Weighted influence in q-voter model of opinion formation

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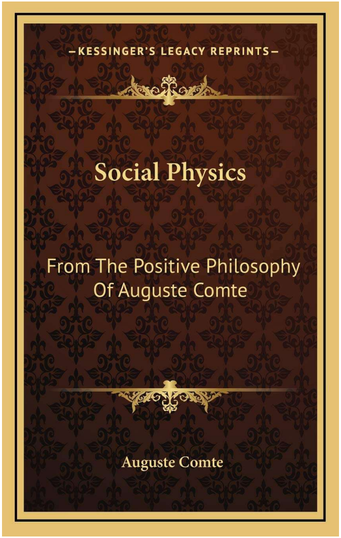
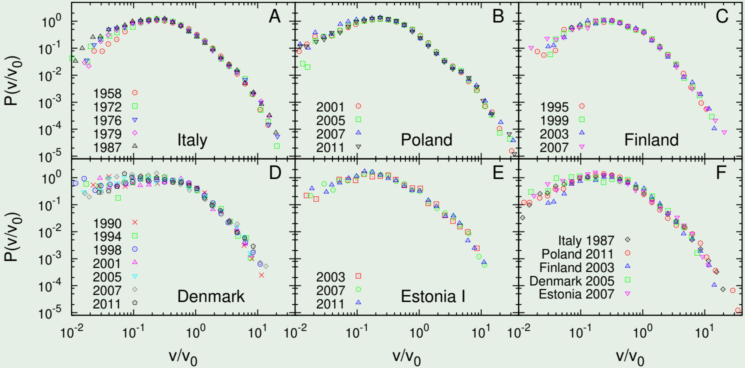


Auguste Comte

19 January 1798 - 5 September 1857

Comte believed that society could be studied in the same way as the natural world, and that social phenomena could be explained through scientific laws.



Distribution of electoral performance of candidates in proportional elections with open lists. Italy (until 1992), Poland, Finland, Denmark and Estonia (after 2002) follow essentially the same rules.

From Chatterjee *et al.* (2013).

Opinion formation in a society

One of the problems in social phenomena studied extensively is how the society forms opinion on public issues.

Opinion formation - an evolving process

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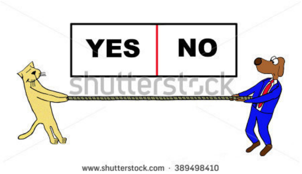

After a lot of debates, discussions etc. people usually arrive at a decision.

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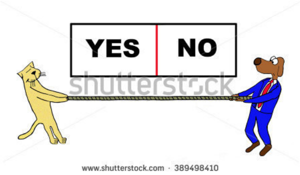

After a lot of debates, discussions etc. people usually arrive at a decision.

Consensus: All/majority of the members agree

Figure courtesy: Internet

Modelling opinion dynamics

Discrete opinion values:

When there is a finite number of opinions possible (e.g, excellent, very good, good, poor etc.), one may attach discrete numbers to each opinion.

Suppose it is a yes/no (binary) case (e.g., Brexit), elections with two contestants etc., opinions are designated conventionally by 0 and 1 or -1 and +1.

Consensus or its absence is a macroscopic feature (average opinion).

Question: **what kind of microscopic interactions led to the result?**

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Question: **what kind of microscopic interactions led to the result?**

Attempt to model opinion formation in a society:

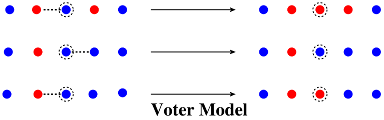
- A set of agents whose states are described by appropriate values that evolve obeying some rules.

- **Interactions in general involve a single/group of individuals influencing the opinion of other agent(s).**

- In the models, interactions are considered among the individuals located on the sites of a lattice or in a virtual space where no spatial picture is used.

Voter model: Simplest possible model with binary variables. Individuals just mimic the choice of others - person A follows the opinion of person B randomly chosen among A's neighbours.

