p-adic adelic metrics, *p*-adic heights, and rational points on curves

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Rational Points on Modular Curves, ICTS
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A brief history of Quadratic Chabauty

$X/\mathbb{Q}$ nice curve. $b \in X(\mathbb{Q})$. $\mathrm{rank}(J(\mathbb{Q})) = r = g$. p good prime.

Landmarks in applying Quadratic Chabauty

1. Integral points on monic odd-degree hyperelliptic curves.
   Balakrishnan-Besser-Mueller $/\mathbb{Q}$, 2016.
   Balakrishnan-Besser-Bianchi-Mueller /number fields, 2020.
2. Rational points for curves when $r < g + \mathrm{rank}(\mathrm{NS}(J)) - 1$.
   Balakrishnan-Dogra, 2016.
   (Motivation: Chabauty-Kim method)
3. Rational points on the cursed curve $X_s(13)$.
   Balakrishnan-Dogra-Mueller-Tuitman-Vonk, 2019.

Quadratic Chabauty wishlist

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Want:

$$h = \sum h_q \colon J(\mathbb{Q}) \to \mathbb{Q}_p \quad \text{such that}$$

- h is a quadratic function. $r = g \Rightarrow h$ can be expanded in an explicit basis of products of single Coleman integrals.
- h_p is an (iterated) Coleman integral.
- For $q \neq p$, h_q takes on finitely many values T on $X(\mathbb{Q}_q)$. Furthermore, $h_q = 0$ if X has potential good reduction at q.
- $h - h_p$ (appropriately extended to a Coleman function) to be a locally non-constant function on $X(\mathbb{Q}_p)$.

A source of analytic functions vanishing on rational points

Balakrishnan-Dogra:
Using Chabauty-Kim theory, can satisfy Quadratic Chabauty wishlist using a function produced from a “nice” correspondence.

In practice: Need explicit non-abelian p-adic Hodge theory!

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Question: Is there an Arakelov-theoretic explanation of the role of a "nice" correspondence?

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Answer: Yes!

p-adic heights for line bundles on abelian varieties

Theorem (Besser-Mueller-S., 2022)

Let $\mathcal{L} \in \mathrm{Pic}(J)$. There is a definition of a canonical p-adic height

$$h_{\mathcal{L}}^{\mathrm{can}} \colon J(\mathbb{Q}) \to \mathbb{Q}_p.$$

Assume that $[\mathcal{L}] \neq 0 \in \mathrm{NS}(J)$ and that $i^*\mathcal{L} \cong \mathcal{O}_X$. Then $h_{\mathcal{L}}^{\mathrm{can}}$ satisfies the conditions in the Quadratic Chabauty wishlist.

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Strategy:

- Define a notion of a p-adic adelic metric associated to a line bundle $\mathcal{L}$ on J equipped with a "curvature form".
- Identify a canonical metric for a given curvature form.
- Show Quadratic Chabauty wishlist is satisfied using properties of the canonical metric for $\mathcal{L}$ with $[\mathcal{L}] \neq 0$ and $i^*(\mathcal{L}) \cong \mathcal{O}_X$.

History of various constructions of p-adic height pairings

NEW! One curvature form to rule them all!

Zarhin, 1987.
Schneider, 1982.
Mazur-Tate, 1983.
Coleman-Gross, 1989.
Nekovář, 1993.

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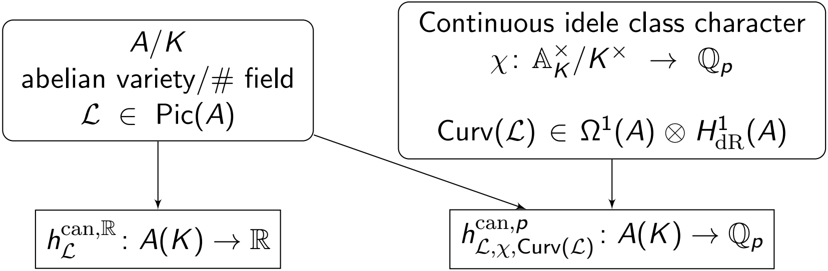
Question: Why a new theory of canonical p-adic heights?

Answer:

- Our construction parallels Zhang's construction of canonical $\mathbb{R}$-valued heights from $\mathbb{R}$-valued adelic metrics.
- Extends Quadratic Chabauty to number fields/bad primes p. Coleman integration $\rightsquigarrow$ Vologodsky integration.
- It connects p-adic heights for various p.
- New way to construct and compute local contributions at finite places to canonical $\mathbb{R}$-valued heights!

