1(E)$ = Hermitian metrics on E_ρ inducing 1 on $\Lambda^n E_\rho$.

An equivariant map $f : \tilde{S} \to X_n^{\mathbb{C}}$ is equivalent to a Hermitian metric

H on E.

3) $\mathfrak{sl}(n,\mathbb{C})$-valued 1-form $\omega = -\frac{1}{2} H^{-1} dH$ induces an iso. between $f^* T X_n^{\mathbb{C}} \to S$ and the space $\mathrm{End}_0^H(E)$ of H-self adjoint traceless endomorphisms of E.

$T = T^{*_H} = H^{-1} \bar{T}^T H$.

Derivative of f or H is $\Psi_H \in \mathrm{End}_0^H(E)$.

Note $\mathrm{End}_0^H(E)^{\mathbb{C}} = \mathrm{End}_0(E) =$ traceless ~~endpomorphisms~~ endomorphisms.

Define connection on E by
$$\nabla_H = D - \Psi_H,$$
extends to $\mathrm{End}_0^H(E)$.

<u>Exercise</u>: ∇_H on $\mathrm{End}_0^H(E)$ is the pullback of the L.C. connection on $X_n^{\mathbb{C}}$.

Decompose $T^* S^{\mathbb{C}} = (T^*S)^{1,0} \oplus (T^*S)^{0,1}$,
$$\Psi_H = \Psi_H^{1,0} + \Psi_H^{0,1}.$$
<u>Rmk.</u> $\Psi_H^{0,1} = (\Psi_H^{1,0})^{*_H}$

f harmonic iff $\nabla^{0,1} \partial f = 0$

iff $\nabla_H^{0,1}\Psi_H^{1,0}=0$.

KM thm $\Rightarrow$ $\nabla_H^{0,1}$ induces complex structure on E.

Def'n: A $SL(n,\mathbb{C})$-Higgs bundle $(E,\bar\partial_E,\phi)$ on S is a hol. v. bundle $(E,\bar\partial_E)\to S$ with $\phi\in\Omega^{1,0}(\operatorname{End}E)$ s.t. $\bar\partial_E\phi=0$ called the Higgs field.

$\deg E=0$

Equivariant harmonic map $\tilde S\to X_n^{\mathbb{C}}$ gives rise to a Higgs bundle on S, $(E_\rho,\nabla_H^{0,1},\Psi_H^{1,0})$

Flatness of D + holomorphicity of $\Psi_H^{1,0}$ is expressed via Hitchin's self-duality eqn's $F(\nabla_H)+[\Psi_H^{1,0},(\Psi_H^{1,0})^{*_H}]=0$.

Higgs bundles $(E,\bar\partial_E,\phi)$, when does it come from a harmonic map.

Given $(E,\bar\partial_E)$, Hermitian metric

H on E, $\exists!$ connection ∇_H, Chern connection, s.t. $\nabla_H H = 0$, $\nabla_H^{0,1} = \bar\partial_E$.

We want to find H s.t.

$$F(\nabla_H) + [\phi, \phi^{*_H}] = 0. \qquad (*)$$

$\Rightarrow$ $D = \nabla_H + \phi + \phi^{*_H}$ is flat, get holonomy rep ρ, for which H induces a ρ-equivariant map

<u>**Def'n**</u>: $(E, \bar\partial_E, \phi)$ is stable if for any ϕ-inv. proper hol. subbundle $F \subset E$, $\deg F < 0$.

<u>**Thm.**</u> (Hitchin 1986, Simpson 1988)
$(E, \bar\partial_E, \phi)$ is stable and has no non-trivial automorphisms (simple) iff one can find unique H solving S.D. eqns $(*)$.

Non-abelian Hodge correspondence

$$\begin{tikzcd}[column sep=huge] \left\{ \begin{array}{l} \pi_1\Sigma_g \to SL(n,\mathbb{C}) \\ \text{irreducible} \end{array} \right\} /\!\!/_{\text{conj.}} \arrow[r, leftrightarrow] \arrow[r, bend right, "\text{harmonic map D.C.L.}"'] & \left\{ \begin{array}{l} \text{stable} \\ SL(n,\mathbb{C})\text{-Higgs} \\ \text{bundles} \end{array} \right\} /\!\!/_{\text{iso.}} \arrow[l, bend right, "\text{S.D. Hitchin, Simpson}"'] \end{tikzcd}$$