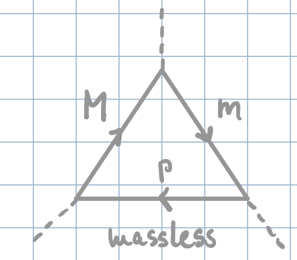
Problem Set 1 (10 April 2024)

Exercise 1:

Perform the region analysis for the integral:

$$I = \mu^{2\epsilon} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 (p^2 - M^2)(p^2 - m^2)} \quad ; \quad M^2 \gg m^2$$



(zero ext. momenta)

Solution:

We first derive the exact expression for the integral, using that:

$$\frac{1}{p^2 (p^2 - M^2)(p^2 - m^2)} = \frac{1}{M^2 m^2 p^2} + \frac{1}{M^2 - m^2} \left[\frac{1}{M^2} \frac{1}{p^2 - M^2} - \frac{1}{m^2} \frac{1}{p^2 - m^2} \right]$$

We now employ a standard formula for one-loop integrals.

One-loop master integral:

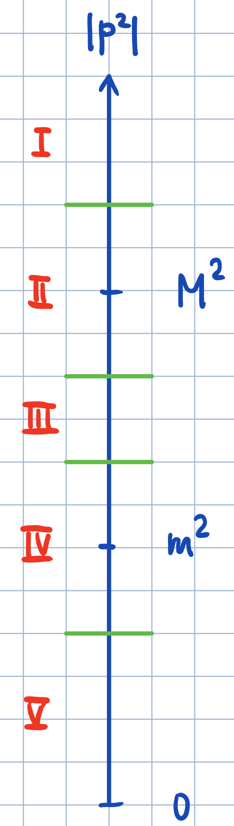
$$\int \frac{d^D \ell}{(2\pi)^D} \frac{1}{(\ell^2 - \Delta)^n} = (-1)^n \frac{i}{(4\pi)^{D/2}} \frac{\Gamma(n - D/2)}{\Gamma(n)} \Delta^{D/2 - n}$$

This leads to the exact answer:

$$I = \frac{i}{(4\pi)^{2-\epsilon}} \Gamma(\epsilon) \frac{\mu^{2\epsilon}}{1-\epsilon} \frac{(M^2)^{-\epsilon} - (m^2)^{-\epsilon}}{M^2 - m^2} = - \frac{i}{16\pi^2} \frac{\ln M^2/m^2}{M^2 - m^2} + \mathcal{O}(\epsilon)$$

$$= - \frac{i}{16\pi^2} \frac{1}{M^2} \left[1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \dots \right] \ln \frac{M^2}{m^2} + \mathcal{O}(\epsilon)$$

We now performs the region analysis in dimensional regularization. We need to consider 5 regions in total:



I. $|p^2| \gg M^2 \gg m^2$

$$I_I = \mu^{2\epsilon} \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2)^3} \left[1 + \frac{M^2}{p^2} + \dots \right] \left[1 + \frac{m^2}{p^2} + \dots \right]$$

$$= 0 \qquad \text{(scaleless integrals)}$$

II. $|p^2| \sim M^2 \gg m^2$

$$I_{II} = \mu^{2\epsilon} \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2)^2 (p^2 - M^2)} \left[1 + \frac{m^2}{p^2} + \frac{m^4}{(p^2)^2} + \dots \right]$$

The integrals can be evaluated using a master formula.

Useful integral:

$$\int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 - M^2} \frac{1}{(p^2)^a} = - \frac{i}{(4\pi)^{D/2}} \Gamma\left(1 - \frac{D}{2}\right) (M^2)^{\frac{D}{2} - 1 - a}$$

This leads to:

$$I_{II} = \frac{i}{(4\pi)^{2-\epsilon}} \frac{\Gamma(\epsilon)}{1-\epsilon} \left(\frac{\mu^2}{M^2} \right)^{\epsilon} \frac{1}{M^2} \left[1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \dots \right]$$

We can not only evaluate the leading term, but also power corrections in $m^2/M^2 \ll 1$.

III. $M^2 \gg |p^2| \gg m^2$

$$I_{III} = \mu^{2\epsilon} \int \frac{d^Dp}{(2\pi)^D} \frac{1}{(p^2)^2} \frac{1}{(-M^2)} \left[1 + \frac{p^2}{M^2} + \dots\right]\left[1 + \frac{m^2}{p^2} + \dots\right]$$

$$= 0 \quad \text{(scaleless integrals)}$$

IV. $|p^2| \sim m^2 \ll M^2$

$$I_{IV} = \mu^{2\epsilon} \int \frac{d^Dp}{(2\pi)^D} \frac{1}{p^2(p^2-m^2)} \frac{1}{(-M^2)} \left[1 + \frac{p^2}{M^2} + \frac{(p^2)^2}{M^4} + \dots\right]$$

$$= -\frac{i}{(4\pi)^{2-\epsilon}} \frac{\Gamma(\epsilon)}{1-\epsilon} \left(\frac{\mu^2}{m^2}\right)^{\epsilon} \frac{1}{M^2} \left[1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \dots\right]$$

V. $|p^2| \ll m^2 \ll M^2$

$$I_{V} = \mu^{2\epsilon} \int \frac{d^Dp}{(2\pi)^D} \frac{1}{p^2} \frac{1}{(-m^2)(-M^2)} \left[1 + \frac{p^2}{M^2} + \dots\right]\left[1 + \frac{p^2}{m^2} + \dots\right]$$

$$= 0 \quad \text{(scaleless integrals)}$$

Notice that non-zero contributions arise only if the loop momentum is assumed to be commensurate with one of the external scales, $|p^2| \sim M^2$ or $|p^2| \sim m^2$. Note also that in all cases we integrate over <u>all</u> loop momenta after the expansions of the integrand have been performed. The sum $I_{II} + I_{IV}$ precisely reproduces the exact result on page A.1, including the power-suppressed corrections.