Clifford and Sudbury 1973



Some basic models of opinion dynamics

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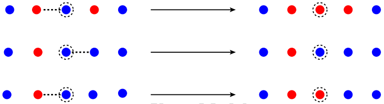
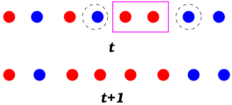


Voter Model

Sznajd model: based on social validation. A single person cannot influence the opinion of others, a group can

Sznajd-Weron and Sznajd (2000)

Dynamical rule in one dimension: a pair of neighbouring spins are picked up. If they are in the same state, their neighbours are made to follow their state.



Sznajd model

Conservation in Voter model dynamics

Let the fraction of agents f_+ $(f_- = 1 - f_+)$ have positive (negative) opinions initially.

If out of n neighbours, n_+ have positive opinion, the probability for an agent to have a positive opinion is $n_+/n \approx f_+$ in a large enough system.

Take e.g., a $\{+, -\}$ pair.

It can become remain $\{+, -\}$ or become $\{+, +\}$, $\{-, +\}$ or $\{-, -\}$ following all possible interactions and probabilities given in terms of f_+, f_-.

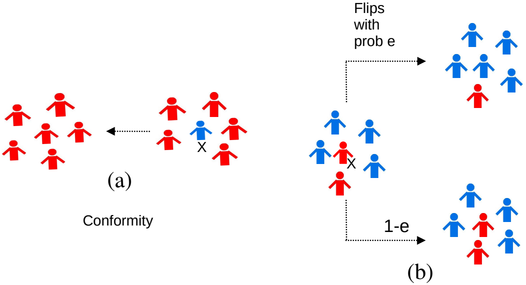
Important result: The ensemble averaged density of the two opinions remains constant: $\frac{df_+}{dt} = 0$ in any dimension.

For a particular case, the system reaches consensus (either all positive/negative) in dimensions $d \leq 2$.

Nonlinear q voter model

Castellano et al 2009

- L agents with binary opinions (*positive* or *negative*) on a complete graph.
- The focal agent (*randomly chosen*) is influenced by q neighbours (*also randomly chosen*) $\Rightarrow$ q-panel
- If the q-panel is **unanimous**, the focal agent conforms to that opinion.
- Otherwise, the focal agent flips with probability ϵ

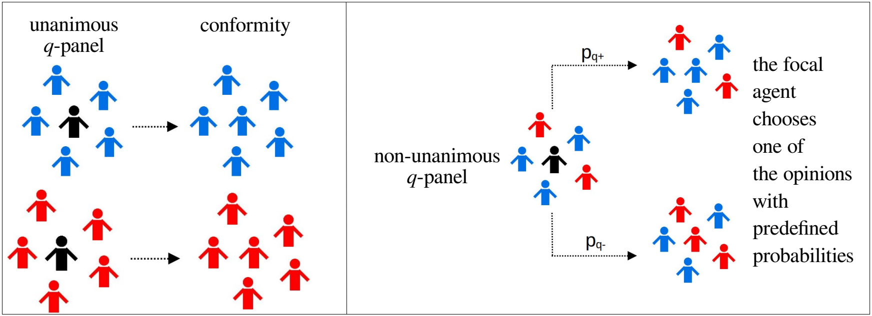


Equivalent to the voter model for $q = 1$, any ϵ.
For $q = 2, \epsilon = 0$ equivalent to the Sznajd model.
Many variants of the model. Can be obtained from a generic model (ongoing work; M. Doniek, P. Mullick, K. Sznajd-Weron, PS).

The actual composition of the q panel, when not unanimous, is not considered.

