Lectures bouncing Cosmology

– Introduction

• Amazing Standard Model (Hot BB) that did not even exist 100 years ago

successes!

- * Recession of galaxies ⇒ expansion
- * Isotropy of CMBR
- * Dark energy & acceleration of expansion
- * Abundance of light elements
- * Large Scale structure formation & numerical simulations
- + initial conditions

⇒ Working phenomenological description from a fraction of a second after "birth" (big bang) until now!

• Problems:

Inflation provides solution

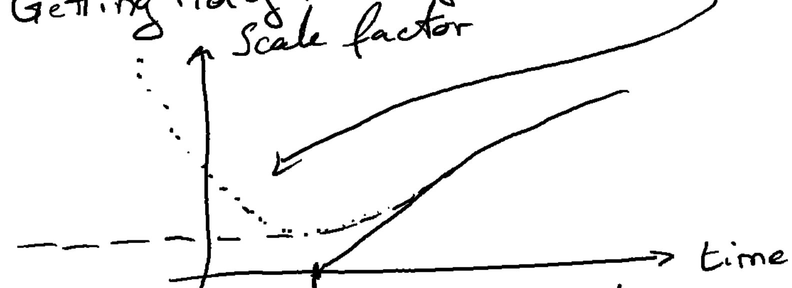
- * Vanishingly small spatial curvature
- * Lack of exotic relics (monopoles in GUT, SUSY particles...)
- * Seeds for Large Scale Structure formation
- (* Baryogenesis ⇒ Particle physics pb?)
- * Initial Singularity ⇒ ! remains...
  ⇒ tiny horizon

Inflation has problems of its own:

- – Super planckian excursions
- – η – problem
- – Eternal inflation

$\boxed{\oplus \text{ Singularity}}$

Getting rid of the singularity ⇒ BOUNCE



Possible issues to be addressed:

- → Conditions for a GR bounce: ~~NEC~~
- → Instabilities
- → Matching conditions for perturbations
- → Implementation in accepted theories (string theory, quantum gravity...)

I - Models and requirements

[I-0] Notations & Conventions

4D - spacetime + GR

$$S = \frac{c^4}{16\pi G_N}\int \sqrt{-g}\, d^4x\,(R - 2\Lambda) + S_{mat.}$$

$$\Rightarrow \boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

Metric ansatz: homogeneous & isotropic

= Friedman-Lemaître-Robertson-Walker (FLRW)

$$ds^2_{FLRW} = -dt^2 + a^2(t)\,\gamma_{ij}\, dx^i dx^j$$

$$= a^2(\eta)\left(-d\eta^2 + \gamma_{ij}\, dx^i dx^j\right)$$

a: scale factor; η: conformal time

$$\gamma_{ij} = \frac{\delta_{ij}}{\left(1 + \frac{1}{4} K \delta_{mn} x^m x^n\right)^2}$$

$$\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2$$

K: spatial curvature $K = 0, \pm 1$ (flat, open, closed)

Natural units

$$\hbar = c = 8\pi G_N = 1$$

Planck mass $M_p = G_N^{-1/2}$ dimensionless

Matter = fluid (ok for background, even for scalar field...)

$$T_{\mu\nu} = (\rho + P) u_\mu u_\nu + P g_{\mu\nu}$$

$g^{\mu\nu} u_\mu u_\nu = -1$
timelike vector

$$H = \frac{\dot a}{a} = \frac{1}{a}\frac{da}{dt}$$

Einstein eqs:

$$\begin{cases} H^2 + \dfrac{K}{a^2} = \dfrac{1}{3}\rho \\[2mm] \dot H + H^2 = \dfrac{\ddot a}{a} = -\dfrac{1}{6}\underbrace{(\rho + 3P)}_{<0 \text{ for inflation}} \end{cases}$$

Strong Energy condition violation

Conformal time

$$d\eta = a\,dt$$

$$\Rightarrow \begin{cases} \mathcal{H}^2 + K = \dfrac{1}{3}\rho a^2 \\[2mm] \mathcal{H}' = -\dfrac{1}{6} a^2 (\rho + 3P) \end{cases}$$

$$\mathcal{H} = \frac{a'}{a} = \frac{1}{a}\frac{da}{d\eta} = \frac{1}{a^2}\frac{da}{dt} = \frac{H}{a}$$

Fluid conservation

$$\nabla_\mu T^{\mu\nu} = 0 \Rightarrow \dot{\rho} + 3H(\rho + P) = 0 = \rho' + 3\mathcal{H}(\rho+P)$$

$$\Rightarrow \rho = \rho_{ini}\, e^{-3\int [1+w(a)]\, d\ln a} \xrightarrow[w = C^{te}]{} \rho_{ini}\left(\frac{a}{a_{ini}}\right)^{-3(1+w)}$$

$$P = w(\rho)$$

Scalar field w/ canonical kinetic term + potential:

$$\mathcal{L}[\phi(x)] = -\frac{1}{2}\underbrace{(\partial\phi)^2}_{g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)} - V(\phi)$$

Stress-energy tensor

$$T_{\mu\nu} = -2\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} + \mathcal{L}\, g_{\mu\nu}$$

$$\Rightarrow T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi + V(\phi)\right]g_{\mu\nu}$$

Homogeneity + isotropy $\Rightarrow \phi = \phi(t)$

$\Rightarrow$ Set $\quad \partial_\mu\phi = \dot{\phi}\, u_\mu$

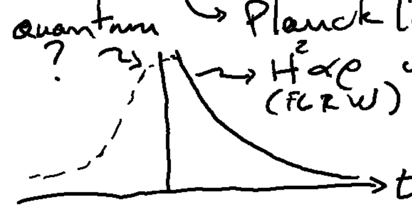
$$\Rightarrow T_{\mu\nu} = \underbrace{\dot{\phi}^2}_{\rho+P} u_\mu u_\nu + \underbrace{\left(\frac{1}{2}\dot{\phi}^2 - V\right)}_{P} g_{\mu\nu}$$

$$\begin{cases} \rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) = \frac{1}{2a^2}\phi'^2 + V(\phi) \\ P = \frac{1}{2}\dot{\phi}^2 - V(\phi) = \frac{1}{2a'^2}\phi'^2 - V(\phi) \end{cases}$$

I.1) Overview of models

→ Mostly semi classical scalar fields in GR (+ extensions) or string th.

→ Quantization of gravity (canonical, LQG, ...) = original motivation for PBB scenario



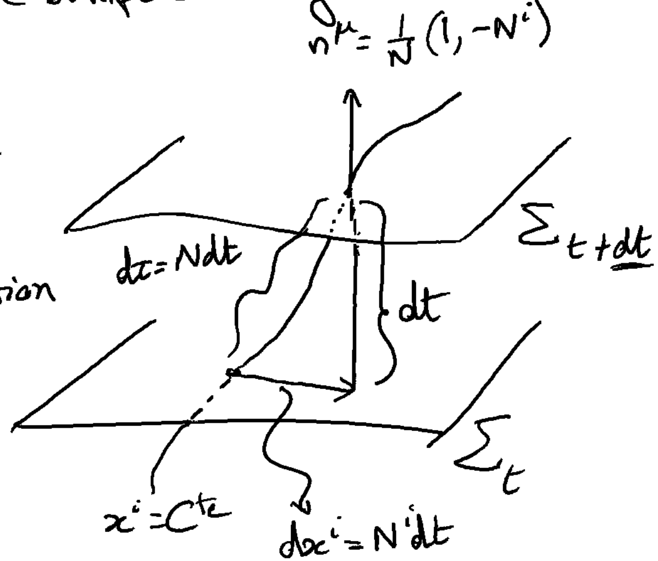
$\Rightarrow$

Canonical quantum cosmology

= the simplest way to build a bounce?

space time

3+1 decomposition



$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

$$= -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

lapse function

induced metric
= 1^{st} fundamental form

Shift vector

Extrinsic curvature = 2^{nd} fundamental form

$$K_{ij} = -\nabla_j^{(h)} n_i = \left[\nabla_i^{(h)} N_j + \nabla_j^{(h)} N_i - \dot{h}_{ij}\right]$$

Action:

$$S_{EH} = \frac{1}{2}\int_{\mathcal{M}} \sqrt{-g}\,(R - 2\Lambda)\, d^4x + \underbrace{\int_{\partial\mathcal{M}} \sqrt{h}\, K\, d^3x}_{\text{Gibbons-Hawking surface term (Einstein 1916)}} + \underset{\text{matter}}{S_m[\phi, \dots]}$$

