Adaptive Measurements for Profit and Pleasure

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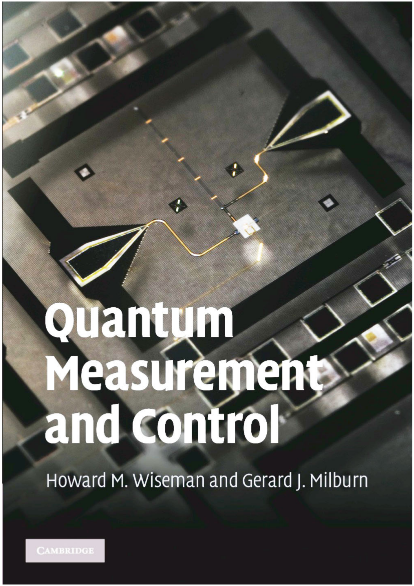
Measurement and Control



- The obvious reason to combine measurement and control is feedback, to purposefully **change** the average system evolution.
- Non-trivial even classically.
- *cf.* adaptive measurement — controlling future measurements on the basis of the results of past ones, to **obtain better data**, leaving the average system evolution **unchanged**.
- Classically, a non-problem if measurements can be perfect, but non-trivial in the quantum case.

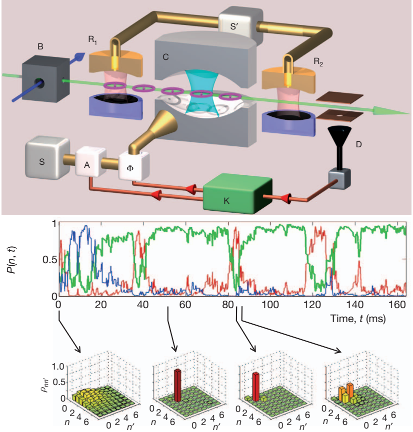
Measurement and Control

LETTER

doi:10.1038/nature10376

Real-time quantum feedback prepares and stabilizes photon number states

Clément Sayrin[1], Igor Dotsenko[1], Xingxing Zhou[1], Bruno Peaudecerf[1], Théo Rybarczyk[1], Sébastien Gleyzes[1], Pierre Rouchon[2], Mazyar Mirrahimi[3], Hadis Amini[2], Michel Brune[1], Jean-Michel Raimond[1] & Serge Haroche[1,4]



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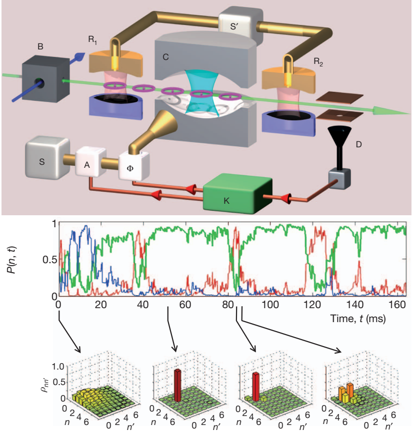

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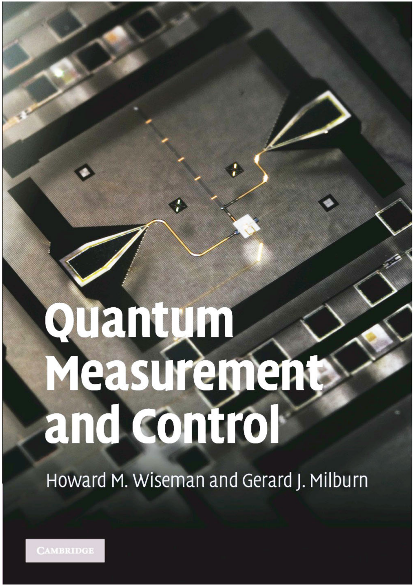

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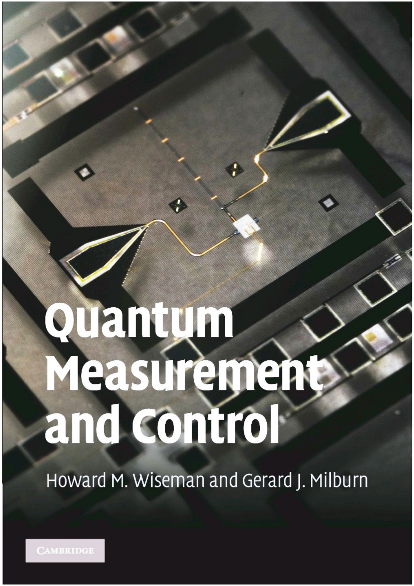

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Outline

1. Formally Defining Adaptive Measurements
2. Adaptive Measurements for Profit
   - Doing some things better
   - Doing some things *much* better
   - Doing some things *perfectly*
   - Doing some things *uniquely*
3. How Big a Brain does it take to Track an Open Quantum System?
   - Quantum Jumps: The Old Quantum Theory
   - Quantum Jumps: The Modern Understanding
   - Physically Realizable Ensembles
   - For a qubit, a two-state classical memory is all it takes ...
   - A Fuller Answer
4. Conclusion
   - Summary and Questions

General Quantum Measurements

- For simplicity, restrict to *efficient* measurements (those that take pure states to pure states).
- A measurement $\mathfrak{M}$ is described by a set $\{(r, \hat{M}_r) : r\}$ of *outcomes* and *measurement operators*.
- The unnormalized state *conditioned* on outcome r is $\tilde{\rho}'_r = \hat{M}_r \rho \hat{M}_r^\dagger$.
- The probability for result r is $P_r = \mathrm{Tr}[\tilde{\rho}'_r] = \mathrm{Tr}[\rho \hat{M}_r^\dagger \hat{M}_r]$.
- The only other restriction on $\mathfrak{M}$ is $\sum_r \hat{M}_r^\dagger \hat{M}_r = I$.
- The *unconditioned* post-measurement state is $\rho' = \mathcal{T}\rho$, where $\mathcal{T}\bullet = \sum_r \hat{M}_r \bullet \hat{M}_r^\dagger$ is a CPTP map.
- A measurement is called *complete* if $\forall\ r\ \ \hat{M}_r = \hat{U}_r \hat{\pi}_r$, where $\hat{\pi}_r$ is a rank-one projector.