Pratik Mullick and PS 2024, arXiv:2409.09817

- L agents with binary opinions (*positive* or *negative*) on a complete graph.
- The focal agent (*randomly chosen*) is influenced by a random q panel
- q-panel unanimous $\Rightarrow$ the focal agent conforms to that opinion.
- **Otherwise, the focal agent chooses the positive opinion with probability p_{q+} and the negative opinion with probability p_{q-}**



Model definition

- n is the number of agents with positive opinion in the q-panel
- p and $1-p$ are considered as the influential power of positive and negative opinions
- p_{q+}, (p_{q-}) are the weighted average of the influential powers

$$\boxed{p_{q+} = \frac{np}{p(2n-q)+(q-n)}; \qquad p_{q-} = \frac{(q-n)(1-p)}{p(2n-q)+(q-n)}}$$

- $p = 1/2$, any $q \Rightarrow$ all agents are equally influential, $p_{q+} = n/q \Rightarrow$ **mean field voter model with q neighbours**

Results: mean field calculations

We take the system size $L \to \infty$ but keep q finite.

The master equation for f_+, the fraction of positive opinions

$$\frac{df_+}{dt} = -\omega_{+\to-} f_+(t) + \omega_{-\to+} f_-(t)$$

Transition rates ω are given by

$$\omega_{+\to-} = \underbrace{(1-f_+)^q}_{\text{unanimous q panel}} + \underbrace{\sum_{n=1}^{q-1} p_{q-} \binom{q}{n} {f_+}^n (1-f_+)^{q-n}}_{\text{all other cases}},$$

$$\omega_{-\to+} = \underbrace{f_+^q}_{\text{unanimous q panel}} + \underbrace{\sum_{n=1}^{q-1} p_{q+} \binom{q}{n} {f_+}^n (1-f_+)^{q-n}}_{\text{all other cases}}$$

Mean field rate equation

$$\frac{df_+}{dt} = \sum_{n=0}^{q} p_{q+} \binom{q}{n} {f_+}^n (1 - f_+)^{q-n} - f_+.$$

$f_+ = 0,\ 1$ are trivial fixed points (no dynamical evolution).

For $p = 1/2$, $p_{q+} = n/q$, one gets

$$\frac{df_+}{dt} = 0$$

irrespective of the value of q (conservation) as expected

All points are fixed points here.

Consistent with a mean field voter model with q neighbours; any value of f_+ is a fixed point here.

Results: mean field calculations for small q

$$\boxed{\boldsymbol{q}\mathbf{=2}}$$

$$\frac{df_+}{dt} = f_+(1-f_+)(2p-1)$$

No fixed point other than $f_+ = 0, 1$ for any $p \neq 0.5$
Solution:

$$f_+ = \frac{Ae^{(2p-1)t}}{1+Ae^{(2p-1)t}},$$

where $A = \frac{f_+(0)}{1-f_+(0)}$.

Linear stability analysis (LSA) about the fixed points f_+^*:

$$f_+ = f_+^* + \delta$$

$$|\delta| \propto e^{\pm(2p-1)t}$$

where the $+$ $(-)$ sign is for the fixed point $f_+^* = 0$ $(f_+^* = 1)$.

Implies that whenever the initial configuration is biased towards the positive (negative) opinion, the final outcome would be a positive (negative) consensus for $p > 0.5$ $(p < 0.5)$.

Results: mean field calculations for small q

$$\boxed{\boldsymbol{q = 3}}$$

Again only two fixed points at $f_+^* = 0, 1$ for $p \neq 0.5$.

LSA: $f_+ = f_+^* + \delta$

For $f_+^* = 0$,

$$\delta \propto e^{\frac{4p-2}{2-p}t}$$

and for $f_+^* = 1$ (for which δ is negative),

$$|\delta| \propto e^{-\frac{4p-2}{1+p}t}.$$

Qualitatively the results are same as $q = 2$.

In fact due to the symmetry $p \to 1 - p$, $f_+ \to 1 - f_+$, that the flows are to 1 and 0 for $p > 0.5$ and $p < 0.5$ is expected.

Results: mean field calculations for large q

$$\frac{df_+}{dt} = \sum_{n=0}^{q} p_{q+} \binom{q}{n} {f_+}^n (1-f_+)^{q-n} - f_+.$$

An additional assumption is made for large values of q as calculations become cumbersome with the original equation.

Simplified mean field theory (SMF)

Replace n by its average value which is qf_+.

SMF is correct for $q \to \infty$.