$K = h^{ij} K_{ij} = K^i{}_i$

$h = \det(h_{ij})$

Compact hypersurfaces $\hookrightarrow \int_{\partial\mathcal{M}} = 0$

$\underline{3+1}$:

$$S = \int dt \left[\frac{1}{2}\int d^3x\, N\sqrt{h}\left(K_{ij}K^{ij} - K^2 + {}^3R - 2\Lambda\right) + L_m\right]$$

Arnowitt-Deser-Misner (= ADM)

$$S = \int dt\, L = \int dt\,(L_{grav} + L_m)$$

Canonical momenta:

$$\pi^{ij} := \frac{\partial L}{\partial \dot{h}_{ij}} = -\frac{1}{2}\sqrt{h}\,(K^{ij} - h^{ij}K)$$

$$\pi^\phi := \frac{\partial L}{\partial \dot\phi} = -\sqrt{h}\, n^\mu \partial_\mu \phi = -\frac{1}{N}\sqrt{h}\left(\dot\phi - N^i \frac{\partial\phi}{\partial x^i}\right)$$

$$\pi^0 := \frac{\partial L}{\partial \dot N} \approx 0 \quad \& \quad \pi^i := \frac{\partial L}{\partial \dot N_i} \approx 0$$

weak equality

Primary constraints

Gravitational Hamiltonian:

$$H = \int d^3x \left(\pi^0 \dot{N} + \pi^i \dot{N}_i + \pi^{ij} \dot{h}_{ij} \right) - L_{grav}$$

$$= \int d^3x \left(\pi^0 \dot{N} + \pi^i \dot{N}_i + N \mathcal{H} + N^i \mathcal{H}_i \right)$$

$\mathcal{H}_i \hookrightarrow -2\sqrt{h}\, \nabla_j \left(\frac{\pi^{ij}}{\sqrt{h}} \right)$

$\mathcal{H} = \frac{1}{\sqrt{h}} \left(h_{ik} h_{jl} - \frac{1}{2} h_{ij} h_{kl} \right) \pi^{ij} \pi^{kl} - \sqrt{h} \left({}^3R - 2\Lambda \right)$

$\mathcal{H} \approx 0$ & $\mathcal{H}_i \approx 0$ $\oplus$ dynamical eqs. for h_{ij}

$\Longleftrightarrow$ Einstein equations

$\longrightarrow$ Hamiltonian formulation of GR.

Quantization?

* Configuration space (superspace)

$$\text{conf} := \{ h_{ij}(x^i), \phi(x^i) \,|\, x \in \Sigma \} / \text{Diff}(\Sigma)$$

$\text{Diff}(\Sigma)$: space of diffeomorphisms of Σ.

Wave functional

"$\Psi[h_{ij}(x), \phi(x)] \sim \langle h_{ij}, \phi | \Psi \rangle$"

(by analogy w/ ordinary QM)

Quantization à la Dirac

$$\pi^{ij} \longrightarrow -i \frac{\delta}{\delta h_{ij}}$$

$$\pi^{\phi} \longrightarrow -i \frac{\delta}{\delta \phi}$$

$$\left. \begin{aligned} \pi^0 &\longrightarrow -i \frac{\delta}{\delta N} \\ \pi^i &\longrightarrow -i \frac{\delta}{\delta N_i} \end{aligned} \right\} \text{Primary constraints} \Rightarrow \frac{\delta \Psi}{\delta N} = 0 = \frac{\delta \Psi}{\delta N_i}$$

$\longrightarrow$ Wave functional does not depend on N & N_i

$\hookleftarrow$ Not actual degrees of freedom

Relevant degrees of freedom $= h_{ij}, \phi$.

Momentum constraint:

$$\mathcal{H}^i \approx 0 \Longrightarrow i \nabla_j^{(3)} \frac{\delta \Psi}{\delta h_{ij}} = \hat{T}^{0i} \Psi$$

Configurations related by a coordinate transformation yield $\Longleftrightarrow$ wave functionals

$\hat{T}^{0i}$: Stress energy from L_m

Hamiltonian constraint $\mathcal{H} \approx 0$ classically

$\Rightarrow \quad \hat{\mathcal{H}}\Psi = 0$ Wheeler–DeWitt

$$\left[-2G_{ijkl}\frac{\delta^2}{\delta h_{ij}\,\delta h_{kl}} + \sqrt{h}\left(-{}^3R + 2\Lambda + \hat{T}^{00}\right)\right]\Psi = 0$$

$\frac{1}{2\sqrt{h}}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right)$: DeWitt metric

Issues $\frac{\delta^2}{\delta h\,\delta h}$ badly defined (factor ordering...
needs regularisation & renormalise)

$\hat{\mathcal{H}}\Psi = 0$: Timeless Schrödinger equation.

Simple example : $S_m = \int d^4x\sqrt{-g}\,P(w,\phi)$

perfect fluid

equation of state.

Minisuperspace = restrict attention to FLRW:

$\phi \to$ velocity potential

$$ds^2 = -N^2(\tau)d\tau^2 + a^2(\tau)\gamma_{ij}dx^i dx^j$$

Canonical transformations

Time

$$\Rightarrow \quad H = N\left(-\frac{P_a^2}{4a} - Ka + \frac{P_T}{a^{3w}}\right)$$

choose a^{3w} !

$$\Rightarrow \quad i\frac{\partial\psi}{\partial T} = \frac{1}{4}\frac{\partial^2\psi}{\partial x^2} \quad \text{free particle}$$

$$x \propto a^{\frac{3}{2}(1-w)}$$

Solution:

Gaussian Wave Packet

$$\psi = A(x,T)\,e^{iS(x,T)}$$

$$A \propto \sqrt{1+(T/T_0)^2}\;e^{-\frac{T_0 x^2}{T_0^2+T^2}}$$

$$S \to \frac{Tx^2}{T_0+T^2} + \frac{\pi}{4} + \frac{1}{2}\arctan\left(\frac{T_0}{T}\right)$$

Transition amplitude between $H<0$ & $H>0$

or various ways to calculate trajectory

$$a(T) = \langle\hat{a}\rangle_\psi,\ \sqrt{\langle\hat{a}^2\rangle},\ dBB,\ |q(\tau),p(\tau)\rangle.$$

$$\Rightarrow \quad a \sim a_B\sqrt{1+(T/T_0)^2}$$

Bounce avoiding the singularity.

Loop quantum gravity $\Rightarrow$ similar scenarios (other variables = Ashtekar...)

$\oplus$ $\exists$ energy scale $E_{LQG} \lesssim E_P$ at which Lorentz violated

$$\left.\begin{array}{l} \rightsquigarrow \gamma_1, E_1 \\ \rightsquigarrow \gamma_2, E_2 \end{array}\right\} \quad \Delta t_{\text{arrival}} \propto \left(\frac{\Delta E}{E_{LQC}}\right)^{\alpha}$$

α → not well known.

⟹ Constraints on E_{LQC}.

Bounce implemented phenomenologically:

$$H^2 \sim \frac{1}{3}\rho - \rho^2/\rho_{LQC}$$

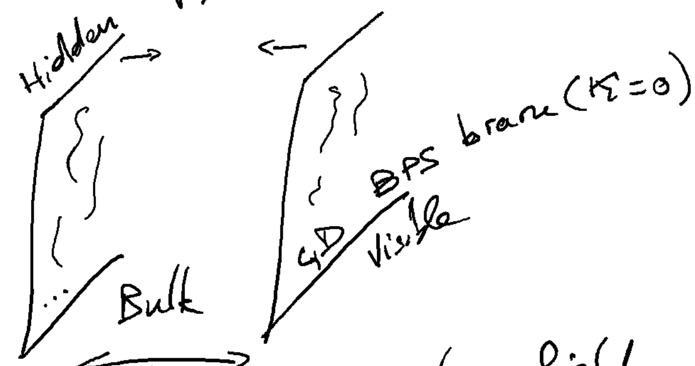
same as in string theory for a brane but negative sign (⟺ timelike extra dim!)

Debates on perturbations...

Other example of quantum gravity approache: the Ekpyrotic scenario = branes in more than 4D...

+ Cyclic extension.