Adaptive Measurements

- For a complete measurement, $\tilde{\rho}'_r = M_r \rho M_r^\dagger \propto U_r \pi_r U_r^\dagger$, is determined solely by r.
  $\implies$ later measurements give no more information about ρ.
- For *incomplete* measurements, making a second measurement may yield more information. And so on ....
- If the experimenter has:
  1. The ability to perform a *sequence* $\{\mathfrak{M}_n\}$ of *incomplete* measurements.
  2. *Restrictions* on the class of each measurement,
  3. But still with some *choice* as to what measurement to make at step n

  then the optimal choice for the second measurement will depend in general on the *result* of the first measurement, and so on.
- Making this so realizes an adaptive measurement.
- If the restrictions on $\mathfrak{M}^n$ includes that $\mathcal{T}^n$ is *fixed*, then the protocol is **purely** an adaptive measurement, as $\rho^N = \mathcal{T}^N \cdots \mathcal{T}^1 \rho^0$.

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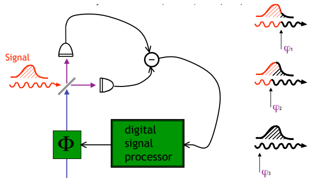
Adaptive Homodyne on Coherent States



1. single-shot *estimation of a fixed phase*
   - **Theory** [HMW, *PRL* (1995)]: predicted improvement offered by adaptive measurement in V (*i.e.*, MSE):

     $$50\% \text{ reduction in } V(\phi).$$

   - **Expt** [Armen *et al.*, *PRL* (2002)]: measured 40% reduction in V.
2. continuous *tracking of a diffusing phase*
   - **Theory** [Berry & HMW, *PRA* (2002)]: predicted improvement

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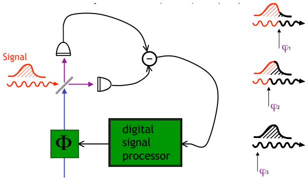

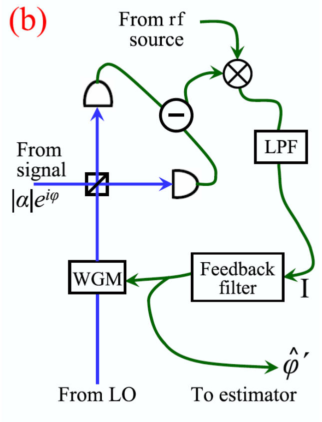

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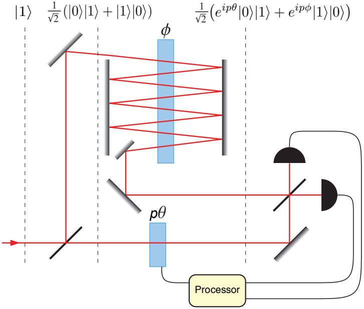
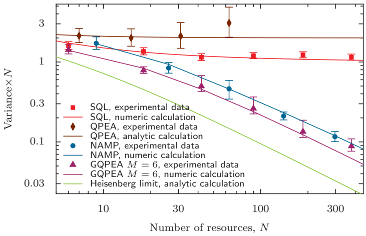
Adaptive Single-photon Multipass Interferometry





- Fix $N = \#$ photon-passes.
- **Theory** and **Expt** [Higgins, Berry, Bartlett, HMW & Pryde, *Nature* (2007); *NJP* (2009)].
- Heisenberg limit:

$$V \approx 10/N^2$$

- Best known nonadaptive scheme (2009)

$$V \approx 20/N^2.$$

- Best known adaptive scheme (2007)

$$V \approx 15/N^2.$$

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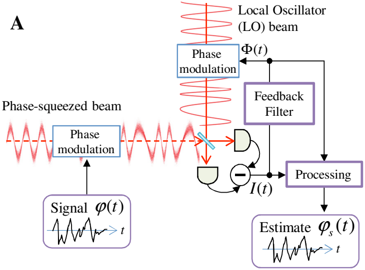
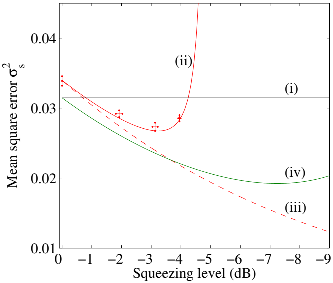
Adaptive Homodyne *Tracking* on Squeezed States





- Allow *arbitrary squeezing* on a coherent beam, on which is imprinted a *diffusing phase* which is continuously *tracked*.
- **Theory** [Berry & HMW, *PRA* (2006, Erratum 2013)]:

$$V_{\text{heterodyne}} = O((\mathcal{N}/\kappa)^{-1/2})$$
$$V_{\text{adaptive}} = O((\mathcal{N}/\kappa)^{-2/3})$$

  Here κ = diffusion rate, $\mathcal{N}$ = photon flux (including flux due to squeezing).

- **Expt** [Yonezawa *et al.*, *Science* (2012)]: measured 15% reduction in V below the coherent-state-limit (for given $\mathcal{N}$), at optimal degree of squeezing.

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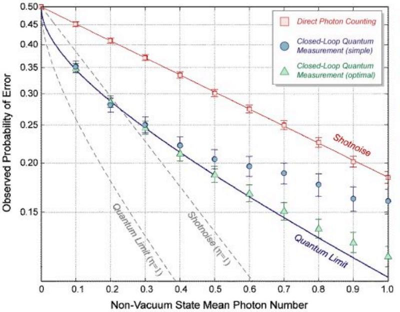
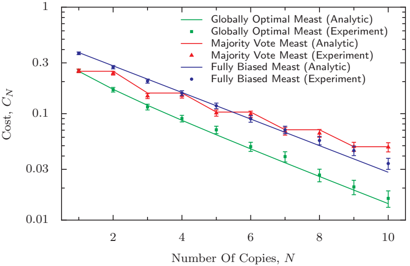
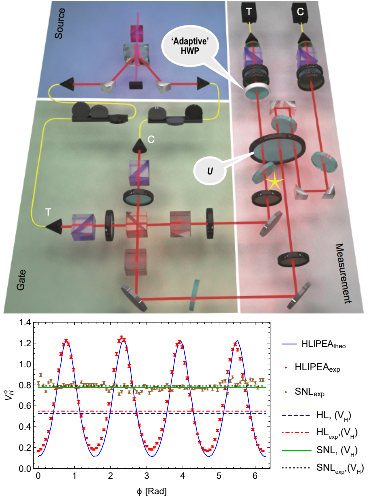
Helstrom-Limited Nonorthogonal State Discrimination





