Uniform Versions of Shafarevich Conjecture

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Let K be a number field, $\bar{K}$ be an alg. closure of K and $G_K := \mathrm{Gal}(\bar{K}/K)$. Let g, n, m be positive integers and ℓ be a prime number. Let $S \supset S_\infty$ be a finite set of places of K. Let $S_\ell := S \cup \{v|\ell\}$. Let F be a non-archimedean local field with Char(F)=0 and residue characteristic ℓ.

Theorem (Shafarevich Conjecture, proved by Faltings, 1983)
Given K, g, S; there are only finitely many isomorphism classes of abelian varieties defined over K of dimension g with good reduction outside S.

Finiteness criteria for (potential) equivalence

Lemma (finiteness criteria)

Given K, S, F, ℓ, n; $\exists$ a finite set T of finite places of K ($S \cap T = \emptyset$) with the following property:
suppose $\rho_1, \rho_2 : G_K \longrightarrow GL_n(F)$ are two continuous semisimple representations of G_K.

- *(Faltings',1983) If they are unramified outside S_ℓ and satisfy*

$$Tr\, \rho_1(F_v) = Tr\, \rho_2(F_v), \quad for\ v \in T;$$

  Then $\rho_1 \cong \rho_2$.

- *(Rajan,—, 2023) If $\exists$ $m > 0$ s. t. for $v \notin S_\ell$, I_v acts by scalar matrices of order dividing m and the repns satisfy:*

$$Tr\, \rho_1(F_v^m) = Tr\, \rho_2(F_v^m), \quad for\ v \in T.$$

  Then ρ_1, ρ_2 are potentially isomorphic, i.e., $\rho_1|_{G_L} \cong \rho_2|_{G_L}$ for some finite extension L/K.

Theorem (Rajan,—,2023)
Fix K, S, ℓ, w, n, d, F. There are, up to pot. equiv., only fin. many cont., s.s. Gal. repns. $\rho : G_K \to GL_n(F)$, s. t. I_v, $v \notin S_\ell$ acts by scalar matrices, and pure of deg. at most d and wt. w.

Now using the following:

- (Faltings', 1983) $V_\ell(A)$ is semisimple and $\mathrm{Hom}(A, B) \otimes \mathbb{Q}_\ell \simeq \mathrm{Hom}_{G_K}(V_\ell(A), V_\ell(B))$; for any ab. var. $A/K, B/K$.
- (Faltings', 1983) Each K-isog. class of ab. var. over K only contains fin. many K-isom. classes.
- (Masser-Wüstholz, 1993; Bost, 1996) For any ab. var. A/K, there are only a fin. num. of $\bar{K}$-isom. classes of ab. var. B/K and $\bar{K}$-isog. to A.

We get

Theorem (—, Rajan, 2023)
There are, up to $\bar{K}$-isomorphism, only fin. many ab. var. of dim. g defined over K, s. t. I_v, $v \notin S$ acts by scalars $\{\pm 1\}$ on $V_\ell(A)$ for ℓ coprime to the res. char. at v.

Totally Bad Reduction of abelian varieties

Definition

An ab. var. A_v over K_v is said to have totally bad reduction *(resp.* purely additive reduction*) if $\tilde{\mathcal{A}}_v^0$ (connected component containing identity of the special fibre $\tilde{\mathcal{A}}_v$ of the Néron model $\mathcal{A}_v$ at v) is an affine group scheme (resp. unipotent group scheme) over k_v.*

Chevalley decomposition over k_v:

$$0 \to T \times U \to \tilde{\mathcal{A}}_v^0 \to B \to 0.$$

Suppose A is an elliptic curve, then A has good red. at v (resp. semi-stable red., additive red.) if and only if $\tilde{A}_v$ is an elliptic curve (resp. $\tilde{A}_v^0 \cong \mathbb{G}_m$, $\tilde{A}_v^0 \cong \mathbb{G}_a$). Hence tot. bad red. is same as bad red. and purely additive red. is same as additive red.

- *If A_v has tot. bad red. and acquires good red. over a quad. extn of K_v then for ℓ coprime to the res. char. of K_v, I_v acts via scalars $\{\pm 1\}$ on $V_\ell(A_v)$.*

Summary

Fix K, g, $S \supseteq S_\infty$. For any fin. set of places S' of K containing S, let $\mathcal{X}(K, g, S')$ and $\bar{\mathcal{X}}(K, g, S')$ denote the K-isomor. classes and $\bar{K}$-isomor. classes of the ab. var. A/K of dim. g resp., satisfying the following:

(a) for any place $v \notin S'$, A has good red. at v,

(b) for any place $v \in S' \setminus S$, A_v has either good red. or tot. bad red. and acquires good red. over a quad. extn of K_v.

For fixed K, g and S, the union

$$\bar{\mathcal{Y}}(K, g, S) := \bigcup_{S' \supseteq S} \bar{\mathcal{X}}(K, g, S')$$

is finite.