Initial ekpyrotic:



5^{th} dim ⟺ scalar field

$$S_5 \propto \int_{\mathcal{M}_5} d^5x \sqrt{-g_5}\left[\frac{1}{2}R_{(5)} - \frac{1}{2}(\partial\phi)^2 - e^{2\phi}F^2\right]$$

↓ compactification (integrate out d.o.f)

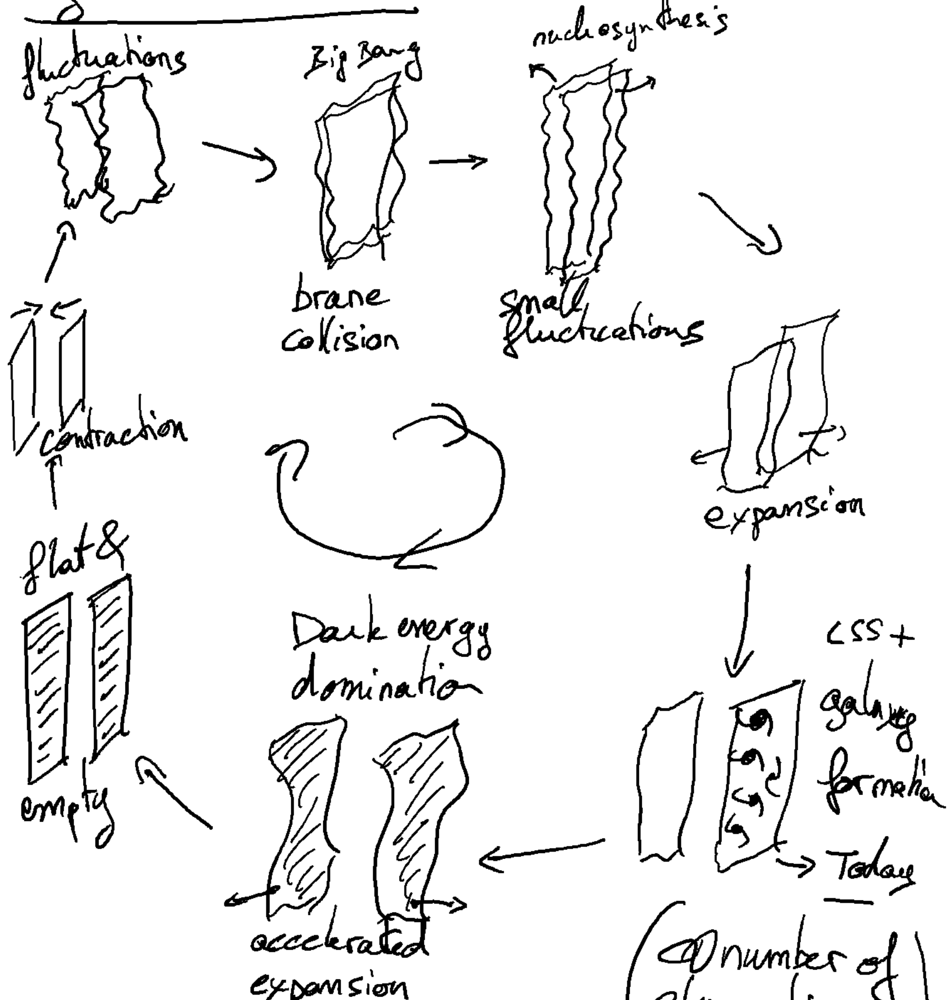
$$S_4 \propto \int_{\mathcal{M}_5} d^4x \sqrt{-g}\left[\frac{1}{2}R_{(4)} - \frac{1}{2}(\partial\varphi)^2 - V(\varphi)\right]$$

$V(\varphi) \to -V_1 e^{-\alpha\varphi}$

⟹

$\boxed{\text{fissues}}$

Cyclic extension:

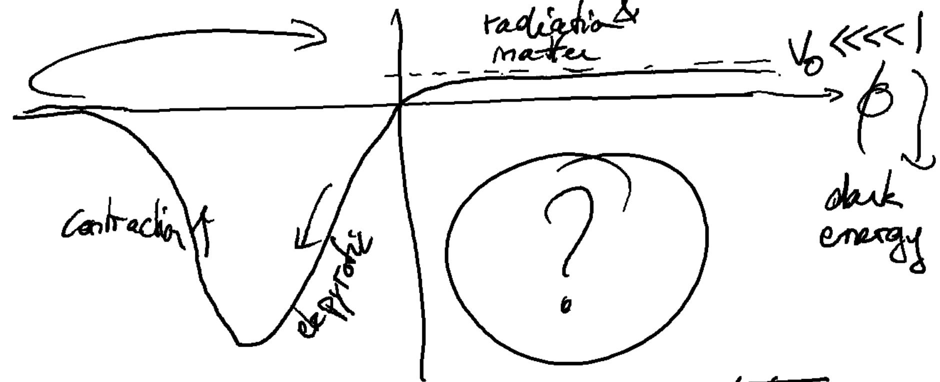


(number of ekpyrotic issues)

Eternal cycles...
(epi-cycles?)

Potential in "one field" cyclic

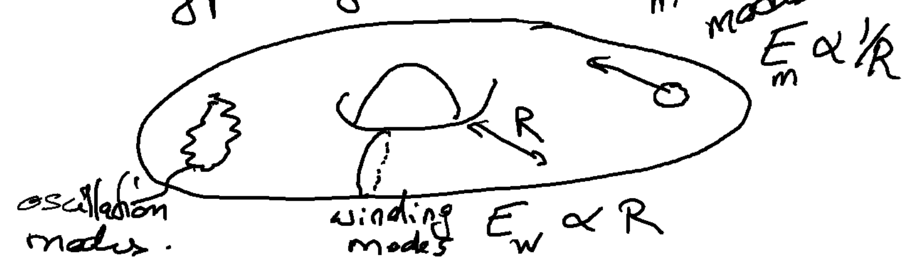
(new ekpyrotic, 2 fields, conversion adiabatic → entropic modes... Phoenix model...)



Another string-based model:

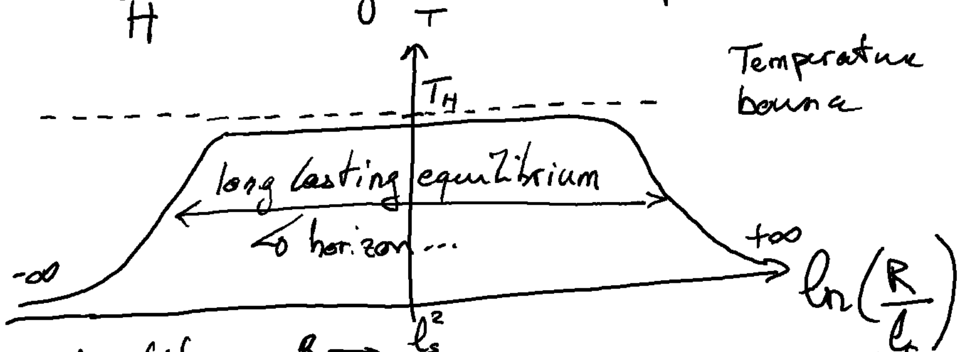
String gas cosmology.

- Space = 9D torus radius R
- 3 types of modes:



⇒ ∃ maximum temperature for a gas of strings in thermal equilibrium

T_H = Hagedorn temperature.



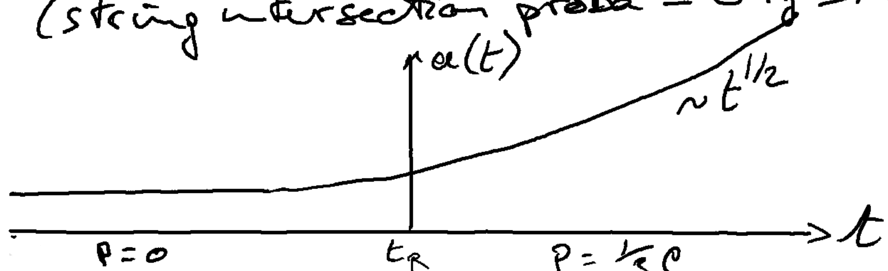
T-duality: $R \longrightarrow \frac{l_s^2}{R}$