1. single-shot discrimination of $|\alpha\rangle$, $|0\rangle$ with minimal (Helstrom) error.
   - *Restricting* to photon counting and displacement, *adaptive* displacement is necessary & sufficient.
   - **Theory**: Dolinar, IBM (1973)
   - **Expt**: ~~Cook, Martin & Geremia, Nature (2007).~~
2. Helstrom-limit discrim. of n-qubit product states $|\theta\rangle^{\otimes n}$, $|-\theta\rangle^{\otimes n}$.
   - *Restricting* to single-qubit operations, *adaptive* measurements is necessary & sufficient.
   - **Theory**: Acìn *et al.*, *PRA* (2005)
   - **Expt**: Higgins *et al.*, *PRL* (2009).

Heisenberg-Limited Interferometry using Multiphoton Entanglement and Multipasses



- Fix the total number of photon-passes N.
- **Theory** [HMW *et al.*, *IEEE* (2009), based on Griffiths and Niu, *PRL* (1996)]:

  Asymptotically achieves $V_{\rm HL} = \pi^2/N^2$.

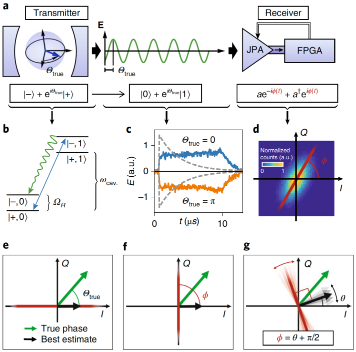
  For $N = 3$:

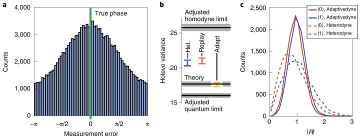
$$V_{\rm adaptive} = V_{\rm HL} \approx 0.53$$
$$V_{\rm nonadaptive} \approx 0.65$$

- **Expt** [Daryanoosh, Slussarenko, Berry, HMW, Pryde, *Nature Comms.* (2018)]:

$$V_{\rm adaptive} \approx 0.55$$

Perfect Phase Measurement in $\{|0\rangle, |1\rangle\}$ subspace





- Aim: projection onto canonical phase states $|\varphi\rangle = (|0\rangle + e^{i\varphi}|1\rangle)/\sqrt{2}$, using only homodyne measurement.
- **Theory** [HMW, *PRL* (1995)]:

$$2\pi\, P_{\text{rnd-homodyne}}(\varphi) = 0.80|\langle\varphi|\psi\rangle|^2 + 0.20$$
$$2\pi\, P_{\text{heterodyne}}(\varphi) = 0.88|\langle\varphi|\psi\rangle|^2 + 0.12$$
$$2\pi\, P_{\text{adaptive}}(\varphi) = 1.00|\langle\varphi|\psi\rangle|^2 + 0.00.$$

- **Experiment** [Martin, Livingston, Hacohen-Gourgy, HMW & Siddiqi, *Nature Phys.* (2020)]: measured 15% reduction in V below the measured heterodyne limit.

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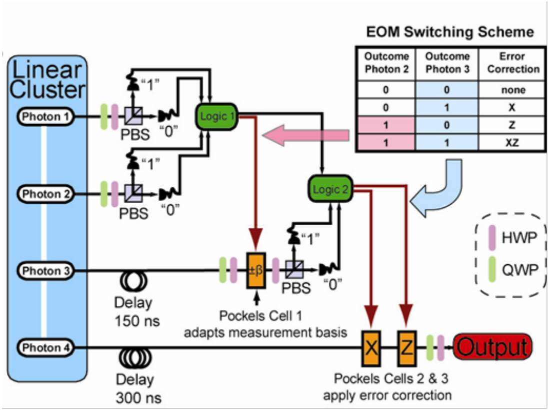
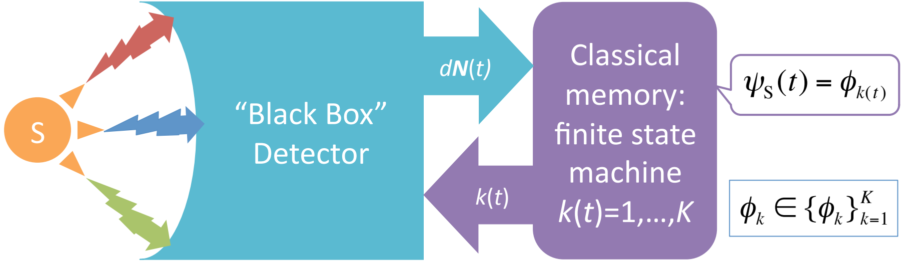
Measurement-Based Quantum Computing



- **Theory**: Raussendorf and Briegel, *PRL* (2001).
- **Experiment**: Prevedel *et al.* (Vienna), *Nature* (2007).
- **Industry**: Ψ-Quantum, Xanadu, others.

Tracking an Open Quantum System with a Finite Classical Memory



Theory: Karasik and HMW, *PRL* (2011); *ibid. PRA* (2011). Warszawski and HMW, *NJP* (2019); *ibid.*, *Quantum* (2019).

But this is **for Pleasure ...**

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The First Quantum Dynamics (Einstein, 1917)

On the quantum theory of Radiation

A. Einstein, Phys. Z. 18, 121 (1917).

2. Hypotheses on the radiative exchange of energy

Let Z_n and Z_m be two quantum-theoretically possible states of the gas molecule, whose energies are ε_n and ε_m, respectively, and satisfy the inequality $\varepsilon_m > \varepsilon_n$. Let us assume that the molecule is capable of a transition from state Z_n into state Z_m with an absorption of radiation energy $\varepsilon_m - \varepsilon_n$; that, similarly, the transition from state Z_m to state Z_n is possible, with emission of the same radiative energy. Let the radiation absorbed or emitted by the molecule have frequency ν which is characteristic for the index combination (m, n) that we are considering.

