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# Positivity Constraints on EFT

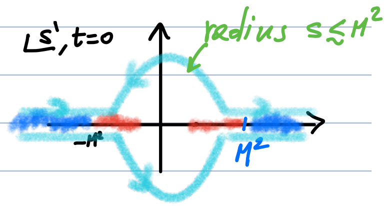
We have seen in L3 that positivity bounds follow by S-matrix principles encoding causality/analyticity and unitarity.
On the other hand we have seen in L2 that subluminality/causality enforces also non-trivial bound on the size of Wilson coefficients, can we obtain the same using S-matrix argument? Yes!

## — Positivity & Theory of Moments —

Let's work again first in the simplest case: forward scattering of massless scalar, assuming weakly coupled UV-completions

(1) $$M(s,t,u) = c_2 \frac{(s^2+t^2+u^2)}{2} + c_3\, stu + c_4 \left(\frac{s^2+t^2+u^2}{2}\right)^2 + \dots$$

and define our dispersive observables "the arcs"




(2) $$\boxed{a_n(s) \equiv \oint \frac{ds'}{2\pi i}\frac{1}{s'}\frac{M(s')}{s'^{\,n+2}}}$$

The ARCs at $t=0$

$M(s') \equiv M(t \to 0)$

where $n \geq 0$ to ensure convergence at $s' \to \infty$ according to Froissart
The (2) provides an IR-representation of the Wilson coefficients (that survive in the forward limit)

$M(s,t=0) = c_2 s^2 + c_4 s^4 + \dots$
only even powers.
$a_{n\text{-odd}} = 0$ at $t=0$

(3) $$\boxed{a_n(s) = c_{n+2} \quad \text{if } n \in \text{even}}$$

IR-representation Wilson coeffic.

( tree-level. If loop non-linear map $a_n \Leftrightarrow c_n$ )

On the other hand, analyticity allows a UV-representation

(4) $$a_n(s) = \frac{2}{\pi}\int_s^\infty \frac{ds'}{s'}\frac{\operatorname{Im} M(s')}{s'^{\,n+2}} \underset{s'=s/x}{=} \frac{1}{s^{n+2}}\frac{2}{\pi}\int_0^1 \frac{dx}{x}\, x^{n+2}\operatorname{Im} M(s/x)$$

that records them in terms of moments of a positive measure thanks to unitarity:

(5) $$\boxed{\begin{aligned} &a_n(s)\cdot s^{n+2} = \int_0^1 dx\, \mu(x)\, x^n \equiv \langle x^n\rangle \\ &\mu(x) \equiv x \operatorname{Im} M(s/x) \geq 0 \end{aligned}}$$

($n \in$ even)

$$\Longrightarrow \boxed{c_{n+2}\, s^{n+2} = \langle x^n \rangle = \mu_n}$$

IR-Representation

Remarks:

- The measure $\mu(x)$ has support on $(0,1) \subset \mathbb{R}$
- As long as one is happy with tree-level statements, i.e. neglecting calculable IR-loops, one can just set $s = M^2$ (if UV-theory weakly coupled). That is, we assume couplings sufficiently small that we can make a trustworthy argument to the desired $n$ (i.e. $n \lesssim \frac{\log g_*^2/16\pi^2}{\log s/M^2}$ for fixed couplings $g_*^2 \sim c_n$ and fixed $s/M^2$).

Notice that $x < 1 \Rightarrow x^{n+2} < x^{n+1} < x^n$ i.e.

(6) $$\boxed{\mu_{n+2} < \mu_{n+1} < \mu_n}$$ arcs are monotonically decreasing

$$\Downarrow$$

(7) $$c_{n+4}\, s^{n+4} < c_{n+2}\, s^{n+2} \Longrightarrow \boxed{\frac{c_{n+4}}{c_{n+2}}\, s^2 < 1 \quad \begin{matrix} n \geq 0 \\ \text{even} \end{matrix}}$$

Which is basically an upper bound on how large can be taken, i.e. a statement on the cutoff $M^2$:

(8) $$\boxed{M^4 < \underbrace{c_{n+2} / c_{n+4}}_{n \in \text{even} \quad n>0}}$$ upper bound cutoff!