Using $n = qf_+$ and appropriate values of p_{q+} and p_{q-}, the transition rates contain only two terms

$$\omega_{-\to+} = f_+^q + [1 - f_+^q - (1-f_+)^q] \times \frac{pf_+}{(1-f_+)(1-p) + f_+p}$$

$$\omega_{+\to-} = (1-f_+)^q + [1 - f_+^q - (1-f_+)^q] \times \frac{(1-p)(1-f_+)}{(1-f_+)(1-p) + f_+p}.$$

Now the rate equation simplifies to

$$\frac{df_+}{dt} = f_+^q(1-f_+) - (1-f_+)^q f_+ + (1-f_+)f_+ \frac{\{1-f_+^q-(1-f_+)^q\}(2p-1)}{(1-p)(1-f_+)+pf_+}.$$

Apart from $f_+^* = 0, 1$ a third fixed point is obtained.

The third fixed point in general, depends on both p and q.

Also, if one puts $f_+ = f_- = 0.5$ and $p = 0.5$ in the above equation, one gets $\frac{df_+}{dt} = 0$, which implies that for any q this is the third fixed point.

For $q \to \infty$,

$$\frac{df_+}{dt} = (1-f_+)f_+ \frac{(2p-1)}{(1-p)(1-f_+)+pf_+}$$

such that the fixed points are again simply $f_+^* = 0, 1$ and for $p = 0.5$, all points are fixed points.

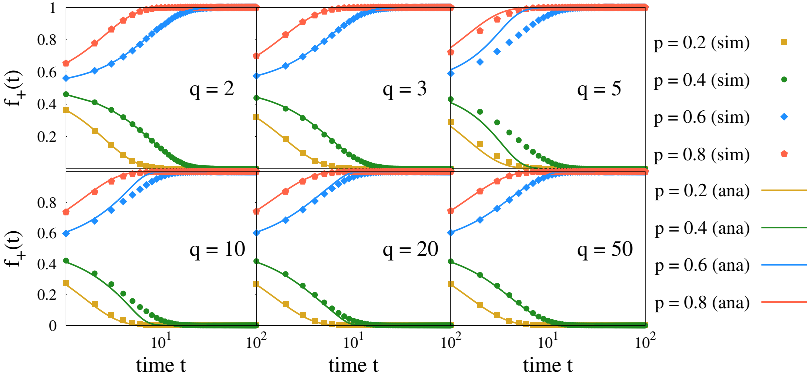
LSA: $f_+ = 1 - |\delta_0| e^{-\left(\frac{2p-1}{p}\right)t}$ near $f_+^* = 1$

Again, for $p > 0.5$, remains 1. Near $f_+^* = 0$, $f_+ \propto e^{\left(\frac{2p-1}{1-p}\right)t}$

This vanishes for $p < 0.5$.

Does the third fixed point really exist?

Trajectories of f_+



- For $q = 2, 3$ exact expression for $\frac{df_+}{dt}$ was used.
- For $q \geq 4$, SMF agrees better with MC results as q increases.

Results: Monte Carlo simulations

- For $q = 2$, $t \gg 1$,

$$f_+(t) = 1 - \alpha \exp(-\beta t) \quad \text{or} \quad f_+(t) = \alpha' \exp(-\beta' t) \qquad \text{with } \beta, \beta' = |2p - 1|$$

- Based on this, we conjecture, that for any q

$$\begin{aligned} f_+(t) &= \alpha \exp(-\beta t), \quad \text{for } p < 0.5 \\ &= 1 - \alpha' \exp(-\beta' t), \quad \text{for } p > 0.5. \end{aligned}$$