For the laws governing this transition, we introduce a few hypotheses which are obtained by carrying over the known situation for a Planck resonator in classical theory to the as yet unknown one in quantum theory.

(a) *Emission of radiation.* According to Hertz, an oscillating Planck resonator radiates energy in the well-known way, regardless of whether or not it is excited by an external field. Correspondingly, let us assume that a molecule may go from state Z_m to a state Z_n and emit radiation energy $\varepsilon_m - \varepsilon_n$ with frequency μ, without excitation from external causes. Let the probability $\mathrm{d}W$ for this to happen during the time interval $\mathrm{d}t$, be

$$\mathrm{d}W = A_m^n \,\mathrm{d}t, \qquad \text{(A)}$$

where A_m^n is a constant characterising the index combination under consideration.

$$P(Z_m, t + dt | Z_n, t) = \kappa_{mn} dt$$

$$\kappa_{mn} = (N_m^n + 1) A_m^n \text{ for } \varepsilon_m > \varepsilon_n$$

$$\kappa_{mn} = N_m^n A_m^n \text{ for } \varepsilon_m < \varepsilon_n$$

$$N_{mn} = N_{\text{Planck}}(\varepsilon_m - \varepsilon_n)$$

Classical master equation:

$$\dot{p}_m = \sum_n \kappa_{mn}(p_n - p_m).$$

Ergodic (unique steady state):

$$\lim_{t\to\infty} p_n(t) = p_{\text{Boltzmann}}(\varepsilon_n)$$

Bohr's and Einstein's Quantum Jumps

"The passing of the systems between different stationary states ... cannot be treated [using] ordinary mechanics ... [and] is followed by the emission of a homogeneous radiation, for which [$\Delta E = h\nu$]. [This] is in obvious contrast to the ordinary ideas of electrodynamics, but appears necessary in order to account for the experimental facts. (Bohr, 1913).

"The weakness of the theory [is] that it leaves the moment and direction of the elementary processes to 'chance'." (Einstein, 1917).

- The transitions may be stochastic, but they correspond to physical events: absorption from, or emission into, the radiation bath.
- Thus the state Z_n of the atom at any time is knowable in principle by monitoring the bath.
- If the atom can be approximated as having finitely many (D) levels, then a finite (D-state) classical memory is all that is required to keep track of the atomic state.

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Unravelling Quantum Master Equations

- If we ignore the bath then even if both system and bath are initially pure, the system state will decohere:

$$|\Psi(0)\rangle = |\phi(0)\rangle_{\rm env} \otimes |\psi(0)\rangle_{\rm sys} \to |\Psi(t)\rangle = \exp\left(-i\hat{H}_{\rm tot}t\right)|\Psi(0)\rangle$$
$$\text{(pure)}\ |\psi(0)\rangle_{\rm sys} \to \rho_{\rm sys}(t) = {\rm Tr}_{\rm env}[|\Psi(t)\rangle\langle\Psi(t)|]\ \text{(mixed)}$$

- If the Born-Markov approximation is valid, $\rho_{\rm sys}(t)$ obeys a master equation of the Lindblad form:

$$\dot\rho(t) = \mathcal{L}\rho(t) \equiv [-i\hat{H}, \rho] + \sum_{\ell=1}^{L}\mathcal{D}[\hat{c}_\ell]\rho.$$

- *If* it is valid *then* it is also the case that the bath can be measured repeatedly, on a time scale which is short compared to the interesting system evolution, *without invalidating the master equation*.
- This is called monitoring the system. If the monitoring is perfect, then this produces a stochastic *pure* conditioned system state $|\psi_{\rm c}(t)\rangle$:

$$\mathrm{E}[|\psi_{\rm c}(t)\rangle\langle\psi_{\rm c}(t)|] = \rho(t) = \exp(\mathcal{L}t)|\psi(0)\rangle\langle\psi(0)|.$$

Rediscovering quantum jumps

- Consider the master equation with a single decoherence channel:

$$\rho(t+dt) = \rho(t) - i[\hat{H}, \rho(t)]dt + [\hat{c}\rho(t)\hat{c}^\dagger - \tfrac{1}{2}\hat{c}^\dagger\hat{c}\rho(t) - \tfrac{1}{2}\rho(t)\hat{c}^\dagger\hat{c}]dt.$$

- If $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$ then this can be rewritten to $O(dt)$ as

$$P_0(dt)|\psi_0(t+dt)\rangle\langle\psi_0(t+dt)| + P_1(dt)|\psi_1(t+dt)\rangle\langle\psi_1(t+dt)|,$$

where

$$|\psi_0(t+dt)\rangle = \left(1 - i\hat{H}dt - \tfrac{1}{2}\hat{c}^\dagger\hat{c}dt\right)|\psi(t)\rangle/\sqrt{P_0(dt)}$$

$$|\psi_1(t+dt)\rangle = \sqrt{dt}\,\hat{c}|\psi(t)\rangle/\sqrt{P_1(dt)}$$

$$P_1(dt) = 1 - P_0(dt) = \langle\psi(t)|\hat{c}^\dagger\sqrt{dt}\,\sqrt{dt}\,\hat{c}|\psi(t)\rangle$$

- $P_1(dt) = O(dt) \implies$ "1" events are non-null "detections".

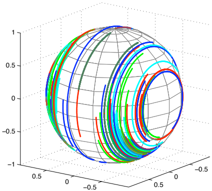
$$|\psi_0(t+dt)\rangle \approx |\psi(t)\rangle \text{ (no detection } \implies \text{ smooth evolution)}$$

$$|\psi_1(t+dt)\rangle \not\approx |\psi(t)\rangle \text{ (detection } \implies \text{ quantum jump).}$$

General Properties of Conditional Evolution

Unlike the quantum jumps in Einstein's thermal equilibrium model,

1. The post-jump $|\psi_1(t+dt)\rangle$ depends on the pre-jump $|\psi(t)\rangle$
2. Jumps don't take you to an orthogonal state: $\langle\psi_1(t+dt)|\psi(t)\rangle \neq 0$
3. Even with no jump, you don't stay fixed: $|\psi_0(t+dt)\rangle \neq |\psi(t)\rangle$



In general, the long-time *conditioned state* $|\psi_c(t)\rangle$ explores some manifold within Hilbert space, so

$$\rho_{ss} = \int d\mu_{ss}(\phi)|\phi\rangle\langle\phi|.$$

Thus, even for a D-dimensional Hilbert space, a classical memory of *infinite* size would be required to keep track of which $|\phi\rangle$ pertains.