Remark:

- the $c_n$'s in (1) are dimensionful: if rescaling them in units of $M$, $c_n = g_n / M^{2n}$, one gets that (7) is equivalent to

(9) $$M(s) = g_2 \frac{s^2}{M^4} \Big( 1 + \underbrace{\tfrac{g_4}{g_2} \left(\tfrac{s}{M^2}\right)^2}_{<1} + \underbrace{\left(\tfrac{g_6}{g_2}\right) \left(\tfrac{s}{M^2}\right)^4}_{<1} + \dots \Big)$$

$\Rightarrow$ $\boxed{\text{Supersoft ruled out!}}$ (for Wilson coefficients that survive in $M(s)$ for $t \to 0$)

(i.e. theory running faster than $E^4$)

Despite there would be a symmetry to realize it, namely

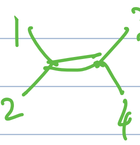
$\pi \to \pi + \frac{1}{n!} c_{\mu_1 \dots \mu_n} x^{\mu_1 \dots \mu_n}$

- From (8) $\Rightarrow$ no matter how small the coupling, $\mu \ll 1$, the cutoff is (bounded by) two consecutive Wilson coefficients

(10) $$M(s) = \varepsilon s^2 \left(1 + \frac{s^2}{M^2} + \dots \right)$$

$\varepsilon \sim 10^{-13}$

$\hookrightarrow$ still amplitude breaks at $s = M^2$

Compare it with:  $= \frac{g_*^2}{s - M^2} = -\frac{g_*^2}{M^2} \left( 1 + \frac{s}{M^2} - \left(\frac{s}{M^2}\right)^2 + \dots \right)$

Now that we know that Wilson coefficients are moments we can say much more.

For instance, we can define a scalar product on the space of (real) polynomials on $(0,1)$:

(11) $$\langle p_1(x) \,|\, p_2(x) \rangle = \int_0^1 p_1(x)\, p_2(x)\, d\mu$$

$$\Downarrow$$

(12) Cauchy-Schwarz $$\underbrace{|\langle x^n | x^m \rangle|^2}_{\left(c_{n+m+2}\, s^{n+m+2}\right)^2} \leq \underbrace{\langle x^n | x^n \rangle}_{c_{2n+2}\, s^{2n+2}} \underbrace{\langle x^m | x^m \rangle}_{c_{2m+2}\, s^{2m+2}}$$

(13) $$\boxed{c^2_{n+m+2} \leq c_{2n+2}\, c_{2m+2} \qquad n, m = \text{even}}$$

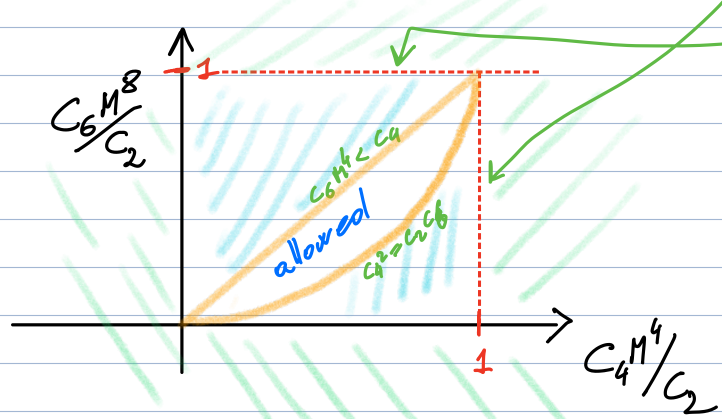
Simplest Non-Linear Bound

Example:

(14) $c_4^2 \leq c_2\, c_6$ $\quad (n=0,\ m=2) \quad \Longrightarrow \quad \underbrace{c_6/c_2 \geq \left(c_4/c_2\right)^2}$ parabola

which together with (8), $c_4 M^4 < c_2$ and $c_6 M^4 < c_4 < c_2/M^4$

(15)




(to be contrasted with just $c_i \geq 0$ that would allowed all first quadrant)

In our collaboration, 2011.00037, we refer to (15) as "the banana plot"

