Counting rational points on modular curves of genus 0 over number fields

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Overview of this talk

- Motivation and recent history
- Weighted projective spaces
- Height functions
- Applications to modular curves
- Work in progress

Parts of this are joint work with Filip Najman and with Irati Manterola Ayala.

Motivation and recent history

Why count points on curves of genus 0?

- Quantify how ‘common/rare’ it is for elliptic curves to admit a given level structure
- Depends on the ‘fine structure’ of a (modular) curve (moduli stack, not coarse moduli space)
- Prove instances of the ‘stacky’ Batyrev–Manin conjecture for stacks occurring ‘in nature’

Note: no knowledge of stacks required for this talk!

Motivation and recent history

For an elliptic curve E over $\mathbf{Q}$, choose a short Weierstraß model

$$E\colon y^2 = x^3 + ax + b$$

with $\quad a, b \in \mathbf{Z}, \quad 4a^3 + 27b^2 \neq 0, \quad \nexists\, p \text{ prime} : p^4 \mid a, p^6 \mid b.$

We call such a pair (a, b) *primitive*. Define the *height* of E as

$$H(E) = \max\{|a|^3, |b|^2\}.$$

Remark

We could also count by other measures of ‘complexity’, such as Faltings height, discriminant or conductor. The above height is the most convenient for our purposes.

Counting elliptic curves over $\mathbf{Q}$

Question 0

How many elliptic curves of bounded height are there?

Answer: count primitive integral points (a, b) in the box $[-T^{1/3}, T^{1/3}] \times [-T^{1/2}, T^{1/2}]$.

By a classical sieve argument, a fraction $1/\zeta(10)$ of the points is primitive.

Theorem

As $T \to \infty$, we have

$$\#\{(a, b) \in \mathbf{Z}^2 \text{ primitive} \mid |a| \leq T^{1/3}, |b| \leq T^{1/2}\} \sim \frac{4}{\zeta(10)} T^{5/6}.$$

Elliptic curves over $\mathbf{Q}$ with given torsion

Question 1

What happens if we restrict to elliptic curves with given level structure, for example torsion subgroup C_n $(1 \leq n \leq 10$ or $n = 12)$ or $C_2 \times C_n$ $(n = 2, 4, 6, 8)$?

Theorem [Harron and Snowden, 2017]

If Γ is one of the above torsion groups,

$$\#\{E \mid E(\mathbf{Q})_{\text{tor}} \simeq \Gamma, H(E) \leq T\} \asymp T^{1/d(\Gamma)}$$

for some explicit $d(\Gamma) \in \mathbf{Q}_{>0}$.
Moreover, for $\Gamma = C_1, C_2, C_3$, the left-hand side is asymptotically equivalent to an explicit constant times $T^{1/d(\Gamma)}$.

Counting elliptic curves with given torsion

Idea of Harron and Snowden's proof

- Construct families

$$\mathcal{E}\colon y^2 = x^3 + f(t)x + g(t)$$

with $f, g \in \mathbf{Q}[t]$ admitting an inclusion of Γ.

- Estimate how many $t \in \mathbf{Q}$ give rise to elliptic curves over $\mathbf{Q}$ with height $\leq T$.

Weighted projective spaces

Question 2

Can we determine the growth rate, or even prove asymptotic formulae, for elliptic curves over number fields (with given level structure) with respect to a suitable notion of height?

We first need such a ‘suitable notion of height’!

Natural setting: heights on stacks, in particular weighted projective spaces.

Weighted projective spaces

Definition

Let $(w_0, \dots, w_n)$ be an $(n+1)$-tuple of positive integers. The *weighted projective space* with weights w is the quotient stack

$$\mathbf{P}(w) = [\mathbf{G}_{\mathrm{m}} \backslash \mathbf{A}^{n+1}_{\neq 0}]$$

for the weight w action

$$\lambda \cdot_w (x_0, \dots, x_n) = (\lambda^{w_0} x_0, \dots, \lambda^{w_n} x_n).$$

We will mostly be interested in the set of K-points for a field K:

$$\mathbf{P}(w)(K) = K^\times \backslash (K^{n+1} - \{0\}).$$

Important remark: If $K \to L$ is a field extension, the induced map $\mathbf{P}(w)(K) \to \mathbf{P}(w)(L)$ is not injective in general.

Height functions

Let K be a number field and w an $(n+1)$-tuple of positive integers.