Question

Can we control the way the system jumps (without changing the average evolution), so that it is restricted to **finitely many states***?*

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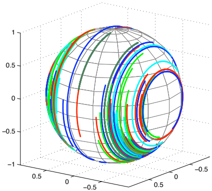

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Thus, even for a *D*-dimensional Hilbert space, a classical memory of *infinite* size would be required to keep track of which $|\phi\rangle$ pertains.

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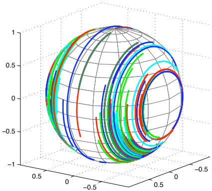
Can we control the way the system jumps (without changing the average evolution), so that it is restricted to **finitely many states***?*

General Properties of Conditional Evolution

Unlike the quantum jumps in Einstein's thermal equilibrium model,

1. The post-jump $|\psi_1(t+dt)\rangle$ depends on the pre-jump $|\psi(t)\rangle$
2. Jumps don't take you to an orthogonal state: $\langle\psi_1(t+dt)|\psi(t)\rangle \neq 0$
3. Even with no jump, you don't stay fixed: $|\psi_0(t+dt)\rangle \neq |\psi(t)\rangle$



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- Because the dynamics is Markovian, the average system dynamics $\dot{\rho} = \mathcal{L}\rho$ is unchanged by any processing of the system output fields prior to detection (it is just a change of basis).

- In quantum optics terms, we can put the output fields through a passive interferometer, also introducing local oscillator fields.

- To attain all possible unravellings, it is necessary to process the output fields adaptively. That is, the monitoring scheme chosen at time t is determined by the record prior to time t.

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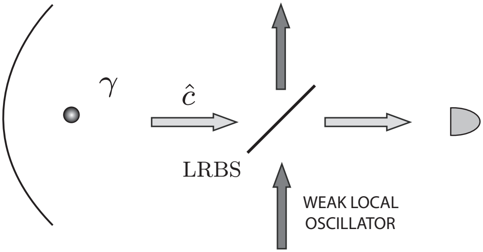

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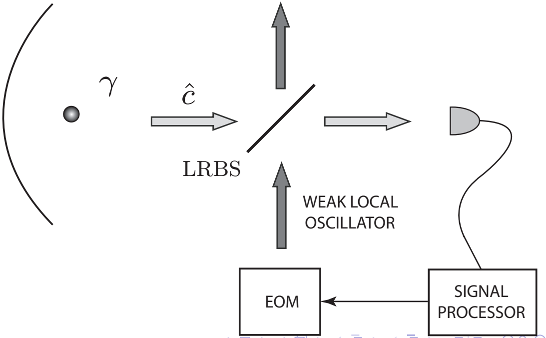

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1 Formally Defining Adaptive Measurements

2 Adaptive Measurements for Profit
- Doing some things better
- Doing some things *much* better
- Doing some things *perfectly*
- Doing some things *uniquely*

3 How Big a Brain does it take to Track an Open Quantum System?
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Not all Ensembles are Physically Realizable

- Restrict to *ergodic* master equations: $\rho_{\rm ss} = \lim_{t\to\infty} e^{\mathcal{L}t}\rho(0)$.
- We say that an ensemble $\{|\phi_k\rangle\}_{k=1}^{K}$ **represents** $\rho_{\rm ss}$ iff

$$\exists \text{ positive weights } \{\wp_k\} \text{ such that } \rho_{\rm ss} = \textstyle\sum_{k=1}^{K} \wp_k |\phi_k\rangle\langle\phi_k|.$$

- We say that an ensemble $\{|\phi_k\rangle\}_{k=1}^{K}$ is **physically realizable** (PR) in steady-state if there exists a way (which could be adaptive) to monitor the bath such that, *for all long times t*, $|\psi_{\rm c}(t)\rangle = |\phi_k\rangle$ for some k.
- **Theorem** (Wiseman & Vaccaro, *PRL* (2001)): the ensemble $\{|\phi_k\rangle\}_{k=1}^{K}$ is physically realizable in s.s. iff there exists $\kappa_{jk} > 0$:

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The Bloch Representation

- For a qubit we use the Bloch or SU(2) representation,

$$\rho = \frac{1}{2}\left(I + x\hat{\sigma}_x + y\hat{\sigma}_y + z\hat{\sigma}_z\right).$$

- Then defining $\mathbf{r} = (x, y, z)^\top$, $\dot{\rho} = \mathcal{L}\rho$ becomes

$$\dot{\mathbf{r}} = A\mathbf{r} + \mathbf{b},$$

  where we require A to be Hurwitz so that $\mathbf{r}_{\rm ss} = -A^{-1}\mathbf{b}$.

- We seek a PR ensemble $\{\mathbf{r}_k\}_{k=1}^K$. That is $K^2 - K$ rates $\kappa_{jk} > 0$ and K 3-vectors $\mathbf{r}_k$ satisfying

$$\forall k \;\; \mathbf{r}_k \cdot \mathbf{r}_k = 1$$

$$\forall j \;\; A\mathbf{r}_j + \mathbf{b} = \sum_{k=1}^{K} \kappa_{jk}(\mathbf{r}_k - \mathbf{r}_j).$$

- This is $4K$ *quadratic* equations in $K^2 + 2K$ unknowns. Solutions may exist for $K \geq 2$ but for arbitrary K it is still NP-complete.