## General strategy

given positive measure $d\mu(x)$ in $(0,1)$ we can test it against positive polynomials $p_n(x)$ of order $M$ to get relations among the first $M+1$ moments:

$$(16)\qquad \int_0^1 d\mu(x) \underbrace{\sum_n^M g_n x^n}_{p_M(x)>0} \Rightarrow \sum_n g_n \langle x^n \rangle \geq 0 \Rightarrow \sum_n g_n c_{n+2} s^{n+2} \geq 0$$

so the varying $g_n$ (while keeping $p_M>0$) we reduce the allowed space of $c_{n+2}$. ↘

$$\boxed{\text{Positivity Bounds} \Longleftrightarrow \text{Space of Positive Polynom. in } I=(0,1)}$$

This is a problem of algebraic geometry: find positive polyn. in domain $D$ defined by other polynomials $Q_j>0$:

(in our case $Q_1 = x$ $\quad Q_2 = 1-x \quad Q_1>0 \quad Q_2>0$ define $D=I=(0,1)$)

Solution "Sum of square theorem" (Schmuedgen 1991)

$$P(x) \geq 0 \text{ in } D = \{x \in \mathbb{R}^n / Q_i(x) \geq 0\}$$

$$\Updownarrow$$

$$(17)\qquad P(x) = \sum_{i_1\ldots} \overbrace{Q_{i_1\ldots}(x)\; Q_{i_n}(x)}^{\text{positive in } D}\; \overbrace{q_i^2(x)}^{\text{positive everywhere}}$$

Example:

$$(18)\qquad \begin{cases} P_{2M}(x) \geq 0 \quad \text{in} \quad (0,1) = \{x / x>0,\ 1-x>0\} \\ \Downarrow \\ P_{2M} = q_{1M}^2(x) + x\, q_{2\,M-1}^2 + (1-x)\, q_{3\,M-1}^2 + (1-x)\,x\, q_{4\,M-1}^2(x) \end{cases}$$

(19) $\int d\mu \; 1 = \mu_0 > 0 \,, \quad \underbrace{\int d\mu \; x^n = \mu_n \geq 0}_{\text{positivity}} \,, \quad \int d\mu \; x^n (1-x) = \underbrace{\mu_n - \mu_{n+1} \geq 0}_{\text{monotonicity}}$

(20) $\int d\mu \, (\underbrace{x^{2n} g_n)^2}_{\text{square}} \geq 0 \iff g_{2n} \, g_{2m} \, \mu_{2n+2m} \geq 0 \quad \forall \, g_{2n}, g_{2m}$

(21) $\boxed{\text{Henkel} \quad (H)_{nm} = \mu_{2n+2m} = c_{2n+2m+2} \, s^{2n+2m+2}}$ positive definite matrix

(22) $\boxed{\begin{pmatrix} \mu_0 & \mu_2 \\ \mu_2 & \mu_4 \end{pmatrix} \succeq 0 \implies c_2 \cdot c_6 \geq c_4^2}$

in agreement with (14).

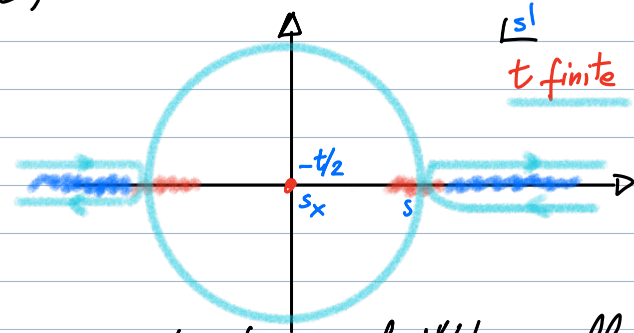
Using (18) one generates all bounds, but if one restricts to the first 3 Wilson coefficients, we optained the optimal bounds already.

What about the Wilson coefficients that are in front to powers of $t^n$? Can we constrain them? You bet!

## — Finite $-t$ Positivity Bounds —

Let's assume that amplitude is $s$-analytic for fixed finite $t$, typically it has been proven rigorously for $-\mu_L^2 < t < \mu_R^2$ with $\mu_R^2 \sim O(1)\, m^2$ (e.g. $4m^2$ exactly for identical particles) and $\mu_L^2 \sim O(100)\, m^2$.
(In fact, for $t<0$ maximally analyticity assumes $-t > s$ for scattering the lightest state particles; we will not need this much, but in gravity is required..)