Initial condition = very small & dominated by winding modes

⇒ need to annihilate to grow space

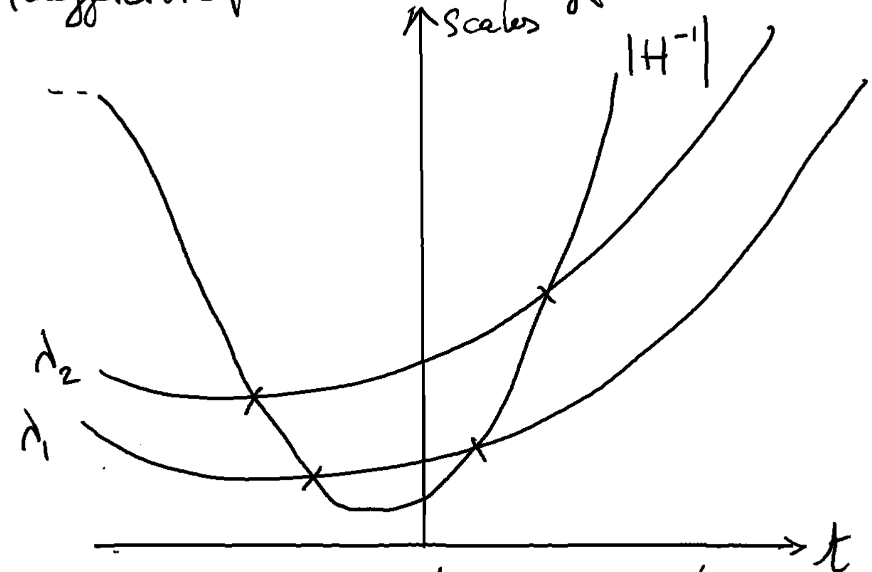
⇒ only 3 spatial dimensions

(string intersection proba = 0 if $D>3$)



Measuring lengths $\neq a(t)$ $\boxed{!}$

(different operators in different phases)



approximately scale invariant spectrum of perturbations...

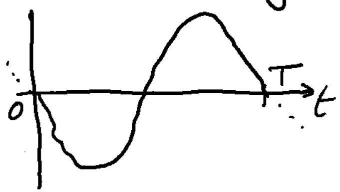
(slightly red though, ok!)

spectrum of gravitational waves = slightly blue!

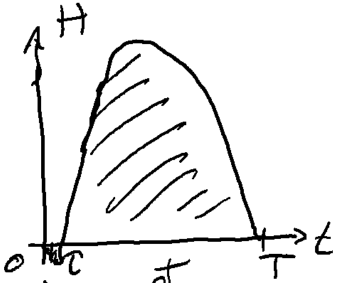
Cyclic cosmology:

H periodic: $H(t+T) = H(t)$

but not symmetric



but rather



$$\left|\int_0^{\tau} H\,dt\right| \ll \int_{\tau}^{T} H\,dt$$

$$H(t) = H_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right)\right]$$

$$\frac{d\ln a}{dt} \Rightarrow a = a_0 e^{H_0 t} \exp\left\{\frac{T}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[a_n \sin - b_n \cos\right]\right\}$$

$$\Rightarrow a(t+T) = e^{H_0 T} a(t) = e^{N} a(t)$$

Assume N large $\Rightarrow$ "solve" entropy pb.

Issue: geodesic completeness

Def: a spacetime is geodesically complete if all geodesic worldlines through spacetime are both future & past infinite as measured by proper time along the worldline from any finite time t:

$$\int_{-\infty}^{t} ds = \int_{t}^{\infty} ds = \infty$$

Example: de Sitter inflation $a = e^{H_0 t}$

$\lim_{t\to-\infty} a(t) = 0$

Comoving observer = constant spatial coordinates

$\Rightarrow ds = dt$ & $t \in [-\infty, +\infty]$

$\Rightarrow \int ds = \infty$ ok

Non comoving geodesic, test particle:

$$ds^2 = -dt^2 + a^2(t)\, d\vec{x}^2$$

$$\Rightarrow \Gamma^0_{ij} = a\dot{a}\,\delta_{ij}$$

$$\& \quad \Gamma^i_{0j} = \Gamma^i_{j0} = \frac{\dot{a}}{a}\delta^i_j$$

$$u_\mu u^\mu = g_{\mu\nu} u^\mu u^\nu = g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$$

$$= -\left(\frac{dt}{ds}\right)^2 + a^2(t)\left|\frac{d\vec{x}}{ds}\right|^2 = -1$$

Geodesic eq:

$$\frac{d^2x^i}{ds^2} + \underbrace{\Gamma^i_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}}_{\underbrace{\Gamma^i_{0j}\frac{dt}{ds}\frac{dx^j}{ds} + \Gamma^i_{j0}\frac{dx^j}{ds}\frac{dt}{ds}}_{= 2\frac{\dot{a}}{a}\frac{dt}{ds}\frac{dx^i}{ds}}} = 0$$

$$\dot{a}\frac{dt}{ds} = \frac{da}{ds} \quad \Rightarrow \quad \frac{d}{ds}\left[a^2(t)\frac{d\vec{x}}{ds}\right] = 0$$

Integrate ⇒ constant along geodesic

$$a^2\left|\frac{d\vec{x}}{ds}\right| = v_0 \quad \text{constant}$$

$$\Rightarrow \left(\frac{d\vec{x}}{ds}\right)^2 = v_0^2 a^{-4}$$

$$\Rightarrow \frac{dt}{ds} - \underbrace{a^2\left(\frac{d\vec{x}}{ds}\right)^2}_{v_0^2 a^{-2}} = 1$$

$$\Rightarrow ds = \frac{dt}{\sqrt{1+v_0^2 a^{-2}(t)}}$$

de Sitter $\quad a = e^{H_0 t}$

$$\Rightarrow \underbrace{\int_{-\infty}^{0}\frac{dt}{\sqrt{1+v_0^2 e^{-2H_0 t}}}}_{\int_{-\infty}^{0} ds \,<\, \infty \;!} = \frac{1}{2H_0}\ln\left(\frac{\sqrt{1+v_0^2}+1}{\sqrt{1+v_0^2}-1}\right)$$

geodesically incomplete

Robert Geroch
Ann. Phys. 48, 526 (1968)

Generalize for arbitrary inflation

= Borde, Guth & Vilenkin

PRL 90, 151301 (2003)

Cyclic ekpyrotic

= Same (what dominates is $e^{H_0 t} \ldots \exp\{\sin \& \cos\}$ do not change)

Kinney & Stein 2110.15380

Brilliant, jealous I didn't think of it!

A short history: Predate inflation by 50 yrs!

Richard Tolman Phys Rev 3, 1758 (1931)

Georges Lemaître, Annales Société Scientifique de Bruxelles A53, 51 (1933)

Reprinted GRG 29, 641 (1997)

$S_{n+1} \gg S_n$ straightforward thermodyn!

- Parker & Fulling 1973

Massive scalar field & large occupation number $\langle a^\dagger a \rangle \gg 1$

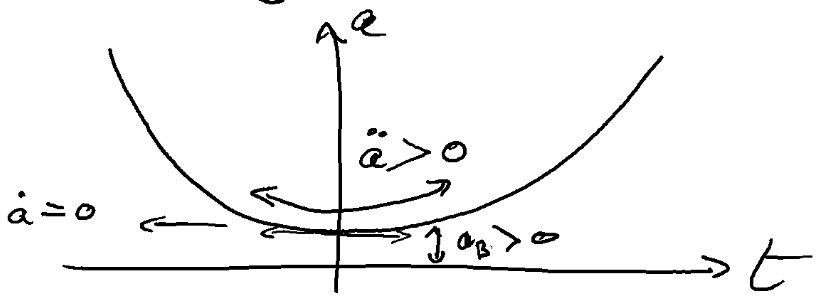
- Starobinsky 1978

+ ... Plethora of models

I.2) Getting a bounce!

→ How? technicalities

→ Why? solving puzzles



Friedman eqs:

$$H^2 + \frac{K}{a^2} = \frac{1}{3}\rho$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3P)$$

↳ $> 0 \Rightarrow \rho + 3P < 0 \Rightarrow \cancel{SEC}$

$\dot{a} = 0 \Rightarrow \boxed{\frac{K}{a_B} = \frac{1}{3}\rho_B}$ → Flat case $\Rightarrow \rho_B = 0$

→ $K > 0$

$$\dot{H} = \frac{K}{a^2} - \frac{1}{2}(\rho + P)$$

→ $K = 0 \Rightarrow \underbrace{\rho + P < 0}_{\cancel{NEC}}$

n^μ = arbitrary null vector

$g_{\mu\nu} n^\mu n^\nu = 0$

Null Energy Condition: $T_{\mu\nu} n^\mu n^\nu \geq 0$

⇒ True for all known acceptable classical fluids.

$$T_{\mu\nu} = (\rho + P) u_\mu u_\nu + P g_{\mu\nu}$$

u: timelike

$$\Rightarrow (\rho + P)\underbrace{(u \cdot n)^2}_{>0} \geq 0$$

$$\boxed{\rho + P \geq 0}$$

Scalar field

$\rho + P = \dot{\phi}^2 > 0$

positive definite

~~NEC~~ necessary for $K=-1$ & 0

$K=1$: unclear

ex: Scalar field + $K=1$

$\exists$ bouncing solutions) a set of measure zero in the space of initial conditions

$$\rho_B = \rho(a_B) = \frac{3K}{a_B^2}$$

$$P_B = P(a_B) < -\frac{K}{a_B^2}$$

$\Rightarrow$ Potentially leads to instabilities

Example: a hydrodynamical fluid

$$a = a_B + b\eta^{2n} + d\eta^{2n+1} + e\eta^{2n+2} + \dots$$

+ Evolution of Bardeen potential

(see D. Wands lectures and later...)