Two-State Jumping ($K = 2$)

- For $K = 2$ the dynamical constraints imply

$$A(\mathbf{r}_1 - \mathbf{r}_2) = -(\kappa_{12} + \kappa_{21})(\mathbf{r}_1 - \mathbf{r}_2).$$

- **Lemma** If A is 3×3 and Hurwitz then it has at least one real, negative eigenvalue. That is, $\exists \mathbf{u} : A\mathbf{u} = -\lambda \mathbf{u}$, $\lambda < 0$.
- **Theorem** There always exists a two-state jumping solution

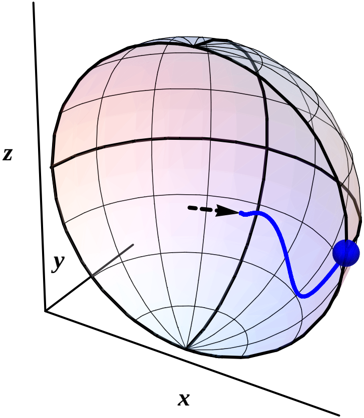
$$\begin{aligned} \mathbf{r}_1 &= \mathbf{r}_{ss} + \varepsilon_1 \mathbf{u} \\ \mathbf{r}_2 &= \mathbf{r}_{ss} - \varepsilon_2 \mathbf{u} \\ \kappa_{12} &= \wp_2 |\lambda| \; ; \quad \wp_2 = \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2} \\ \kappa_{21} &= \wp_1 |\lambda| \; ; \quad \wp_1 = \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \end{aligned}$$

- This hold regardless of the number of jump operators $\{\hat{c}_l\}_{l=1}^{L}$.

Example: Resonance Fluorescence

$$\mathcal{L}\rho = \mathcal{D}[\tfrac{\hat\sigma_x - i\hat\sigma_y}{2}]\rho - i[\tfrac{\Omega}{2}\hat\sigma_x, \rho]$$

Ω=1



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- For $|\Omega| < 0.25$, other $K = 2$ ensembles exist.
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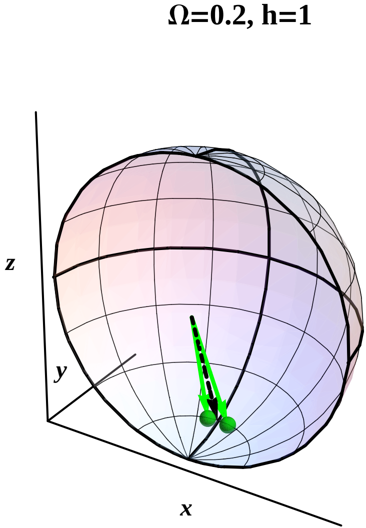

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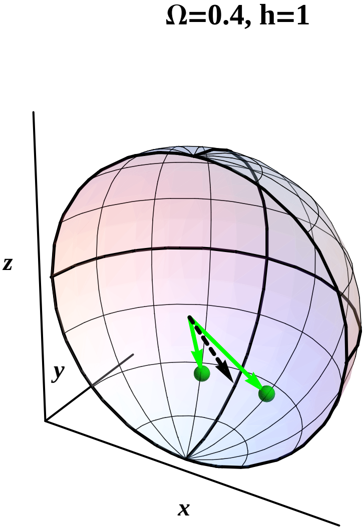

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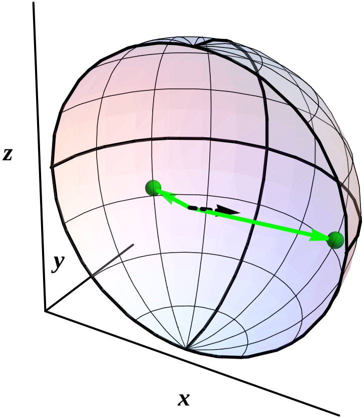
Ω=1, h=1



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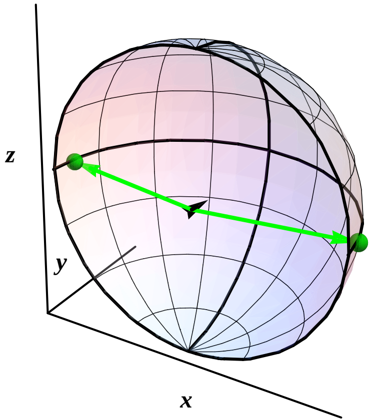

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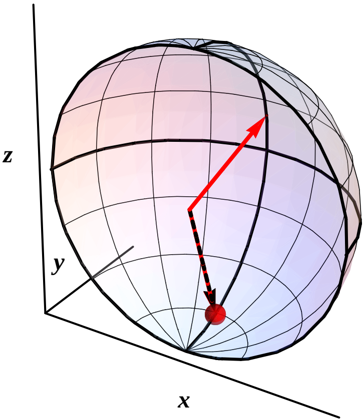
Ω=0.2, h=0.03



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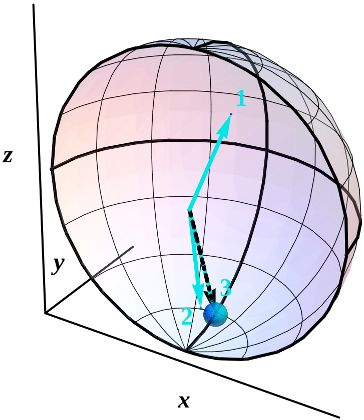
Ω=0.2, h=0.0822



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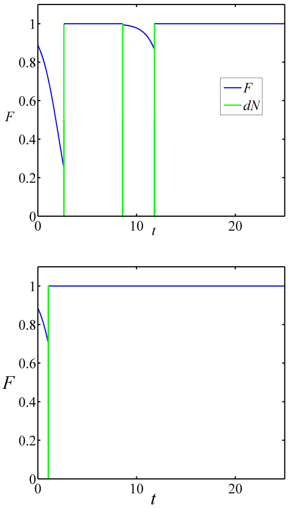
Stability. (Karasik & Wiseman, *PRA*, 2011.)



- We have *analytically proven* the *mean square stability* of *all* the $K = 2$ and $K = 3$ schemes presented. That is,

$$\lim_{t\to\infty} \text{Expected}\left[|\langle\psi_c(t)|\phi_{k(t)}\rangle|^2\right] = 1,$$

  with $k(t)$ a function of the record alone.
- However, *some* of these schemes have deterministically *unstable* stages.
- Even those that are piecewise deterministically stable can suffer a drop in fidelity upon a jump.
- Proving the stability of *all* finite-K schemes for an arbitrary system is an open problem.

