The Bigeodesic Problem

Riddhipratim Basu
(ICTS-TIFR)

ICTS Kolmogorov Symposium
25 April 2023

Geodesics / Bigeodesics

- Consider a geodesic metric space.
- A ray is called a semi-infinite geodesic if every finite segment is a geodesic between its endpoints.
- A bi-infinite line is called a bigeodesic if the same holds true.
- Straight lines are examples of bigeodesics in $\mathbb{R}^2$.

Do bigeodesics exist in random metric spaces?

- Consider first passage percolation on $\mathbb{Z}^d$, $d \geq 2$.
- Let $\{t_e\}_{e \in E(\mathbb{Z}^d)}$ denote a configuration of i.i.d. edge weights. (assume positive, cts, high moments for convenience)
- Define the length of a path γ by

$$\ell(\gamma) = \sum_{e \in \gamma} t_e$$

- Define a random metric on $\mathbb{Z}^d$ by

$$T(u,v) = \min_{\gamma : u \to v} \ell(\gamma)$$

a random distortion of the graph distance.

Do bigeodesics exist in random metric spaces?

- An old question attributed to Furstenberg (Kesten '86)
- Do bigeodesics exist in FPP on $\mathbb{Z}^d$?
- Connection to uniqueness of ground states in an Ising model with i.i.d. coupling (for $d=2$).
- By a 0-1 law, the probability of existence of a bigeodesic is either 0 or 1.
- So it is equivalent to study whether bigeodesics through 0 exist w.p. >0.

Partial progress

- The main question still remains open.
- There is a heuristic by Chuck Newman which suggests that the answer should be No for all dimensions $d < d_c$; the upper criti dimension; i.e. for all dimensions where the fluctuation exponent for the metric $\chi > 0$.

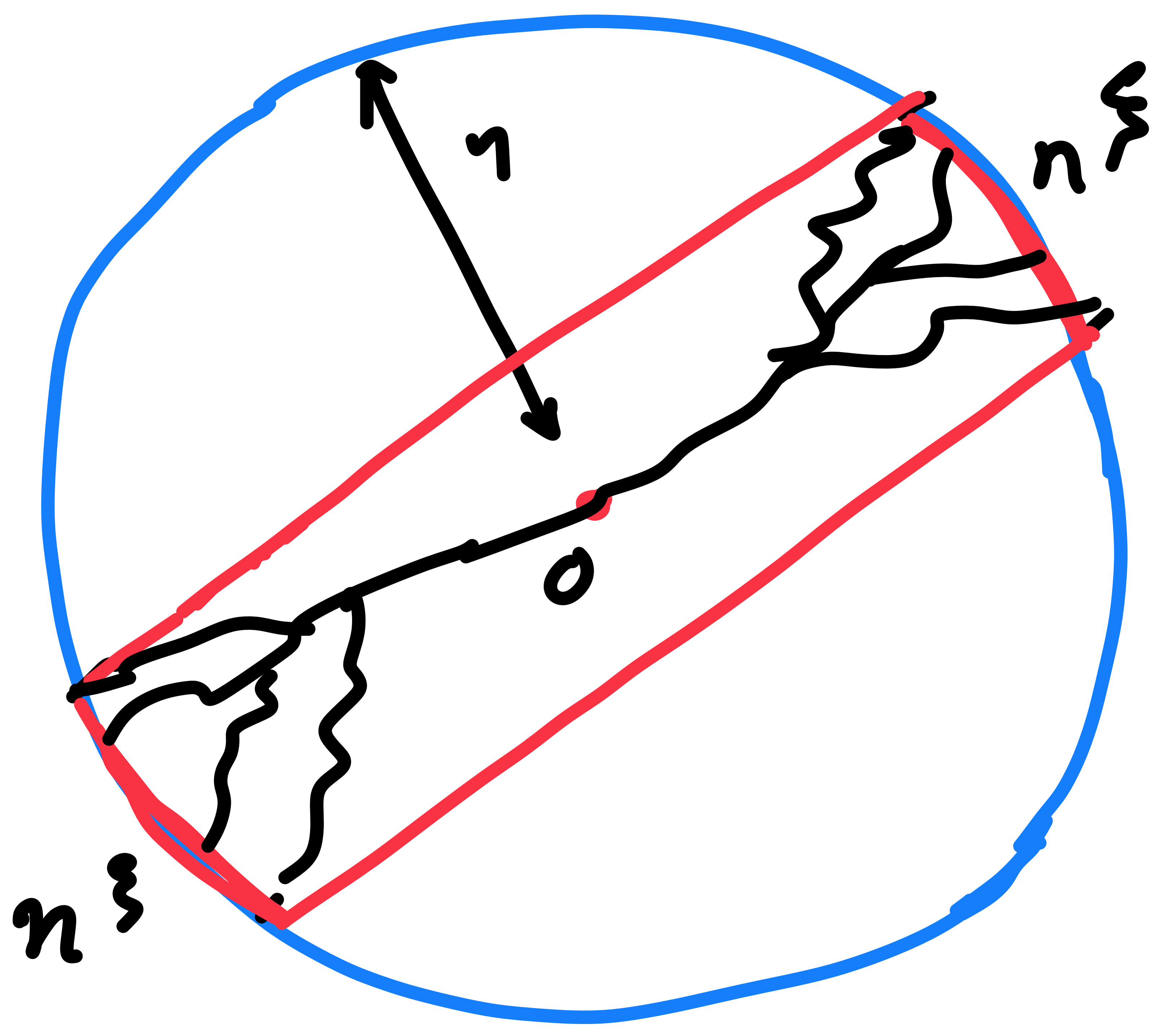
(this means that the FPP distance between two points $\sim n$ distance apart has fluctuations $\sim n^{\chi}$)