Definition

The *scaling ideal* of $x \in K^{n+1} - \{0\}$, denoted by $\mathfrak{I}_w(x)$, is the unique smallest fractional ideal $\mathfrak{a}$ of K such that

$$x \in \mathfrak{a}^{w_0} \times \cdots \times \mathfrak{a}^{w_n}.$$

The *Archimedean height* of $x \in K^{n+1} - \{0\}$ is the real number

$$H_{w,\infty}(x) = \prod_{v \text{ inf. place of } K} \max_{0 \leq i \leq n} |x_i|_v^{1/w_i}.$$

Remark: can also define the scaling ideal locally at finite places.

Height functions

Definition [Deng, 1998]

The *(weight w) height function* on $\mathbf{P}(w)(K)$ is defined by

$$\begin{aligned} H_{w,K} \colon \mathbf{P}(w)(K) &\longrightarrow \mathbf{R}_{>0} \\ [x] &\longmapsto \frac{H_{w,\infty}(x)}{N(\mathfrak{I}_w(x))}. \end{aligned}$$

Example: Take $x = (2^5 \cdot 3^2, 2^7 \cdot 3^{-4}) \in \mathbf{P}(4,6)(\mathbf{Q})$. We compute

$$\mathfrak{I}_{(4,6)}(x) = \frac{2}{3}\mathbf{Z},$$

$$H_{(4,6),\infty}(x) = (2^5 \cdot 3^2)^{1/4},$$

$$H_{w,\mathbf{Q}}(x) = 2^{1/4} \cdot 3^{3/2}.$$

Counting points on weighted projective spaces

Theorem [Deng, 1998]

We have

$$\#\{x \in \mathbf{P}(w)(K) \mid H_{w,K}(x) \leq T\} \sim C_{w,K} T^{|w|} \quad \text{as } T \to \infty,$$

where $C_{w,K}$ is an explicit constant involving quantities like the discriminant, regulator and class number of K.

Remarks:

- Deng placed a restriction on w (easily removed)
- Generalises a result of Schanuel on $\mathbf{P}^n(K)$
- Special case of *heights on stacks* introduced more recently by Ellenberg, Satriano and Zureick-Brown and by Darda

Heights of elliptic curves over number fields

Definition [B., Najman, 2022]

The *height* of an elliptic curve E over a number field K is

$$H_K(E) = H_{(4,6),K}(c_4(E), c_6(E)).$$

This generalises (essentially) the earlier definition over $\mathbf{Q}$.

Remark: the logarithm of $H_K(E)$ is similar to the Faltings height, but the difference is unbounded.

First applications to modular curves

Now consider $n \geq 1$ and a subgroup $G \subseteq \mathrm{GL}_2(\mathbf{Z}/n\mathbf{Z})$, for simplicity with surjective determinant, such that the compactified moduli stack X_G is isomorphic to $\mathbf{P}(w_0, w_1)$ for some $w_0, w_1 \geq 1$.

Then we have a commutative square

$$\begin{array}{ccc} X_G & \xrightarrow{\sim} & \mathbf{P}(w) \\ {\scriptstyle \pi_G}\downarrow & & \downarrow{\scriptstyle \phi} \\ X(1) & \xrightarrow{\sim} & \mathbf{P}(4,6) \end{array} \qquad \begin{array}{c} (a, b) \\ \\ \\ (c_4, c_6) \end{array}$$

of algebraic stacks over $\mathbf{Q}$.

There are homogeneous polynomials f_0, f_1 of degrees $4e$, $6e$, for some $e \geq 1$, such that ϕ is given by

$$c_4 = f_0(a, b), \quad c_6 = f_1(a, b).$$

Example: $X_1(3)$

The modular curve $X_1(3)$ fulfills the assumptions:

$$\begin{array}{ccc} X_1(3) & \xrightarrow{\sim} & \mathbf{P}(1,3) \\ \pi\downarrow & & \downarrow\phi \\ X(1) & \xrightarrow{\sim} & \mathbf{P}(4,6) \end{array} \qquad \begin{array}{c} (a = a_1, b = 6a_3) \\ \\ (c_4, c_6) \end{array}$$

The morphism ϕ has $e = 1$ and is given by

$$c_4 = a^4 - 4ab, \quad c_6 = -a^6 + 6a^3b - 6b^2.$$

First applications to modular curves

Let $N_{G,K}(T)$ be the number of isomorphism classes of pairs (E, α) with E an elliptic curve over K and α a G-level structure on E such that $H_K(E) \leq T$.