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Revisiting the Counting Argument

- Recall: finding a PRE means solving the "Wiseman–Vaccaro equation"

$$\forall j\ \mathcal{L}|\phi_j\rangle\langle\phi_j| = \sum_{k=1}^{K} \kappa_{jk} \left(|\phi_k\rangle\langle\phi_k| - |\phi_j\rangle\langle\phi_j|\right).$$

  for an ensemble $\{|\phi_k\rangle\}_{k=1}^{K}$ and positive rates $\{\kappa_{jk}\}$.
- Karasik–Wiseman: $K_{\min} = (D-1)^2 + 1$ to expect solutions.
- Warszawsiki & HMW, *Quantum* (2019) revisited this, but now taking into account the number L of Lindblad operators.
- By a much more complicated parameter-counting argument, we claim that, iff $L < D - 1$, there is a correction to Karasik–Wiseman:

$$K_{\min} = (D-1)^2 + 1 + (2D - 2L - 1).$$

- Note that still $(D-1)^2 + 1 \leq K_{\min} \leq D^2 - 1$.
- But if $\mathcal{L}$ has dynamical symmetries, this may reduce $K_{\min}$.

Revisiting the Counting Argument

- Recall: finding a PRE means solving the "Wiseman–Vaccaro equation"

$$\forall j \; \mathcal{L}|\phi_j\rangle\langle\phi_j| = \sum_{k=1}^{K} \kappa_{jk} \left(|\phi_k\rangle\langle\phi_k| - |\phi_j\rangle\langle\phi_j|\right).$$

  for an ensemble $\{|\phi_k\rangle\}_{k=1}^{K}$ and positive rates $\{\kappa_{jk}\}$.
- Karasik–Wiseman: $K_{\text{min}} = (D-1)^2 + 1$ to expect solutions.
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Questions we posed

Q1 Are there MEs for which $K > D$ is provably necessary for a PRE?

Q2 Is an ensemble size of $K = (D-1)^2 + 1$ (as suggested by Karasik–Wiseman) provably inadequate for some systems?

Q3 Does the refined parameter counting heuristic reliably predict whether PREs are feasible for a ME of a given form?

- Q3a Does the heuristic accurately predict the impossibility of PREs when the number of parameters is less than the number of constraints? (i.e. for ensembles smaller than the determined threshold?)
- Q3b Does the heuristic accurately predict the possibility of PREs when the number of parameters is equal to the number of constraints?
- Q3c Does the heuristic accurately predict the necessity of PREs when the number of parameters is equal to the number of constraints?

Answer 1

Q1 Are there MEs for which $K > D$ is provably necessary for a PRE?

- Yes.
- Hence **open quantum systems can be harder to track than open classical systems.**
- Our investigation was done for a random selection of 20 MEs in $D = 3$.
- For each of these MEs, we obtained a computational proof that $K = D$ PREs cannot exist.
- This was in the form of a Hilbert Nullstellensatz certificate of infeasibility for the equations governing PREs[1].
- This was expected from the Karasik–Wiseman argument that $K \geq (D-1)^2 + 1$ is required.

[1] taking up to 2.5 days of cluster computation each

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Answer 2

Q2 Is an ensemble size of $K = (D-1)^2 + 1$ (as suggested by Karasik–Wiseman) provably inadequate for some systems?

- Yes.
- Our investigation was carried out for a random selection of 10 MEs in $D = 3$ with $L = 1$ Lindblad.
- For each of these MEs, we obtained a computational proof (Hilbert Nullstellensatz) that PREs with $K = 5$ cannot exist.
- This is as expected from Warszawski–Wiseman's refined parameter counting argument, which says that, in this case, $K_{\min} = 8$, in contrast to Karasik–Wiseman's $K_{\min} = 5$.
- Hence **our refined parameter counting argument is supported**.

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Answer 3

Q3 Does the refined (Warszawsi–Wiseman) parameter counting heuristic reliably predict whether PREs are feasible for a ME of a given form?

- It seems that way.

Q3a The non-existence of PREs when $K < K_{\min}$ is supported (Q2).

Q3b To look for PREs when $K = K_{\min}$ beyond $D = 2$ we have to simplify our system by introducing *symmetry*.

- Restricting to **re3its**, our argument says $K_{\min} = 4$ (which is < 5).
- From 80 randomly selected MEs, we found $K = 4$ PREs for 6 of them, using extended polynomial homotopy continuation methods.

Q3c We were able to find PREs in 100% of cases with $K > K_{\min}$.

- But we were restricted to $D = 2$, because the difficulty of finding PREs scales $\sim \exp(D^4)$.
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Outline

1 Formally Defining Adaptive Measurements

2 Adaptive Measurements for Profit
- Doing some things better
- Doing some things *much* better
- Doing some things *perfectly*
- Doing some things *uniquely*

3 How Big a Brain does it take to Track an Open Quantum System?
- Quantum Jumps: The Old Quantum Theory
- Quantum Jumps: The Modern Understanding
- Physically Realizable Ensembles
- For a qubit, a two-state classical memory is all it takes ...
- A Fuller Answer

4 Conclusion
- Summary and Questions

Adaptive Measurements for Profit

- Adaptive measurements comprise a form of measurement-based control, which is distinct from feedback and which has **no analogue classically for perfect measurements**.
- They have countless quantum applications, and many experimental demonstrations, including
  - Phase estimation
    - static phase for fixed mean photon number $\bar{n}$ (coherent or squeezed)
    - static phase for fixed maximum photon number n (especially $n = 1$)
    - dynamic (diffusing) phase for fixed photon flux over diffusion rate $\mathcal{N}/\kappa$ (coherent or squeezed)
    - static interferometric phase for fixed photon-passes M (single photon or entangled)
  - State discrimination of a fixed number of non-orthogonal states.
  - Measurement-based quantum computing.