Newman's Heuristic

- It is belived that there exists another exponent ξ (transversal fluctuation) with

$$\chi = 2\xi - 1$$

such that geodesics at distance n has n^{ξ} fluctuation and most geodesics coalesce at this scale.



Only opposite pairs contribute essentially because of transversal fluctuations.

$\mathbb{P}(\text{crossing}) \approx n^{-\xi}$

$\mathbb{P}(\exists \text{ a crossing}) \lesssim n^{1-\xi} \cdot n^{-\xi}$

$\to 0$ if $\xi > \frac{1}{2}$.

Same heuristic for general d.

Making this rigorous for exactly solvable models

- It is believed that for $d=2$, $\boxed{\chi = \frac{1}{3}, \ \xi = \frac{2}{3}}$
- KPZ universality class.
- This is rigorously known for a few "exactly solvable" models of last passage percolation.

  iid weights on vertices,
  directed paths
  maximum weight paths.

- LPP model on $\mathbb{Z}^2$ with exponential/geometric weights are exactly solvable.

Rigorous results

- For exponential LPP in $\mathbb{Z}^2$, almost surely there are no non-trivial bigeodesics.

  B., Hoffman, Sly (2022)

  Balazs, Busani, Seppalainen (2021)

- For geometric LPP

  Groathouse, Janjigian, Rassoul-Agha (2021)

- Rigorising Newman's heuristic using various integrable probability estimates.

Conditional results in FPP

- No bigeodesics in a fixed direction in $d=2$ under various hypotheses on the limit shape.

  Licea–Newman '96

  Damron–Hanson '14

  Ahlberg–Hoffman '16

- No bigeodesics in $\mathbb{Z}^d$, $d \geq 2$ under strong unproven hypothesis akin to what is known for exactly solvable models.

  Alexander '20

Some other results

- No bigeodesics in the half-plane FPP.

  Wehr - Woo '98

  Auffinger - Damron - Hanson '13

- Bigeodesics exist for FPP on Cayley graphs of Gromov Hyperbolic groups.

  Benjamini - Tessera '15.

- There exists bigeodesics for FPP in Euclidean lattice for some non i.i.d. passage times

  Boivin - Derrien '02

Thank You

Questions ?