Possible cases:

$$\left.\begin{array}{l} \text{(i) } n>1 \ \& \ K\neq 0 \\ \text{(ii) } n>1 \ \& \ K=0 \\ \text{(iii) } n=1 \ \& \ d\neq 0 \ \forall K \end{array}\right\} \underline{\Phi(t_B)=\infty!}$$

(iv) $n=1$ & $d=0$ $\forall K$ symmetric bounce

Even for $K=+1$, $\exists\, t_{\cancel{NEC}}$;

$$\rho(t_{\cancel{NEC}}) + P(t_{\cancel{NEC}}) = 0$$

(~~NEC~~ for a finite amount of time)

$\Phi(t_B) < \infty$ but $\Phi(t_{\cancel{NEC}}) = \infty$ $\underline{!}$

✓

Possible regularization with entropic perturbations (non adiabatic)

$\Rightarrow$ must be suppressed later (not observed...) + backreaction issues!

Reproducing observations (+ Solving SM puzzles)

Data: - $\Omega_K = \frac{K}{a^2 H^2} = 0 \pm 5\times 10^{-3}$

(Flatness)

- $n_s = 0.96$
- $f_{NL} \sim 0$
- Isocurvature modes $\lesssim 1\%$
- $r = \frac{T}{S} < 0.1$

$\rightarrow$ quantum vacuum fluctuations of single scalar d.o.f.

Compatible w/ Slow Roll inflation

(See W. Kinney's lectures)

Def: redshift $\frac{\lambda_{received}}{\lambda_{emitted}} = 1+z = \frac{a_{received}}{a_{emitted}}$

SM problems:

- $\exists$ Big-Bang singularity

$\bar{t}_i$; $a(\bar{t}) = 0$ set $\bar{t} = 0$ by definition

radiation domination: $a \propto \sqrt{t}$

Particle horizon $\Leftrightarrow \int_{t_i} \frac{dt}{a(t)} < \infty$

(event horizon $\rightarrow \int^{\infty} \frac{dt}{a}$)

$\Longrightarrow$ Divides the particles in the universe between those that have been observed and those that have not.

Horizon size $d_H = a(t) \int_{t_i}^{t} \frac{dt}{a(t)}$

Constant $w = \frac{P}{\rho}$

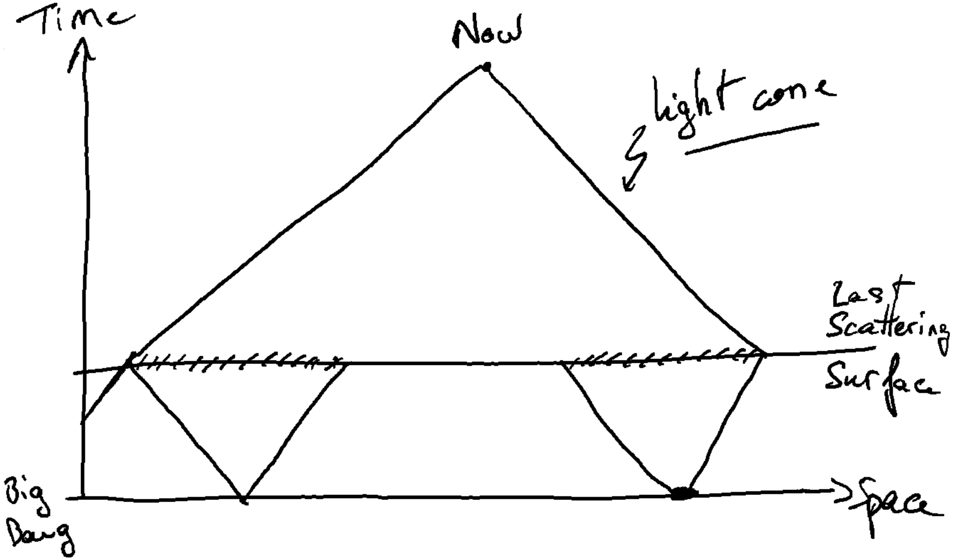
$\Rightarrow d_H = \frac{3(1+w)}{1+3w} t \left[1 - \left(\frac{t_i}{t}\right)^{\frac{1+3w}{3(1+w)}} \right]$

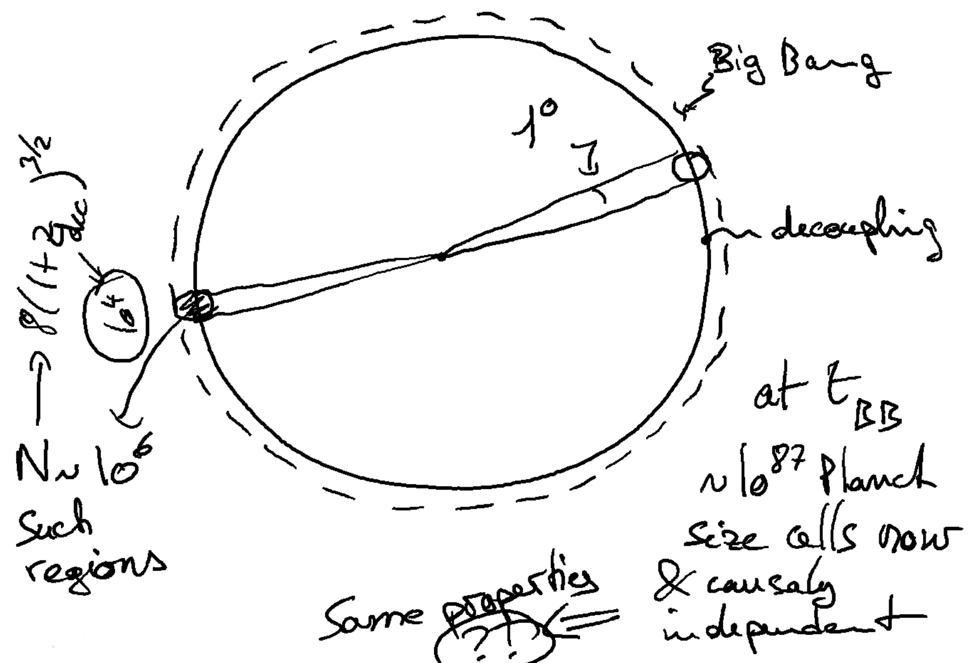
$\boxed{t_{ini} \to -\infty \Rightarrow d_H \text{ large}}$

$N \sim 60$

$z_{max} \sim 10^{28}$

$\xrightarrow[w=1/3]{} 2t\left(1 - \sqrt{t_i/t}\right) < \infty$





Flatness:

$$\frac{d\Omega_K}{d\ln a} = (1+3w)(1-\Omega_K)\Omega_K$$

$$\Downarrow$$

$$\Omega_K = \frac{\Omega_{K0}}{(1-\Omega_{K0})(1+z)^{1+3w} + \Omega_{K0}}$$

Observations: $|\Omega_K| = |\Omega - 1| < 0.01$

$$\Rightarrow \quad |\Omega(z_{eq}) - 1| < 3\times 10^{-5}$$

$$|\Omega(z_{pl}) - 1| < 10^{-60}$$

?

Recall $\dfrac{K}{a^2H^2} = \sum_i \Omega_i - 1 = \Omega - 1$

$$\Omega_i = \frac{\rho_i}{\rho_{crit}} = \frac{H^2(t_0)}{H^2(t)}\Omega_i(t_0)\left(\frac{a}{a_0}\right)^{-3(1+w)}$$

Other potential problems

→ Monopoles in GUT
= original motivation for inflation ?