Adaptive Monitoring for Pleasure

- In semiclassical models (e.g. Einstein's) a D-level open quantum system jumps between the D levels. That is, an observer can keep track of the state using a K-state classical memory with $K = D$.
- For a general ergodic Markovian open quantum system:
  - With a generic monitoring scheme, it is necessary to store real numbers (i.e. the classical memory size $K \to \infty$).
  - By allowing for all possible (in particular, adaptive) monitoring schemes, a finite K should always be sufficient.
  - But by a counting argument, typically $K_{\min} = O(D^2)$.
- For $D = 2$ (a qubit), $K = 2$ (one classical bit) is always sufficient.
- For $D = 3$ we have proven that $K = 3$ is insufficient in general.
  $\Longrightarrow$ **To keep track of an open quantum system you need a bigger brain than you would for an open classical system of the same size.**

Any Questions?

e.g. Given a physically realizable ensemble, can you explicitly construct the (adaptive) monitoring scheme that realizes it?

e.g. Does this generalize to discrete-time evolution (CP-maps)?

e.g. What does it mean to consider all adaptive monitorings?

e.g. What about the Schrödinger-HJW theorem?

e.g. Do these finite-state PREs by adaptive unravellings have any uses?

The Controllable Parameters

- The master equation $\dot{\rho} = \sum_{l=1}^{L} \mathcal{D}[\hat{c}_l]\rho - \mathcal{C}[i\hat{H}]\rho$ is invariant under $\{\hat{c}_l\} \to \{\hat{c}'_m\}$ and $\hat{H} \to \hat{H}'$,

$$\hat{c}'_m = \sum_{l=1}^{L} S_{ml}\hat{c}_l + \beta_m \ , \quad \hat{H}' = \hat{H} - \frac{i}{2}\sum_{m=1}^{M} \tfrac{1}{2}(\beta_m^* \hat{c}'_m - \beta_m \hat{c}'^{\dagger}_m).$$

  Here S is a semi-unitary *matrix* i.e. $\sum_{m=1}^{M} S^*_{l'm} S_{ml} = \delta_{l',l}$.

- Unravelling the master equation $\dot{\rho} = \mathcal{L}\rho$ as

$$\rho + d\rho = dt \sum_{m=1}^{M} \mathcal{J}[\hat{c}'_m]\rho + \left(1 - dt\mathcal{C}[i\hat{H}' + \tfrac{1}{2}\sum_{m=1}^{M} \hat{c}'_m \hat{c}'^{\dagger}_m]\right)\rho$$

  gives **different** conditional evolution, with the **same** average $\rho_{\rm ss}$.

- In quantum optics, S_{ml} describes an interferometer, while β_m describes adding local oscillators before detection.
- For K-state jumping we need K of these: S^k_{ml} and β^k_m, with k chosen adaptively, and with $M \leq \max\{K-1, L\}$.

Keeping track is not the same as Finding

- **Schrödinger-HJW theorem**: If $\rho_{ss} = \mathrm{Tr}_{\mathrm{bath}}\left[|\Psi_{\mathrm{entangled}}\rangle\langle\Psi_{\mathrm{entangled}}|\right]$ then for all pure state weighted ensembles (*not* necessarily orthogonal)

$$\{\wp_b|\phi_b\rangle\langle\phi_b|\}_{b=1}^{B} \text{ such that } \rho_{ss} = \textstyle\sum_{b=1}^{B}\wp_b|\phi_b\rangle\langle\phi_b|,$$

  there exists a bath POVM $\{\hat{E}_b\}_{b=1}^{B}$ such that for

$$\wp_b|\phi_b\rangle\langle\phi_b| = \mathrm{Tr}_{\mathrm{field}}\left[|\Psi_{\mathrm{entangled}}\rangle\langle\Psi_{\mathrm{entangled}}|\hat{E}_b\right].$$

- Does this mean that if one can attain all possible monitorings, one can attain all possible ensembles representing ρ_{ss}, including the diagonal one $\rho_{ss}|\phi_b\rangle = \wp_b|\phi_b\rangle$, $b = 1\ldots D$? No!
- Monitoring means keeping track of the state $|\psi_c(t)\rangle$ for all t.
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The Quantum Optical Theory of Radiation

- Master equation in the Interaction Frame:

$$\dot{\rho} = \sum_{n,m=1}^{D} \kappa_{mn} \left\{ \mathcal{J}[|\varepsilon_n\rangle\langle\varepsilon_m|] - \mathcal{C}[\tfrac{1}{2}|\varepsilon_m\rangle\langle\varepsilon_m|] \right\} \rho.$$

  where $\mathcal{J}[\hat{a}]\rho \equiv \hat{a}\rho\hat{a}^\dagger$, $\mathcal{C}[\hat{b}]\rho \equiv \hat{b}\rho - \rho\hat{b}^\dagger$

- $\implies \rho_{\rm ss} = \sum_{m=1}^{D} p_{\rm Boltzmann}(\varepsilon_m)|\varepsilon_m\rangle\langle\varepsilon_m|$, where $\langle\varepsilon_m|\varepsilon_n\rangle = 0$.

- Say $\rho(t) = |\varepsilon_o\rangle\langle\varepsilon_o|$. Then

$$\begin{aligned}
\rho(t+dt) &= \rho(t) + dt\dot{\rho}(t) \\
&= \sum_{n,m=1}^{D} \kappa_{mn} dt \mathcal{J}[|\varepsilon_n\rangle\langle\varepsilon_m|]\rho(t) + \left[1 - \sum_{n,m=1}^{D} \kappa_{mn} dt \mathcal{C}[\tfrac{1}{2}|\varepsilon_m\rangle\langle\varepsilon_m|]\right] \rho(t) \\
&= \sum_{n=1}^{D} \kappa_{on} dt |\varepsilon_n\rangle\langle\varepsilon_n| + \left[1 - \sum_{n=1}^{D} \kappa_{on} dt\right] |\varepsilon_o\rangle\langle\varepsilon_o| \\
&= \sum_{n=1}^{D} dP_{\rm jump}(o \to n)|\varepsilon_n\rangle\langle\varepsilon_n| + \left[1 - \sum_{n=1}^{D} dP_{\rm jump}(o \to n)\right] |\varepsilon_o\rangle\langle\varepsilon_o|.
\end{aligned}$$