Canonical height machines on abelian varieties



If $\mathcal{L}$ is symmetric, i.e., if $\mathcal{L} \cong [-1]^*(\mathcal{L})$, then $h_{\mathcal{L}}^{\mathrm{can}}$ is quadratic:

$$h_{\mathcal{L}}^{\mathrm{can}}(nP) = n^2 h_{\mathcal{L}}^{\mathrm{can}}(P).$$

If $\mathcal{L}$ is anti-symmetric, i.e., if $\mathcal{L}^{-1} \cong [-1]^*(\mathcal{L})$, then $h_{\mathcal{L}}^{\mathrm{can}}$ is linear:

$$h_{\mathcal{L}}^{\mathrm{can}}(nP) = n^1 h_{\mathcal{L}}^{\mathrm{can}}(P).$$

Let $v \nmid p$ be a finite place of K.
Let $\nu_v = \log ||\cdot||_v$.
Let X/K_v be a projective variety.

Definition: (Inspired by Moret-Bailly, Zhang)
A ($\mathbb{Q}$-valued) metric on a line-bundle $\mathcal{L}$ is a locally constant function (for the analytic topology)

$$\nu\colon \operatorname{Tot}(\mathcal{L}) \setminus \{0\} =: \mathcal{L}^\times \to \mathbb{Q}$$

such that

$$\nu(\alpha w) = \nu_v(\alpha) + \nu(w) \qquad \forall \alpha \in \overline{K_v}^\times, \forall w \in \mathcal{L}_x^\times, x \in X(\overline{K_v})$$

Examples:
Model metrics, admissible metrics (taking "eval.+norm" of fns.).

Curvature forms for line bundles at $v \mid p$

Key ingredient for defining local heights at pplace above p

Definition:
For a place $v \mid p$, a class $\mathrm{Curv}(\mathcal{L}_v) \in \Omega^1(A_v) \otimes H^1_{\mathrm{dR}}(A_v)$ is a curvature form for the line bundle if

$$\Omega^1(A_v) \otimes H^1_{\mathrm{dR}}(A_v) \xrightarrow{\cup} H^2_{\mathrm{dR}}(A_v)$$
$$\mathrm{Curv}(\mathcal{L}_v) \mapsto \mathrm{ch}_1(\mathcal{L}_v).$$

Example: Let X/K be a nice curve of genus $g \geq 1$.
Fix a complementary subspace W to $\Omega^1(X_v)$ in $H^1_{\mathrm{dR}}(X_v)$.
Let $\{\omega_1, \ldots, \omega_g\}$ be a basis for $\Omega^1(X_v)$.
If $\{\overline{\omega_1}, \ldots, \overline{\omega_g}\}$ be the unique dual basis in W (with respect to the cup product pairing). Then

$$2\sum_{i=1}^{g} \omega_i \otimes \overline{\omega_i}$$

is a curvature form for the tangent bundle of X_v.

Proposition: [Besser, *p*-adic Arakelov theory, 2005]
For every curvature form $\mathrm{Curv}(\mathcal{L}_v) \in \Omega^1(A_v) \otimes H^1_{\mathrm{dR}}(A_v)$, there is an associated metric $\log_{\mathcal{L}} \in \mathcal{O}_{\mathrm{Col}}(\mathcal{L}_v^{\times})$, such that, the function $\log_{\mathcal{L}}$ is fiberwise a p-adic logarithm, i.e.,

$$\log_{\mathcal{L}}(\alpha w) = \log_v(\alpha) + \log_{\mathcal{L}}(w) \quad \text{for every } \alpha \in \overline{K_v}^{\times}, w \in \mathcal{L}_x^{\times}.$$

The restriction to X is explicitly described by an iterated integral –

$$\mathrm{Curv}(\mathcal{L}_v) := \sum \omega_i \otimes [\eta_i] \mapsto \sum \int \omega_i \left(\int \eta_i \right) + \int \gamma =: \log_{\mathcal{L}}(s)|_X,$$

where γ is an explicit form "correcting" for the zeroes/poles of s.[1]

[1] Note: There are multiple metrics with the same curvature, but any two such metrics differ by the integral of a holomorphic form.

p-adic adelic-metrics and $\mathbb{Q}_p$-valued heights

Definition: An p-adic adelic metric on a line bundle $\mathcal{L}$ on a projective variety X/K is a collection of metrics

$$\{\nu_v \text{ on } \mathcal{L}_v/X_v/K_v \colon \nu \nmid p \text{ a place of } K\} \cup \{\log_{\mathcal{L}_v} \colon v \mid p\}.$$

such that ν_v is a $\mathbb{Q}$-valued valuation for every $v \nmid p$ and in addition a model-metric for almost every place v.