Motivated by a question of Ihara (1986,1991), Rasmussen-Tamagawa considered the set of K-isomorphism classes $[A]$ of ab. var. A/K,

$$\mathcal{A}(K,g,\ell) := \{[A] : \dim A = g,\ A \text{ has good red. outside } \ell \text{ and } K(A[\ell])/K(\mu_\ell) \text{ is an } \ell\text{-extn}\}.$$

Conjecture (Rasmussen-Tamagawa Conjecture, 2008, 2016)
Given a num. field K, and a pos. int. g; the set $\mathcal{A}(K,g,\ell) = \emptyset$ for sufficiently large ℓ. In particular, $\mathcal{A}(K,g) := \bigcup_\ell \mathcal{A}(K,g,\ell)$ is finite.

- (Rasmussen-Tamagawa 2008, 2016) proved assuming GRH, gave unconditional proof for $K = \mathbb{Q}$ and $g = 1, 2, 3$.
- (uniform version, 2016) $\exists N = N(g,n) > 0$ s. t. $\mathcal{A}(K,g,\ell) = \emptyset$ for K with $[K:\mathbb{Q}] = n$ for $\ell > N$ and assuming GRH they proved it for n is odd. For any n, Bourdon (2015) and Lombardo (2018) proved it for CM ell. curves and ab. var. of CM type resp.
- (Ozeki, 2013) for A/K when the image of $\rho_{A,\ell}$ is abelian.
- (Rasmussen-Tamagawa, 2016) proved that $\mathcal{A}^{\mathrm{ss}}(K,g) := \bigcup_\ell \mathcal{A}^{\mathrm{ss}}(K,g,\ell)$ is finite.

Definition

Let $\mathcal{Z}(K,g)$ be the set of K-isom. classes of ab. var. A/K of dim. g s.t. for any finite place v of K, A_v has either good red., or tot. bad red. and acquires good red. over a quad. extn. of K_v.

Definition

Let $\mathscr{M}(K,g,\ell)$ be a subset of $\mathcal{Z}(K,g)$ consisting of K-isom. classes of those ab. var. A/K for which $K_A(A[\ell])$ is an ℓ-extn. of $K_A(\mu_\ell)$, where $K_A = K(A[12])$. Let $\bar{\mathscr{M}}(K,g,\ell)$ denotes the $\bar{K}$-isom. classes of such ab. varieties.

Note that if a smooth proper var. X/K has semistable red. at u dividing ℓ then the ℓ-adic cohomology group $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ and its dual are 'semistable' at u. For an ab. var. A/K, by Raynaud's criteria, A has semistable red. everywhere over $K(A[12])$.

Theorem (Sarkar, —)
Let K be a number field and g be a positive integer. For sufficiently large prime numbers ℓ, the set $\mathscr{M}(K, g, \ell) = \emptyset$.

Corollary
For a given number field K and a positive integer g, the set $\bar{\mathscr{M}}(K, g) := \bigcup_\ell \bar{\mathscr{M}}(K, g, \ell)$ is finite.

Consider $\mathcal{A}^{\text{pot}}(K, g, \ell) := \mathcal{A}(K, g, \ell) \cap \mathcal{Z}(K, g)$. In fact, $\mathcal{A}^{\text{pot}}(K, g, \ell) \subset \mathscr{M}(K, g, \ell)$ and hence we have

Corollary
For sufficiently large prime numbers ℓ, the set $\mathcal{A}^{pot}(K, g, \ell) = \emptyset$. In particular, $\mathcal{A}^{pot}(K, g) := \bigcup_\ell \mathcal{A}^{pot}(K, g, \ell)$ is finite.

Definition

An E-**rational (**E-**integral) strictly** m-**compatible system** $(\rho_\lambda)_\lambda$ **of** n-**dimensional** λ-**adic representations of** G_K **with defect set** T **and finite** m-**ramification set** S *is a family of continuous (semisimple) representations* $\rho_\lambda : G_K \to GL_n(E_\lambda)$ *such that for any finite place* λ *of* E *not in* T, ρ_λ *satisfy the following conditions:*

1. *the group* $\rho_\lambda(I_v)$ *has order dividing* m *for finite places* $v \notin S_{\ell_\lambda}$*;*
2. *for a finite place* v *of* K *not in* S*, there exists a monic polynomial* $f_v[X] \in E[X]$ *(resp.* $f_v(X) \in \mathcal{O}_E[X]$*) such that for all finite places* $\lambda \notin T$ *of* E *coprime to the residue characteristic of* v*, the characteristic polynomial of* $\rho_\lambda(F_v^m)$ *is equal to* $f_v(X)$*.*

By Raynaud's semistability criteria, A/K has everywhere semistable red. over $K(A[12])$. For any $g > 0$, put $D_g = |GL_{2g}(\mathbb{Z}/3\mathbb{Z})| \cdot |GL_{2g}(\mathbb{Z}/4\mathbb{Z})|$. The degree $[K(A[12]) : K] \mid D_g$.