→ Homogeneity

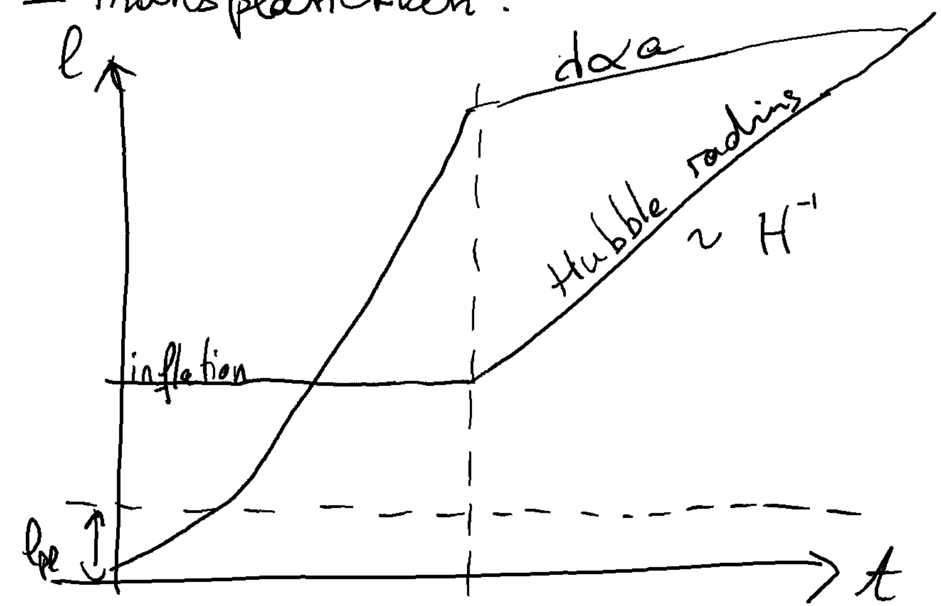
→ Transplanckian
$$\exists t;\ l(t) = l_0 \frac{a(t)}{a_0} \leq l_{pl} \ !$$
⇓
related w/ singularity

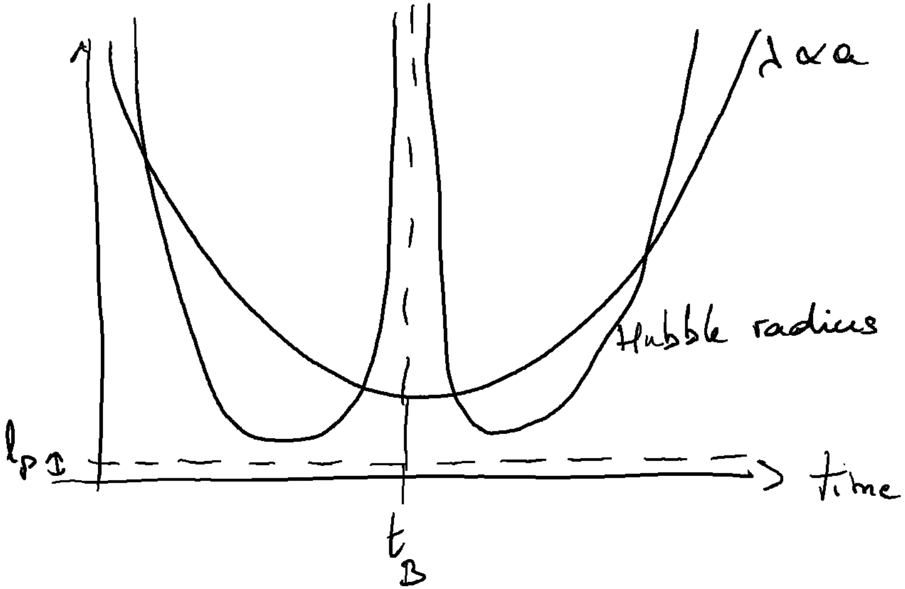
expand on GUT theories $G \to H \to SU_3 \times SU_2 \times U_1 \to \dots$
& Higgs field phases
⇒ Not so much an issue in string theory, not sure why!

Solutions:

- Singularity = trivially ok
(see however cyclic model...)
+ singular bounce !

- Transplanckian:



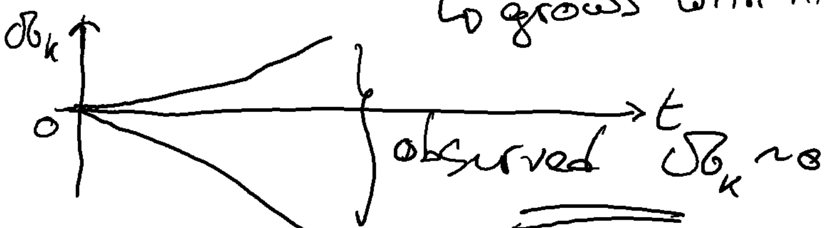


Flatness: $\Omega_K = \dfrac{K}{a^2H^2} = \dfrac{K}{\dot a^2}$

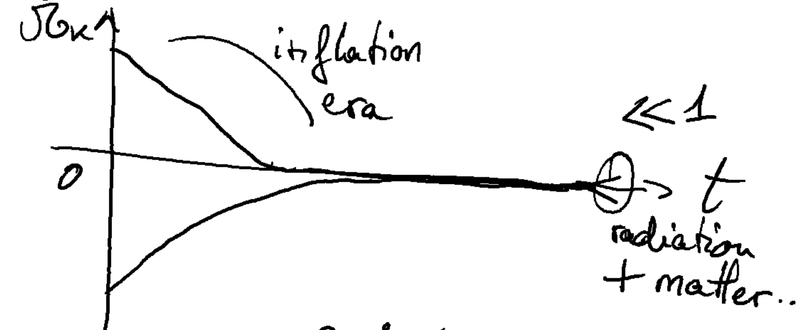
$$\Longrightarrow \quad \frac{d|\Omega_K|}{dt} = -2|K|\frac{\ddot a}{\dot a^3}$$

$\left.\begin{matrix}\ddot a < 0 \\ \dot a > 0\end{matrix}\right\}$ s.r. $\Rightarrow \dfrac{d}{dt}|\Omega_K| > 0$

↳ grows with time

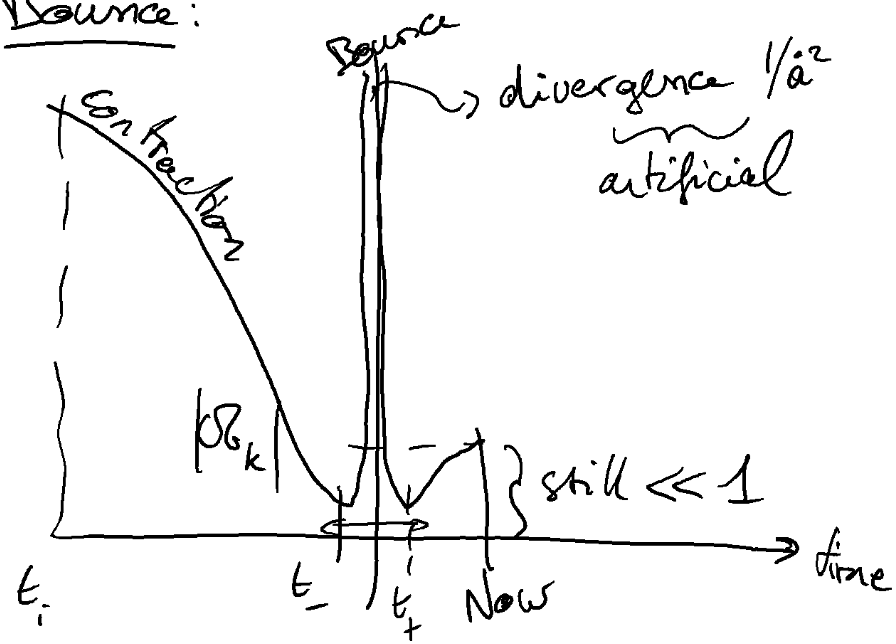


Inflation: $\dot a > 0$ & $\ddot a > 0$



$\ddot a < 0$ & $\dot a < 0$

Bounce:



Contracting universe dominated by fluid X:

$$\Omega(t) = \frac{\Omega_X(t_i)}{\Omega_X(t_i) - \underbrace{|\Omega(t_i)-1|}_{\Omega_K(t_i)}\left(a/a_i\right)^{1+3w_X}}$$

$$\Rightarrow \Omega(t_-) - 1 = \Omega_K(t_-) \simeq \frac{\Omega_K(t_i)}{\Omega_X(t_i)}\left(\frac{a_-}{a_i}\right)^{1+3w_X}$$

$\uparrow$ end of contracting phase.