Definition: The p-adic height function h associated to a p-adic adelic metric on a line bundle $\mathcal{L}$ on X as above is

$$\begin{aligned} h \colon X(K) &\to \mathbb{Q}_p \\ x &\mapsto \sum_{v \nmid p} \nu_v(s(x))\chi_v(\pi_v) + \sum_{v \mid p} \log_{\mathcal{L}}(s(x)), \end{aligned}$$

for some choice of section $s \in \mathcal{L}_x^{\times}$.

Outline

Quadratic Chabauty wishlist + p-adic adelic metrics

Want:

1. A p-adic adelic metric such that the associated height function h is a quadratic function on $J(K)$.
2. For all $v \nmid p$, we want the pull-back of h_v to $X(K)$ under Abel-Jacobi map i to take on finitely many values.
3. Want $h - h_p$ (appr. extd.) to be a locally non-constant Coleman function on $X(\mathbb{Q}_p)$.

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Solution:

Choose a p-adic adelic metric h on $\mathcal{L} \in \mathrm{Pic}(J)$ such that[†]

1. h is a canonical p-adic adelic metric on $\mathcal{L}$.
2. $i^*(\mathcal{L}) \cong \mathcal{O}_X \Rightarrow i^* h_v$ is an admissible metric on $\mathcal{O}_X$.
3. $[\mathcal{L}]$ is nonzero in $\mathrm{NS}(J)$.

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† Observe that these still work over a number field K.

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Canonical metrics on line bundles on abelian varieties

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Step 3: For $v \nmid p$, use the canonical $\mathbb{Q}$-valued valuation appearing in the canonical $\mathbb{R}$-valued Néron-Tate height.

Outline

1. Introduction
2. p-adic adelic metrics and a canonical p-adic height
3. Alternate explanation for Q.C. for rational points
4. Computing local contributions h_q to p-adic heights using q-analytic methods

Extending to bad primes p: replacing Coleman integrals in $\log_{\mathcal{L},p}$ by Vologodsky integrals.

Goal of an integration theory: Define $\int_b^x \omega$ for ω closed.
(More generally, solve a unipotent system of differential equations.)

	Coleman integration	Vologodsky integration
Domain	$V(\overline{\mathbb{Q}_p}), V \subset X^{\mathrm{an}}$ is a wide open	$X(K)$ *Independent of reduction type of X*
Advantage	$\exists$ algos. to compute	Natural extn. of abelian intn.

Structure of X^{an}:
It is a union of basic wide open subspaces, glued along annuli.

Comparison of integrals theorem (Besser-Zerbes, Katz-Litt)

"Vologodsky integrals are local Coleman integrals, patched along annuli, using "harmonic" correction constants"

Compute local height at q for the p-adic height from $\log_{\mathcal{L},q}$

One curvature form to rule them all!

For simplicity, let $K = \mathbb{Q}$.

Let $h^\ell = (h^\ell_v)_v$ be the canonical $\mathbb{Q}_\ell$-valued height.

Question: Is h^ℓ_p related to $\log_{\mathcal{L},p} =: h^p_p$?

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Theorem (Besser-Mueller-S., 2022)

Let $\log_{\mathcal{L},p}$ be the local contribution at p to the canonical p-adic height. Then the function $\mathrm{val}_{\mathcal{L},p} \colon \mathcal{L}^\times(\overline{\mathbb{Q}_p}) \to \overline{\mathbb{Q}_p}$ defined by

$$\mathrm{val}_{\mathcal{L},p} := \left(\frac{d}{d\log(p)} \log_{\mathcal{L},p} \right) \Big|_{\log(p)=0}$$

is a $\mathbb{Q}$-valued valuation ("Vologodsky valuation") appearing in the local contribution at p to the Néron-Tate height.

Advantages of Vologodsky valuations

- Can compute the local contributions at *all* places starting from just the curvature form + Vologodsky integration.
- For a general $\mathcal{L}$ on an abelian variety A, this gives a q-analytic way to compute contributions at finite places to the Néron-Tate height. ("Unified theory of heights")

Remark: Recovers the formula for local heights in Betts-Dogra, currently being implemented on a database of hyperelliptic curves by Betts–Duque-Rosero–Hashimoto–Spelier.