With this analyticity, we can define finite-$t$ arcs centered at the crossing symmetric point $s_\times = 2m^2 - t/2 \underset{m\to 0}{=} -t/2$ ($s_\times \overset{su\text{-crossing}}{\longleftrightarrow} -s_\times - t + 4m^2 = s_\times$) of radius $s + t/2$, to land at $s' = s$:




(23) $$a_n(s,t) = \frac{1}{2\pi i}\oint \frac{M(s',t)}{s'^{\,n+2}}\,\frac{ds'}{s'}$$

These provide IR-representation of linear combinations of Wilson coeff.

IR-representation — $t<0$

(24) $$a_n(s,t) = \begin{cases} n=0, & c_2 - c_3 t + 3c_4 t^2 - 2c_5 t^3 + \dots \\ n=1, & 2c_4 t - 2c_5 t^2 + \dots \\ n=2, & c_4 - c_5 t + \dots \\ \vdots & \vdots \end{cases}$$

($\leftarrow$ odd vanish at $t=0$)

Again, analyticity provides a UV-representation:

(25) $$a_n(s,t) = \frac{1}{\pi}\left(\int_s^\infty \frac{ds'}{s'^{\,n+3}} + \int_{-\infty}^{-s-t}\frac{ds'}{s'^{\,n+3}}\right)\frac{M(s+i\varepsilon,t) - M(s-i\varepsilon,t)}{2i}$$

$s \to -s-t$ in 2° int'
+ crossing $M(s,t) = M(-s-t,t)$

$$= \frac{1}{\pi}\int_s^\infty ds'\, \mathrm{Im}\, M(s',t)\left(\frac{1}{s'^{\,n+3}} + (-1)^n \frac{1}{(s'+t)^{n+3}}\right)$$

Partial Wave Exp.

(26) $$a_n(s,t) = \underbrace{\frac{1}{\pi}\sum_\ell (2\ell+1)}_{\text{Partial waves}} \int_s^\infty ds'\, \underbrace{\mathrm{Im}\, a_\ell(s')}_{\geq 0}\, P_\ell\!\left(1 + \frac{2t}{s'}\right)\left(\frac{1}{s'^{\,n+3}} + (-1)^n\frac{1}{(s'+t)^{n+3}}\right)$$

where the Legendre polynomials, guess what?, are polynomial in $2t/s'$ of order $\ell$: $P_\ell(1+2t/s) = \sum_q^{\ell} \alpha_q (2t/s)^q$ with known $\alpha_q(\ell)$

**Remark:** $P_\ell(1+2t/s)$ can now be negative, but we know the function! We know how negative it can get.

2 strategies:

(a) integrate both sides of (26) with suitably constructed functions $f_\ell(t)$ such that

$$\int_{-s}^{0} dt\, f_\ell(t)\, P_\ell(1+2t/s')\,(\ldots) \geq 0 \quad \forall \ell,\ \forall s'>s \tag{27}$$

this is typically done numerically (e.g. 2102.08951), except in eikonal limit (2211.00085)

(b) defines a 2D-moment problem with respect to a positive measure $\boxed{d\mu(x,\ell) \propto \mathrm{Im}\, a_\ell(s/x)}$ in

$$\underset{\times}{[0,1]} \otimes \{0, 2, 6, 12, \ldots\}_{J^2=\ell(\ell+1)} = \text{support of } d\mu(x, J^2) \tag{28}$$

so that expanding (26) in $t$

$$a_n(s,t)\, s^{n+2} = \sum_{J^2} \int_0^1 \underbrace{d\mu(x,s)}_{\text{unknown but positive}}\, x^n \Big( \overbrace{\beta_{nmk}\, (J^2)^k \left(\tfrac{t}{s} x\right)^m}^{\text{known!}} \Big) \tag{29}$$

↖ Linear combinations of moments!

for instance

$$\boxed{\begin{aligned}
a_0(s,t)\, s^2 &= \sum_{J^2}\int_0^1 d\mu(x,s) \left[1 - \left(\tfrac{3}{2} - J^2\right)\tfrac{t}{s} x + \frac{(J^2-2)(J^2-6)}{4}\left(\tfrac{t}{s}x\right)^2 + \ldots\right] \\
a_1(s,t)\, s^3 &= \sum_{J^2}\int_0^1 d\mu(x,s)\, x \left[2\tfrac{t}{s}x + 2\left(J^2 - 5/2\right)\left(\tfrac{t}{s}x\right)^2 + \ldots\right] \\
a_2(s,t)\, s^4 &= \sum_{J^2}\int_0^1 d\mu(x,s)\, x^2 \left[1 + \left(J^2 - 5/2\right)\left(\tfrac{t}{s}x\right) + \ldots\right] \\
&\ \ \vdots
\end{aligned}} \tag{30}$$