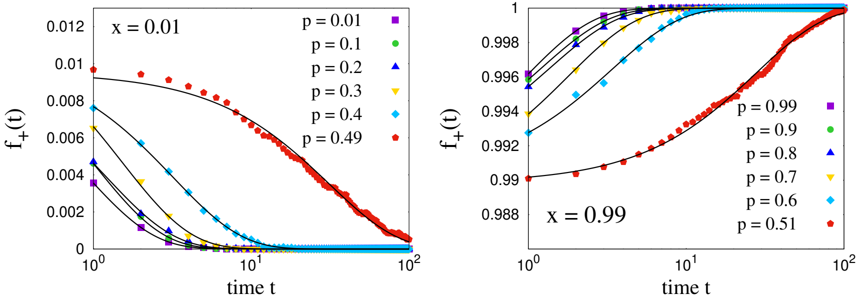
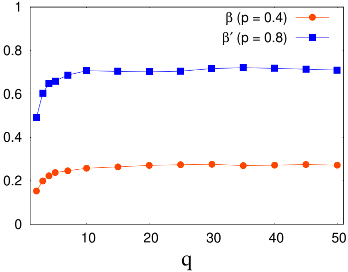
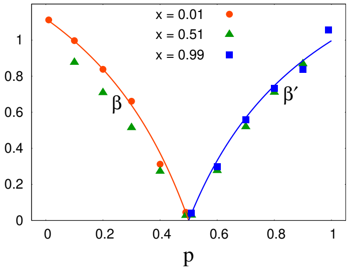


Figure: Data fittings for $q = 50$

- Values of β, β' depend on p, but become q-independent for large q





- Dependence of β, β' on p was found to be (from mean field, LSA)

$$\beta(p) \sim \frac{1-2p}{1-p} \quad \text{for } p < 0.5$$

$$\beta'(p) \sim \frac{2p-1}{p} \quad \text{for } p > 0.5$$

- So there exists a time scale which diverges as $p \to 1/2$ from either side $\Rightarrow$ **dynamical critical point at $p = 1/2$ showing critical slowing down**

Exit probability

Only two stable fixed points: all positive or all negative.
In such systems suppose one **starts** with x fraction of positive opinions.

Exit probability: $E(x)$ The probability that the system ends up in the all positive state.

Voter model: conservation of the densities imply **order parameter** $m = f_+ - f_-$ also conserved

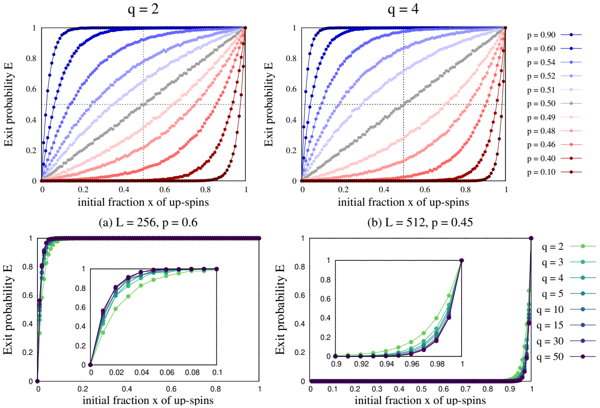
$m(t \to \infty) = m(t = 0) \quad \Longrightarrow$

$$E(x) - (1 - E(x)) = x - (1 - x)$$

Or,

$$\boxed{E(x) = x}$$

Exit probability $E(x)$



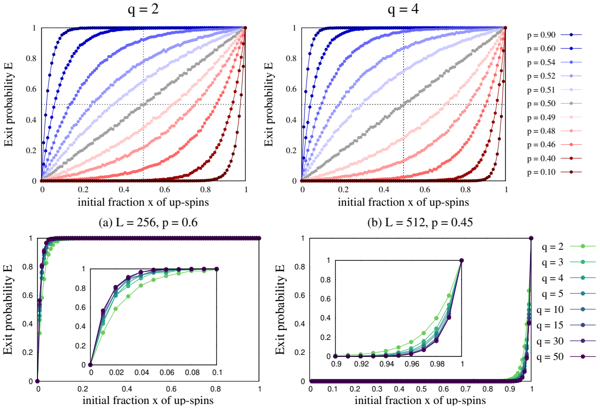
(a) L = 256, p = 0.6

(b) L = 512, p = 0.45

- Results are qualitatively similar across q
- $E(x) = x$ for any q at $p = 1/2$
- Sharp increase of $E(x)$ close to $x = 0$ $(p > 1/2)$ and $x = 1$ $(p < 0.5)$

