Arakelov self-intersection numbers on modular curves

Priyanka Majumder
(Joint works with A. von Pippich, and with D. Banerjee, C. Chaudhuri)

Department of Mathematics

IISER PUNE

Outline

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Its associated modular curve is $X_0(N)$ and we consider $K = \mathbb{Q}$.

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- Let $\omega_{\mathcal{X}_0(N)}$ be the relative dualizing sheaf on $\mathcal{X}_0(N)$. Arakelov defined a metric $\| \cdot \|_{\mathrm{Ar}}$ on $\omega_{\mathcal{X}_0(N)}$. Arakelov self-intersection number of $\omega_{\mathcal{X}_0(N)}$ is given by $\overline{\omega}^2_{\mathcal{X}_0(N)} = \overline{\omega}_{\mathcal{X}_0(N)}.\overline{\omega}_{\mathcal{X}_0(N)} \in \mathbb{R}$, where $\overline{\omega}_{\mathcal{X}_0(N)} = (\omega_{\mathcal{X}_0(N)}, \| \cdot \|_{\mathrm{Ar}})$.

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- $\overline{\omega}^2_{\mathcal{X}_0(N)}$ is independent of the number field K if the minimal regular model $\mathcal{X}_0(N)$ is semi-stable over $\mathcal{O}_K$.

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$$\Delta_{\mathrm{hyp},z} = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = -4y^2\left(\frac{\partial^2}{\partial z \partial \overline{z}}\right).$$

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where $d_z = \left(\partial_z + \overline{\partial}_z\right)$ and $d_z^c = \left(\partial_z - \overline{\partial}_z\right)/(4\pi i)$.

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- Let $\mathcal{G}_{\text{can}}(z,w)$ be the canonical Green's function for $X_0(N)$. Away from the diagonal it is characterized by the differential equation
$$d_z d_z^c\, \mathcal{G}_{\text{can}}(z,w) + \delta_w(z) = \mu_{\text{can}}(z)$$
with the nomalization $\int_{X_0(N)} \mathcal{G}_{\text{can}}(z,w)\, \mu_{\text{can}}(z) = 0$.

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$$\langle D_m, D_m \rangle = -2[K : \mathbb{Q}]\Big(\text{Néron-Tate height of } \mathcal{O}(D_m)\Big),$$

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- In 1998, Michel-Ullmo proved the following

$$h_{NT}\left(\mathcal{O}(D_m)\right) = O\left(\log N \left(\tau(N)\right)^2\right), \quad m \in \{0, \infty\}$$

where $K = \mathbb{Q}$, and $\tau(N) := \sum_{d|N} 1$.

Main Theorem

Theorem 1 (–, A. von Pippich)

Let N be an positive integer, then as $N \to \infty$ we have

$$2g_N(1-g_N)\,\mathcal{G}_{\mathrm{can}}(0,\infty) = 2g_N \log N + o(g_N \log N),$$

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- In 2020, Banerjee-Borah-Chaudhuri proved this for $N = p^2$ with a prime p.

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Theorem 2 (D. Banerjee, C. Chaudhuri, –)

Let p be a prime and $r = 3, 4$. The Arakelov self-intersection number of the relative dualizing sheaf of $\mathcal{X}_0(p^r)$ satisfies

$$\overline{\omega}^2_{\mathcal{X}_0(p^r)} = 3g_{p^r}\log(p^r) + o(g_{p^r}\log p) \text{ as } p \to \infty,$$

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- The special fiber of the regular model $\mathcal{X}_0(p^r)$ over $\mathbb{F}_p$ depends on the parity of $p \bmod 12$, these are $p \equiv 1 \bmod 12$, $p \equiv 5 \bmod 12$, $p \equiv 7 \bmod 12$, and $p \equiv 11 \bmod 12$.