Now, we see that (30) means the arcs are linear combinations of 2D-moments: $\frac{1}{4}\left(\frac{t}{s}\right)^2\left(\mu_{2,2}-8\mu_{2,1}+12\mu_{2,0}\right) + \dots$

(31)
$$a_0(s,t)/s^2 = \mu_{0,0} - \frac{t}{s}\left(\frac{3}{2}\mu_{1,0} - \mu_{1,1}\right) + \dots = s^2\left(c_2 - c_3 t + 3c_4 t^2 + \dots\right)$$
$$a_1(s,t)/s^3 = 2\frac{t}{s}\mu_{2,0} + 2\left(\frac{t}{s}\right)^2\left(-\frac{5}{2}\mu_{3,0} + \mu_{3,2}\right) + \dots = s^3\left(2c_4 t - 2c_5 t^2 + \dots\right)$$
$$a_2(s,t)/s^4 = \mu_{2,0} + \left(\frac{t}{s}\right)\left(-\frac{5}{2}\mu_{3,0} + \mu_{2,1}\right) + \dots = s^4\left(c_4 - c_5 t + \dots\right)$$
$$\vdots$$

where in the last equality we used the IR repr. (24): matching both sides in powers of $t/s$ gives the desired relations among 2D-moments and Wilson coefficients:

Matching to moments leading Wilson

(32) $$\boxed{c_2 s^2 = \mu_{0,0} \qquad c_3 s^3 = +\frac{3}{2}\mu_{1,0} - \mu_{1,1} \qquad c_4 s^4 = \mu_{2,0}}$$

From this immediately follows that $c_3 s^3 < \frac{3}{2}\mu_{1,0} < \frac{3}{2}\mu_{0,0}$ since moments are positive and still monotonic in the first index ($x \in (0,1) \Rightarrow x^{n+1} < x^n \Rightarrow \mu_{n+1,j} < \mu_{n,j}$)

(33) $$\boxed{c_3 s^3 < \frac{3}{2} c_2 s^2}$$ Galileon Bounded Above

Can we also bound it below? since $c_3 s^3 = +3/2\,\mu_{1,0} - \mu_{1,1} > -\mu_{1,1} \rightarrow$ need to bound $\mu_{11}$ (notice we don't have monotonicity in the second index because $J^2$ runs over infinite range). We can bound $\mu_{11}$ using full s-t-u crossing, which gives $c_4 \propto \frac{\partial^2}{\partial t^2} a_0\big|_{t=0}$ and $c_4 = a_2\big|_{t=0}$ ↴

(34) $\frac{c_4}{4}\big(\underbrace{s^2+t^2+u^2}_{\text{fixed by crossing}}\big)^2 \longrightarrow c_4\big(s^4+\dots+3\,\underbrace{s^2t^2}_{}+\dots\big)$

selected by 2 subtractions and 2 t-derivatives

selected by "4 subtractions" i.e. $\mu_{2,0}$

$\Downarrow$

(35) $12\, c_4\, s^4 \overset{(32)}{=} 12\cancel{\mu_{2,0}} \overset{(31)\text{ or }(34)}{=} \mu_{2,2} - 8\mu_{2,1} + \cancel{12\mu_{2,0}}$

Providing a sum-rule among moments, known as

(36) $\boxed{\mu_{2,2} = 8\mu_{2,1}}$ **Null constraint**

simplest example of [illegible]

Now, together with the simplest Henkel matrix constraint (taking $J^2$ contin. on $\mathbb{R}^+ \Rightarrow$ bounds rigorous but suboptimal)