Theorem [B., Najman, 2022]

In the above setting, assume $w = (1, 1)$ or $e = 1$. Then for any number field K,

$$N_{G,K}(T) \asymp T^{1/d(G)},$$

where

$$d(G) = \frac{12e}{w_0 + w_1}.$$

This applies in particular to elliptic curves with an embedding of $C_m \times C_n$ such that $X_1(m, n)$ has genus 0.

More refined results

Question 3

Can the above growth rates $N_{G,K}(T) \asymp T^{1/d(G)}$ be refined to asymptotic formulae $N_{G,K}(T) \sim C_{G,K}T^{1/d(G)}$?

We only get $\asymp$ rather than $\sim$ because we need to compare two different height functions.

In a (mostly) complementary result, Boggess and Sankar determined the growth rate for the number of elliptic curves over $\mathbf{Q}$ with a cyclic n-isogeny for certain values of n. Their methods have similar limitations preventing a refinement from $\asymp$ to $\sim$.

Asymptotic formulae

Cases in which an asymptotic formula was known (all over $\mathbf{Q}$):

- $X(1)$ [Brumer, 1992]
- $X_1(2)$ [Grant, 2000]
- $X_1(3)$ [Harron and Snowden, 2017]
- $X_0(3)$ [Pizzo, Pomerance and Voight, 2020]
- $X_0(4)$ [Pomerance and Schaefer, 2021]

The proofs are increasingly sophisticated and do not seem to generalise easily to other number fields.

Asymptotic formulae

Theorem [B., Manterola Ayala, 2021/23; Phillips, 2022]

Under the same assumptions as before ($G \subseteq \mathrm{GL}_2(\mathbf{Z}/n\mathbf{Z})$ with $X_G \simeq \mathbf{P}(w)$, $w = (1,1)$ or $e = 1$), we have

$$N_{G,K}(T) \sim C_{G,K} T^{1/d(G)}$$

for an explicit constant $C_{G,K}$ involving invariants of the field K and the morphism ϕ.

Idea of proof:

- generalise the strategy of Schanuel and Deng
- use a lattice point counting theorem by Barroero and Widmer for definable sets in an o-minimal structure
- introduce congruence conditions measuring how the scaling ideal changes under the morphism ϕ

Approach using harmonic analysis

The above theorem could probably also be deduced from recent work of Darda on *quasi-toric heights* on weighted projective spaces. This extends work of Batyrev and Tschinkel and of Chambert-Loir and Tschinkel on the Batyrev–Manin conjecture.

Tools: notion of *quasi-toric heights* H; harmonic analysis on $\mathbf{G}_{\mathrm{m}}^n(\mathbf{A}_K)$ with the weight w action of $\mathbf{G}_{\mathrm{m}}(\mathbf{A}_K)$; apply Poisson summation to study the *height zeta function*

$$Z_H(s) = \sum_{x \in K^\times \backslash (K^n - \{0\})} H(x)^{-s}.$$

Via a Tauberian theorem, the behaviour at the pole $s = 1$ translates into an asymptotic for the number of rational points.

Work in progress

An elusive case so far: $X_0(3)$. Not a weighted projective line, but (probably) close enough! (E.g. isomorphic to $\mathbf{P}(2) \times \mathbf{P}(1,3)$.)

Asymptotic formula over $\mathbf{Q}$ [Pizzo, Pomerance and Voight]:

$$N_{X_0(3),\mathbf{Q}}(T) = c_0 T^{1/2} + c_1 T^{1/3} \log T + c_2 T^{1/3} + O(T^{7/24}).$$

Remark: leading term comes from sextic twists of a single elliptic curve with $j = 0$.

One has $X_0(3) \simeq \mathbf{G}_{\mathrm{m}}^2 \backslash \mathbf{A}^3_{\neq 0}$ for the action

$$(\lambda, \mu) \cdot (u, v, w) = (\lambda^2 u, \mu v, \lambda^2 \mu^3 v),$$

and $H_K(E)$ is similar to a quasi-toric height.

Work in progress: generalise Darda's techniques to this quotient, and the result of Pizzo, Pomerance and Voight to number fields.

Thank you for your attention!

Questions?