Beginning of expanding phase = radiation ...

$$\Omega_K(t_+) \simeq \underbrace{\frac{\Omega_K(t_0)}{\Omega_{rad}(t_0)}}_{\to 10^{-4}}\left(\frac{1}{1+z}\right)^2 \qquad \underline{t_0 = \text{Now}}$$

@ nucleosynthesis

$$z_{nuc} = 10^8 \Rightarrow \Omega_K(t_{nuc}) \sim 10^{-15}$$

$$\Delta\Omega_K = \Omega_K(t_+) - \Omega_K(t_-)$$

Calculate using

$$a(t) = a_B\left[1 + \left(\frac{t}{t_c}\right)^2 + \overset{\to\, \mathcal{O}(1)}{\beta}\left(\frac{t}{t_c}\right)^3 + \cdots\right]$$

$$\Rightarrow \Delta\Omega_K \simeq -\frac{3\beta}{2}\left(\frac{t_c}{a_B}\right)^2$$

Assume $N = -\ln\left(\frac{a_i}{a_B}\right)$ "e-folds" of contraction, $N \sim 60$

$z_+ \sim 10^{28}$ (for horizon...)

$$\underbrace{\left(\frac{a_-}{a_i}\right)^{1+3w_X}}_{e^{-2N}} - \frac{3\beta}{2}\left(\frac{t_c}{a_B}\right)^2 \leq \frac{10^6}{z_+^2}$$

demand very short duration:

$$\left(\frac{t_c}{a_B}\right) \lesssim 10^{-25} \quad (!)$$

instantaneous bounce.

A new issue (potentially dramatic):
Shear & BKL instability

Bianchi I metric (example)

$$d\vec{x}^2 = e^{2\theta_x(t)}dx^2 + e^{2\theta_y(t)}dy^2 + e^{2\theta_z(t)}dz^2$$

$$\theta_1 + \theta_2 + \theta_3 = 0$$

$$ds^2 = -dt^2 + a^2(t)\,d\vec{x}^2$$

$\Rightarrow$ Einstein eqs:

$$\begin{cases} H^2 = \frac{1}{3}\rho + \frac{1}{6}\sum_i \dot{\theta}_i^2 = \frac{1}{3}(\rho + \rho_\theta) \\ \dot{H} = -\frac{1}{2}(\rho + P) - \frac{1}{2}\sum_i \dot{\theta}_i^2 \end{cases}$$

$$\ddot{\theta}_i + 3H\dot{\theta}_i = 0 \Rightarrow \dot{\theta}_i \sim a^{-3}$$

$$\underbrace{}_{\rho_\theta + P_\theta} \qquad \rho_\theta = \frac{1}{2}\sum_i \dot{\theta}_i^2$$

$$\Downarrow$$

$$P_\theta = \frac{1}{2}\sum_i \dot{\theta}_i^2 = \rho_\theta$$

$$\Downarrow$$

$$w_\theta = 1$$

$$\rho_\theta \sim a^{-3(1+w_\theta)} \sim a^{-6}$$

$$\rho_m \propto a^{-3}$$

$$\rho_r \propto a^{-4}$$

$$\rho_\theta \propto a^{-6}\Big\}\text{ eventually dominate } a \to 0$$

$\Rightarrow$ BKL instability!

FLRW description no longer valid & singularity reached!

Bianchi IX $\Rightarrow K_0 = 1$

$\Leftrightarrow$ same result, K negligible.

Ekpyrotic solution:

$$\phi \to w_\phi \gg 1 \Rightarrow \rho_\phi \propto \underbrace{a^{-3(1+w_\phi)}}_{\text{dominates!}}$$

Implementing a bounce w/ fields

- Quintom cosmology

$$\mathcal{L}_\phi = -\frac{1}{2}(\partial\phi)^2 - V(\phi) + \underbrace{\frac{1}{2}(\partial\xi)^2}_{\text{Ghost}}$$

$V(\phi) \to \frac{1}{2}\phi^2 m^2 \to \langle\rho\rangle \sim a^{-3}$

or perfect fluid

Ghost $\to \rho \propto a^{-6}$

$\Rightarrow$ instabilities

Lee-Wick model

$$\rho + P = \dot\phi^2 - \dot\xi^2$$

$\Downarrow$

- Ghost condensate

(kinetic analog of Higgs mechanism)

$\hookrightarrow$ dynamical ghost

$$\mathcal{L} = P(X), \quad X = -\frac{1}{2}(\partial\xi)^2 \quad \text{Pressure}$$

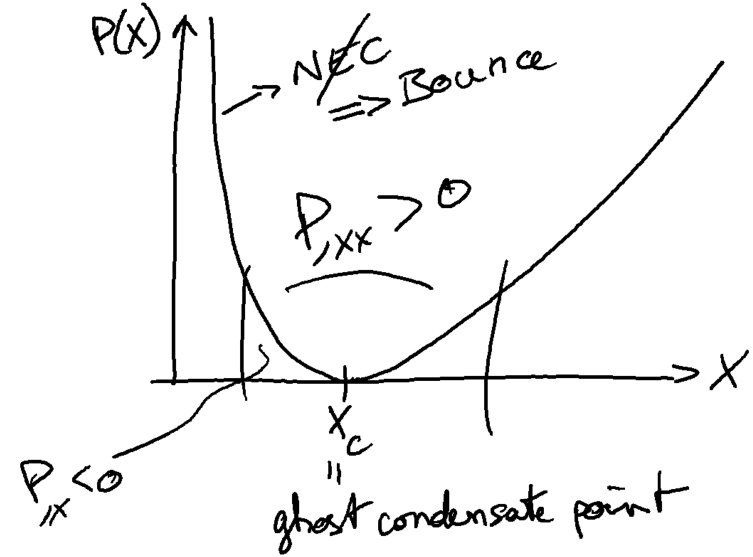
$$\Longrightarrow \frac{d}{dt}\left(a^3 P_{,X}\dot\phi\right) = 0 \qquad P_{,X} = \frac{dP}{dX}$$

$$\boxed{\rho = 2XP_{,X} - P}$$

Assume $\exists X_c ; P_{,X}\big|_{X=X_c} = 0$

$$\Longrightarrow \phi = \sqrt{2X_c}\, t$$

If $P_{,X} < 0$ in $[X_-, X_+]$ finite



$$\rho + P = 2XP_{,X}$$

$\to$ K-bounce

$\to$ new Ekpyrotic

$\to$ Matter bounce

instabilities (no stable Poincaré invariant vacuum)

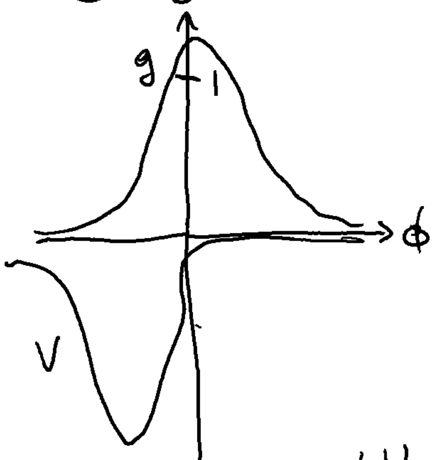
• Galileon & Ghost condensate.

$$\mathcal{L} = K(\phi, X) + G(\phi, X)\,\Box\phi + P_{dust}$$

$$G(\phi,X) \longrightarrow \gamma X$$

$$\downarrow$$

$$[1-g(\phi)]X + \beta X^2 - V(\phi)$$



$$g(\phi) = \frac{2g_0}{e^{-\sqrt{\frac{2}{p}}\phi} + e^{b_g\sqrt{\frac{2}{p}}\phi}}$$

$$V(\phi) = -\frac{2V_0}{e^{-\sqrt{\frac{2}{q}}\phi} + e^{b_V\sqrt{\frac{2}{q}}\phi}}$$

$$|\phi| \gg 1 \quad \& \quad \phi < 0$$

$$\Downarrow$$

$$V \sim V_{ekp} = -2V_0 e^{-\sqrt{\frac{2}{q}}\phi}$$

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi}$$

$$\Rightarrow \quad \phi = \sqrt{2q}\,\ln\left[\sqrt{\frac{2V_0}{q(1-3q)}}\;t\right]$$

$$\Rightarrow w_\phi = -1 + \frac{2}{3q} \sim \frac{2}{3q}, \quad q \ll 1$$

↳ Ekpyrotic phase possible.