Results: Monte Carlo simulations

Exit probability $E(x)$



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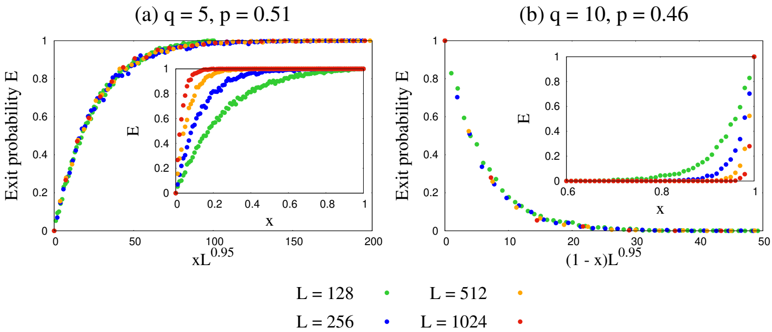
Exit probability results confirm there is no third unstable fixed point.

Finite size scaling



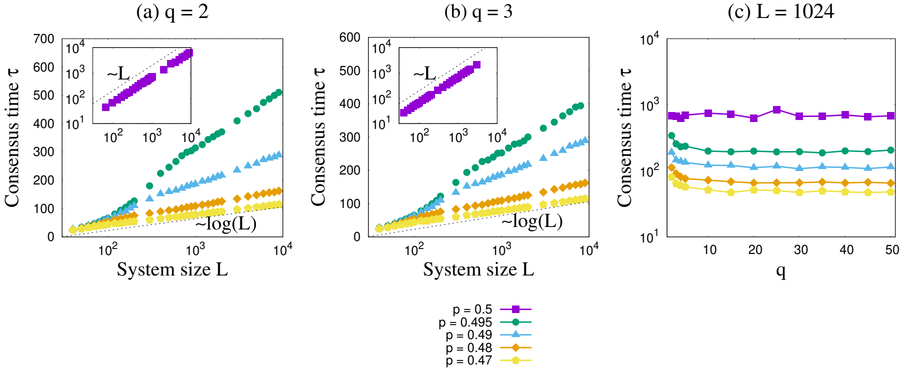
- $E(x)$ was collapsed using the scaling form:

$$E(x) = g_1(xL^{\nu}) \text{ for } p > 1/2$$
$$= g_2((1-x)L^{\nu}) \text{ for } p < 1/2, \quad \nu \approx 0.95$$

- The collapsed data was found to fit well according to

$$g_1(z) = 1 - \exp(-z/b) \quad \text{for } p > 1/2 \quad \text{and} \quad g_2(z) = \exp(-z/b') \quad \text{for } p < 1/2$$

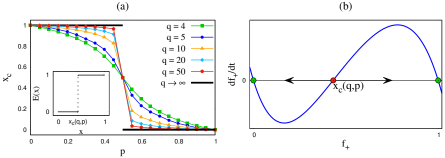
Time to reach consensus τ



- $\tau \sim L$ for $p = 1/2$, but $\tau \sim \log(L)$ for $p \neq 1/2$
- For large q, the values of τ also become independent of q

Comments and conclusions

- Three regimes, $p < 1/2$, $> 1/2$ and $p = 1/2$ for all q.
- $p = 1/2$ claimed to be a dynamic critical point
- Most of the results become independent of q for $q > 20$ approximately.
- Exponent ν is different from that in the variants of the original q voter model
- Although SMF results show a 'fictitious' third (unstable) fixed point, it can be useful. SMF maybe considered as an independent model where $n = qf_+$ is taken (interpreted as the media influence), more realistic behaviour of exit probability



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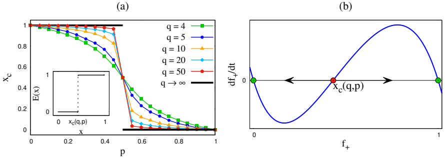

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Thank you for your attention