(37) $\displaystyle\oint d\mu\, (a+bxJ^2)^2 \geq 0 \underset{\forall\, a,\,b}{\Longrightarrow} \begin{pmatrix}\mu_{0,0} & \mu_{1,1}\\ \mu_{1,1} & \mu_{2,2}\end{pmatrix} \succeq 0 \Longrightarrow \boxed{\mu_{1,1}^2 \leq \mu_{0,0}\,\mu_{2,2}}$

gives

(38) $\mu_{1,1}^{\cancel{2}} < 8\,\mu_{0,0}\,\mu_{2,1} \underset{\uparrow\,\text{monotonic in } x}{<} 8\,\mu_{0,0}\,\cancel{\mu_{1,1}} \Longrightarrow \boxed{\mu_{1,1} < 8\,\mu_{0,0}}$

and therefore from (32)

(39) $c_3\, s^3 = +\frac{3}{2}\mu_{1,0} - \mu_{1,1} > -\mu_{1,1} > -8\,\mu_{0,0} = -8\,c_2\, s^2$

we extract as well the lower bound $-8c_2 s^2 < c_3 s^3$:

(39) $\boxed{-8c_2 s^2 < c_3\, s^3 < \frac{3}{2} c_2\, s^2}$

In perfect agreement with the subluminality arguments seen in L2.

Remarks:

- The lower bound in (39) is not optimal because we used only the information of first few moments. Using more moments, e.g. in longer Henkel matrices, $-8c_2 \to -5.2\, c_2$. In any case, Galileon symmetry is forbidden by positivity! (Likewise no supersoft theory too.)

- Method (a) around (27) gives the same result, but requires numerics. Method (a) is however superior when one can't expand around $t=0$, which happens in the presence of IR divergences (like in gravity, where minimal coupling gives $M \sim -s^2/t$ for elastic near-forward scattering)

— Simple Approximate Method —

There is an approximate analytic method that allows to avoid all the gymnastic with moments in strategy (a) or functionals in (b): we know the two sources of t-dependence in (26), e.g for $a_0(s,t)$

(40) $$a_0(s,t) = \frac{1}{\pi}\sum_\ell (2\ell+1)\int_s^\infty ds'\, \mathrm{Im}\, a_\ell(s')\, \underbrace{P_\ell\left(1+\frac{2t}{s'}\right)}_{|P_\ell|\leq 1} \frac{2}{s'^3}\left[\left(1+\frac{1}{(1+t/s')^3}\right)\frac{1}{2}\right]$$

where $P_\ell$ is at worst as negative as $-1$, and where the $[\ldots]$ is monotonic so that it is largest at $s'=s$ i.e. $t/s' \to t/s$

(41) $$|a_0(s,t)| \leq \frac{1}{2}\left(1+\frac{1}{(1+t/s)^3}\right)\frac{1}{\pi}\sum_\ell(2\ell+1)\int_s^\infty ds'\, \mathrm{Im}\, a_\ell(s') = \frac{1}{2}\left(1+\frac{1}{(1+t/s)^3}\right) a_0(s,t=0)$$

(42) $$\boxed{\left|\frac{a_0(s,t)}{a_0(s,t=0)}\right| \leq \frac{1}{2}\left(1+\frac{1}{(1+t/s)^3}\right)}$$ Exact Bound

"Non-forward arcs can't grow too fast with t"

Let's calculate the l.h.s. in the EFT approximately now:

(43) $$\left(1+\frac{c_3}{c_2}t+3\frac{c_4}{c_2}t_+^2\dots\right)^2 \le \left(\frac{1}{2}\left(1+\frac{1}{(1+t/s)^3}\right)\right)^2$$

This allows to immediately exclude Galileon, reasoning by contradiction:

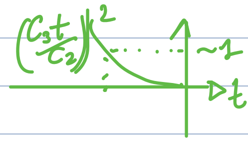
i) assume galileon is good symmetry $\to$ $\frac{c_3}{c_2}t \gg 1$

ii) within galileon EFT, assume higher-$\partial$ are EFT control, $\left|\frac{c_4 t}{c_3}\right| \ll 1, \dots,$

$\Rightarrow$ (43) requires that

(44) $\boxed{\left(\frac{c_3 t}{c_2}\right)^2 \lesssim 1}$ which contradicts our assumptions.

[[ it can be used to determine the t where theory breaks down, i.e. the cutoff, schematically $\left(1+\frac{c_3 t}{c_2}\right)^2$




. Similar logic, more accurate, in 2304.02550 ]]

(if ii) is relaxed, it remains true that l.h.s of (43) is IR calculable in EFT, and one can repeat argument for the first unsuppressed $c_n$-coefficient)