& $a \sim (-t)^p$, $\quad H \sim \frac{p}{t}$

Complete scenario:

1) Pressureless matter contraction

   $w \approx 0$, $\quad a \sim t^{2/3}$

   production of scale invariant spectrum

2) Ekpyrotic contraction

   removes any primordial shear

3) Bounce phase: $g(\phi) > 1$

   for $\phi \in [\phi_-, \phi_+]$

   $\sim -\sqrt{\frac{p}{2}}\ln 2g_0$ $\qquad$ $\sim \sqrt{\frac{p}{2}}\,\frac{\ln(2g_0)}{b_g}$

4) Fast-roll kinetic domination

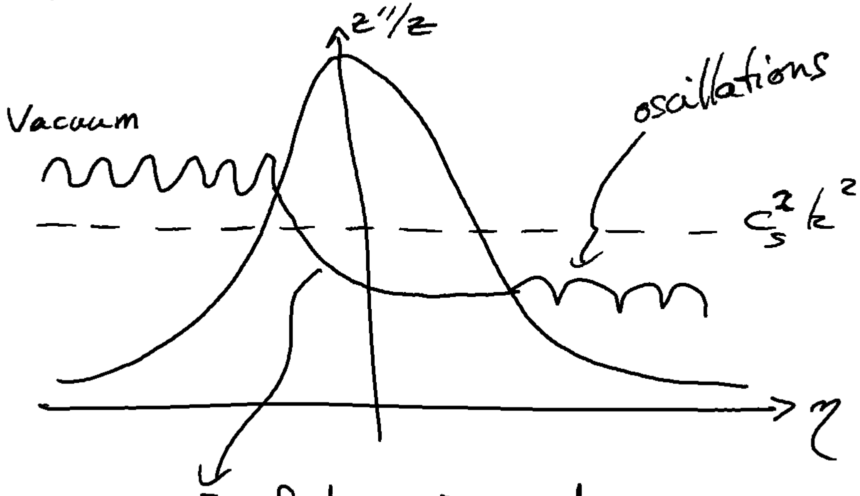
   $V \ll \dot{\phi}^2 \Rightarrow w_\phi \simeq 1 \Rightarrow a(t) \propto t^{1/3}$

Perturbations: See D. Wands lectures

$$v_k'' + \left(c_s^2 k^2 - \frac{z''}{z}\right) v_k = 0$$

Mukhanov-Sasaki

1 scalar field $\quad z = \frac{a\dot{\phi}}{H}$



Similar behavior for tensor modes

$$\mu_k'' + \left(k^2 - \frac{a''}{a}\right)\mu_k = 0$$

Vector modes? Not dynamical $\sim 1/a^2$

$\Rightarrow$ velocity perturbation $V \propto k^2 a^{3w-1}$

issue? Production of $\vec{B}$?

Exponential potential $\quad a \propto |t|^q$

$$\phi = \frac{2q}{t^2}, \quad H = \frac{q}{t}$$

$$z = \frac{a\dot{\phi}}{H} \propto a \propto (-\eta)^{\frac{q}{1-q}}$$

$$\Rightarrow \frac{z''}{z} = \frac{q(2q-1)}{(q-1)^2\eta^2} = \frac{\alpha}{\eta^2}$$

$\Rightarrow$ known solution $\rightarrow$ Bessel functions

$$v_k = \sqrt{-\eta}\, Z_{\left|\frac{1-3q}{2(1-q)}\right|}(-k\eta) \;\&\; \xi = \frac{v}{z}$$

long wavelenght limit

$$\mathcal{P}_\xi(k) = \frac{k^3}{2\pi^2}|\xi_k|^2 \propto k^{\frac{2(2-3q)}{1-q}}$$

deep blue spectrum!

$$\Rightarrow n_s - 1 = \frac{12w}{1+3w} \sim 4 - 2q + \dots$$

actual matter fluid: $w > 0 \Rightarrow n_s > 1$ blue tilt

Scalar field mimicking $\rightsquigarrow$ redish ok

Bounce scenario:

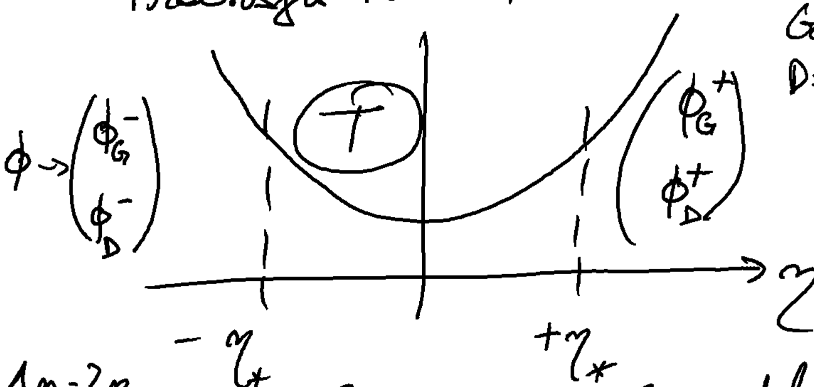
- Needs ekpyrosis to remove shear + another mechanism for perturbations

$\Rightarrow$ Matter contraction

$\Rightarrow$ 2-fields & isocurvature
↳ Curvaton mechanism

- Passing through the bounce?

Transfer matrix

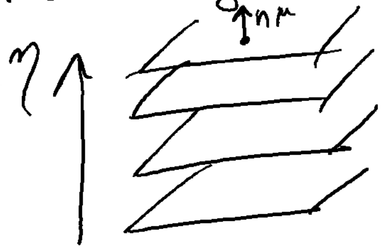


G = growing
D = decaying

$\Delta\eta_B = 2\eta_*$

$$T = \begin{bmatrix} T_{GG}(k) & T_{GD}(k) \\ T_{DG}(k) & T_{DD}(k) \end{bmatrix}$$ model dependent

Possible generic matching conditions?

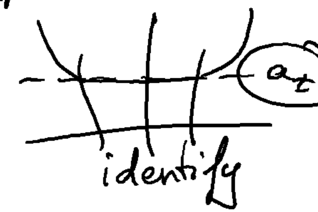


$$q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

Israel junction conditions

$$[q_{\mu\nu}]_\pm = q_{\mu\nu}(\eta_+) - q_{\mu\nu}(\eta_-) = 0$$

$$\Rightarrow \quad [a]_\pm = 0$$



Extrinsic curvature

$$K_{\mu\nu} = \frac{1}{2}\left(q_\mu{}^\sigma \nabla_\sigma n_\nu + q_\nu{}^\sigma \nabla_\sigma n_\mu\right)$$

$$\Rightarrow \quad K^i{}_j = -\frac{\mathcal{H}}{a}\delta^i{}_j$$

$[K^i{}_j]_\pm = 0$ impossible ($\mathcal{H}$ changes sign)

$\Rightarrow \quad [K^i{}_j] = S^i{}_j$ surface stress tensor

$\Rightarrow \quad \mathcal{H}_+ - \mathcal{H}_- = -a_B P_S$ → negative surface tension

Freedom ... model dependent!

Possible instabilities

Ghost condensate

$$\zeta_k'' + 2\frac{z'}{z}\zeta_k' + c_s^2 k^2 \zeta_k = 0$$

$$c_s^2 = \frac{p_{,X}}{p_{,X} + 2X p_{,XX}} \qquad z = a\sqrt{\frac{\dot{H}}{c_s^2 H^2}}$$

$\Rightarrow$ Mukhanov-Sasaki $v_k = z\zeta_k$

$$v_k'' + \left(c_s^2 k^2 - \frac{z''}{z}\right)v_k = 0$$

$$\Rightarrow \zeta_k \sim k^{-1/2} + \sqrt{k}\int z^{-2} d\eta + \cdots$$

(vacuum initial conditions)

$\hookrightarrow$ negligible & decaying in inflation but may be growing and large... spoil prediction (cf. Transfer matrix) $\Rightarrow$ blue spectrum!

$+ (\rho + p) \leq 0 \Rightarrow c_s^2 < 0$ for finite time $\Rightarrow$ exp growth.

Conclusions:

- Bounces solves SM puzzles + new ones.
- Difficult to implement
  - GR + classical / quantum matter
    $\downarrow$
    or modified
  - Instabilities...
    (shear, vectors, curvature)
- Quantum cosmology or string models

$\Rightarrow$ Testbed for theories?

$\hookrightarrow$ quantum theory!

$\Downarrow$

Perturbations = quantum!

background = classical ?