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- As $p \to \infty$ we prove the following

$$\frac{1}{g_{p^r}-1}\left(g_{p^r}\langle V_0 . V_\infty\rangle - \frac{V_0^2+V_\infty^2}{2}\right) = g_{p^r}\log(p^r) + o(g_{p^r}\log p).$$

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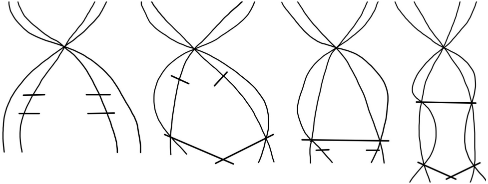

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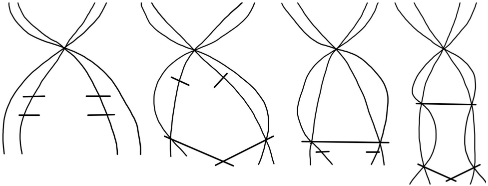

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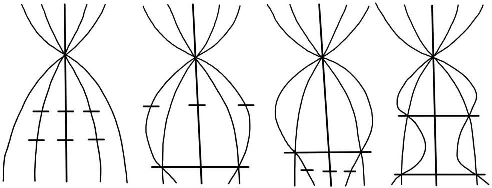

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	$C_{4,0}$	$C_{0,4}$	$C_{3,1}$	$C_{1,3}$	$C_{2,2}$	E	F
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$C_{0,4}$	$\frac{p-11}{12}$	$-\frac{p^4-p^3+10}{12}$	$\frac{p-11}{12}$	$\frac{p^3-p^2-10}{12}$	$\frac{p-11}{12}$	1	1
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F	1	1	1	1	1	0	$-3.$

- In this case the regular model is not minimal.
- After successive blow downs we obtain the minimal regular model.

Intersection matrix for $\mathcal{X}_0(p^r)$

- When $p \equiv 11 \bmod 12$, and $r = 4$, the intersection matrix of the special fibers of the minimal regular model:

	$C'_{4,0}$	$C'_{0,4}$	$C'_{3,1}$	$C'_{1,3}$
$C'_{4,0}$	$-\frac{2p^4-2p^3-p^2+2p-1}{24}$	$\frac{p^2-1}{24}$	$\frac{2p^3-p^2-2p+1}{24}$	$\frac{p^2-1}{24}$
$C'_{0,4}$	$\frac{p^2-1}{24}$	$-\frac{2p^4-2p^3-p^2+2p-1}{24}$	$\frac{p^2-1}{24}$	$\frac{2p^3-p^2-2p+1}{24}$
$C'_{3,1}$	$\frac{2p^3-p^2-2p+1}{24}$	$\frac{p^2-1}{24}$	$-\frac{3p^2+2p-1}{24}$	$\frac{p^2-1}{24}$
$C'_{1,3}$	$\frac{p^2-1}{24}$	$\frac{2p^3-p^2-2p+1}{24}$	$\frac{p^2-1}{24}$	$-\frac{3p^2+2p-1}{24}$.

Intersection matrix for $\mathcal{X}_0(p^r)$

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	$C'_{4,0}$	$C'_{0,4}$	$C'_{3,1}$	$C'_{1,3}$
$C'_{4,0}$	$-\frac{2p^4-2p^3-p^2+2p-1}{24}$	$\frac{p^2-1}{24}$	$\frac{2p^3-p^2-2p+1}{24}$	$\frac{p^2-1}{24}$
$C'_{0,4}$	$\frac{p^2-1}{24}$	$-\frac{2p^4-2p^3-p^2+2p-1}{24}$	$\frac{p^2-1}{24}$	$\frac{2p^3-p^2-2p+1}{24}$
$C'_{3,1}$	$\frac{2p^3-p^2-2p+1}{24}$	$\frac{p^2-1}{24}$	$-\frac{3p^2+2p-1}{24}$	$\frac{p^2-1}{24}$
$C'_{1,3}$	$\frac{p^2-1}{24}$	$\frac{2p^3-p^2-2p+1}{24}$	$\frac{p^2-1}{24}$	$-\frac{3p^2+2p-1}{24}$.

- From this intersection matrix we explicitly compute V_m which are linear combinations of all the irreducible components of the special fiber of the minimal regular model.

References

1. P. Majumder and A.-M. von Pippich, Bounds for canonical Green's functions at cusps, (arxiv.org/abs/2210.04452).
2. D. Banerjee, C. Chaudhuri and P. Majumder, The intersection matrices of $X_0(p^r)$ and some applications, (arxiv.org/abs/2210.08866).

References

1. P. Majumder and A.-M. von Pippich, Bounds for canonical Green's functions at cusps, (arxiv.org/abs/2210.04452).
2. D. Banerjee, C. Chaudhuri and P. Majumder, The intersection matrices of $X_0(p^r)$ and some applications, (arxiv.org/abs/2210.08866).

Thank you for your attention!