- Let S be s. t. outside S, A has pot. good red.Take any ℓ and $v \notin S_\ell$. Let $L = K(A[12])$. Since L/K Galois and $[L:K] \mid D_g$, D_g/f_L is an integer. We get
$$\det(XI_{2g} - \rho_{A,\ell}(F_v^{D_g})) = \det(XI_{2g} - \rho_{A,\ell}(F_{v_L}^{D_g/f_L})).$$
So $(\rho_{A,\ell})_\ell$ give a D_g-compatible system with D_g-ram. set S and defect set empty.

- To any pair (ℓ, A), $A \in \Gamma$ (a fixed $\bar{K}$-isogeny class of ab. varieties A/K of $\dim = g$) attach the ℓ-adic repn $\rho_\ell := \rho_{A,\ell}$. Then $(\rho_\ell)_\ell$ give a D_g^2-compatible system with D_g^2-ram. set S (take as above) and defect set empty. For $A, B \in \Gamma$, $L = K(A[12], B[12])$. By Silverberg, A and B are isogenous over L, hence for $v \notin S_{\ell\ell'}$,
$$\begin{aligned}\det(XI_{2g} - \rho_{A,\ell}(F_{v_L})) &= \det(XI_{2g} - \rho_{B,\ell}(F_{v_L})) \\ &= \det(XI_{2g} - \rho_{B,\ell'}(F_{v_L})) \in \mathbb{Z}[X].\end{aligned}$$
$$\begin{aligned}\det(XI_{2g} - \rho_\ell(F_v^{D_g^2})) &= \det(XI_{2g} - \rho_{A,\ell}(F_{v_L}^{D_g^2/f_L})) \\ &= \det(XI_{2g} - \rho_{B,\ell'}(F_{v_L}^{D_g^2/f_L})) \\ &= \det(XI_{2g} - \rho_{\ell'}(F_v^{D_g^2})).\end{aligned}$$

Lemma

Let $(\rho_\ell)_\ell$ be an $\mathbb{Q}$-integral strictly m-compatible system of n-dimensional continuous, pure semisimple ℓ-adic representations $\rho_\ell : G_K \to GL_n(\mathbb{Q}_\ell)$ with m-ramification set S whose Weil weights are bounded by w. Suppose there exists an infinite set of rational primes Λ satisfying the following conditions:
For any $\ell \in \Lambda$, there exists a Galois extension L^ℓ of K (depending on ℓ) such that $[L^\ell : K] \mid m$ and ρ_ℓ satisfies the following:

1. *$\rho_\ell|_{G_{L^\ell}}$ is unramified at all finite places outside the places of L^ℓ lying above places in S_ℓ,*
2. *(Hodge-Tate) there exists integers $w_1 \leq w_2$ and a place u_ℓ of L^ℓ lying above ℓ such that $\rho_\ell|_{G_{u_\ell}}$ is semistable at u_ℓ and the Hodge-Tate weights of ρ_ℓ at u_ℓ are in $[w_1, w_2]$ for any pair (ℓ, u_ℓ),*
3. *(Potentially Cyclotomic) $\bar{\rho}_\ell^{\mathrm{ss}}|_{G_{L^\ell}} \cong \oplus_{i=1}^n \bar{\chi}_\ell^{a_{\ell,i}}|_{G_{L^\ell}}$, where $a_{\ell,i}$ are integers.*

Then, there exists an integer c independent of ℓ such that ρ_ℓ and $\bigoplus_{i=1}^n \chi_\ell^c$ are potentially equivalent. In particular, the Weil weights of ρ_ℓ and $\bigoplus_{i=1}^n \chi_\ell^c$ are equal and independent of ℓ.

References

- Das, P. and Rajan, C. S., *Finiteness theorems for potentially equivalent Galois representations: extension of Faltings' finiteness criteria*, Proc. Amer. Math. Soc. 151 (2023).
- Das, P. and Rajan, C. S., *A Finiteness theorem for abelian varieties with totally bad reduction*, https://arxiv.org/abs/2110.00870.
- Das, P. and Sarkar, S. *A base change version of Rasmussen-Tamagawa Conjecture*, https://arxiv.org/abs/2208.04170.
- C. Rasmussen and A. Tamagawa, *A finiteness conjecture on abelian varieties with constrained prime power torsion*, Math. Res, Lett. (2008)
- C. Rasmussen and A. Tamagawa, Arithmetic of abelian varieties with constrained torsion, Trans. Amer. Math. Soc.; 2016.
- Y. Ozeki and Y. Taguchi, On congruences of Galois representations of number fields, Publ. Res. Inst. Math. Sci. (2014).
- Ozeki, Yoshiyasu . Non-existence of certain CM abelian varieties with prime power torsion. Tohoku Math. J. (2013).

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