It is instructive to take a closer look at these two integrals. Notice that

$$I_{II} = \mu^{2\epsilon} \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2)^2 (p^2 - M^2)} \left[1 + \frac{m^2}{p^2} + \frac{m^4}{(p^2)^2} + \dots \right]$$

is UV-finite but receives IR-singular contributions from the region where $p^2 \to 0$. Expanding the result in ϵ, we obtain:

$$I_{II} = \frac{i}{(4\pi)^{2-\epsilon}} \frac{\Gamma(\epsilon)}{1-\epsilon} \left(\frac{\mu^2}{M^2} \right)^{\epsilon} \frac{1}{M^2} \left[1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \dots \right]$$

$$= - \frac{i}{16\pi^2 M^2} \left[- \left(\frac{1}{\epsilon_{IR}} - \gamma_E + \ln 4\pi \right) + \ln \frac{M^2}{\mu^2} - 1 + \mathcal{O}(\epsilon) \right] + \mathcal{O}\left(\frac{m^2}{M^4} \right)$$

$$\equiv I_{\text{hard}}$$

Similarly, the integral

$$I_{III} = \mu^{2\epsilon} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 (p^2 - m^2)} \frac{1}{(-M^2)} \left[1 + \frac{p^2}{M^2} + \frac{(p^2)^2}{M^4} + \dots \right]$$

is IR-finite but UV-divergent for $|p^2| \to \infty$. Expanding in ϵ we find:

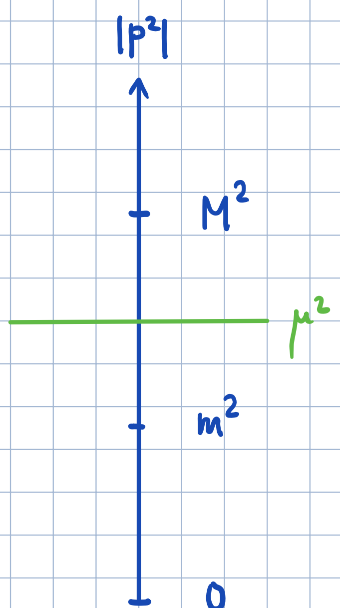
$$I_{IV} = -\frac{i}{16\pi^2 M^2}\left[+\left(\frac{1}{\epsilon_{UV}} - \gamma_E + \ln 4\pi\right) - \ln\frac{m^2}{\mu^2} + 1 + \mathcal{O}(\epsilon)\right] + \mathcal{O}\left(\frac{m^2}{M^4}\right)$$

$$\equiv I_{soft}$$

The divergences arise because after the Taylor expansion I_{II} no longer contains the IR regulator m^2, and I_{IV} no longer contains the UV regulator M^2. In the sum

$$I_{hard} + I_{soft} = -\frac{i}{16\pi^2 M^2}\left[\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} + \ln\frac{M^2}{m^2} + \mathcal{O}(\epsilon)\right] + \mathcal{O}\left(\frac{m^2}{M^4}\right)$$

the divergent terms cancel out. In EFT jargon, the hard contribution is absorbed into a Wilson coefficient, while the soft contribution corresponds to the matrix element of an EFT operator:



hard contr. $\sim \ln\frac{M^2}{\mu^2}$ $\quad\hookrightarrow$ Wilson coefficient $C(M,\mu)$

soft contr. $\sim \ln\frac{\mu^2}{m^2}$ $\quad\hookrightarrow$ EFT matrix element $\langle O\rangle(m,\mu)$

By construction, the dependence on the factorization scale μ cancels between the two contributions!

Exercise 2 :

Evaluate the collinear contribution

$$I_c = i\,\pi^{-D/2}\,\mu^{2\epsilon}\int d^Dk\,\frac{1}{(k^2+i0)\left[(k+p_1)^2+i0\right]\left[2k\cdot p_{2+}+i0\right]}$$

to the Sudakov form factor and give an EFT interpretation.

Solution:

We combine the first two factors in the denominator using a standard Feynman parameter:

$$\frac{1}{(k^2+i0)\left[(k+p_1)^2+i0\right]} = \int_0^1 dx\,\frac{1}{(k^2+2x\,k\cdot p_1+x\,p_1^2+i0)^2}$$

For propagators that are linear in the loop momentum, it is convenient to use Feynman paraters $\lambda \in [0,\infty[$ and the following master formula.

Feynman parameters for linear propagators:

$$\frac{1}{A^a B^b} = \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)}\int_0^\infty d\lambda\,\frac{\lambda^{b-1}}{(A+\lambda B)^{a+b}}$$

linear propagators

This gives:

$$\frac{1}{(k^2+i0)\left[(k+p_1)^2+i0\right]\left[2k\cdot p_{2+}+i0\right]}$$

$$= 2\int_0^1 dx \int_0^\infty d\lambda \, \frac{1}{\underbrace{(k^2+2x\,k\cdot p_1+x p_1^2+2\lambda\, k\cdot p_{2+}+i0)}_{(k+xp_1+\lambda p_{2+})^2-\Delta}{}^3}$$

$$(k+xp_1+\lambda p_{2+})^2-\Delta \; ; \quad \Delta = (xp_1+\lambda p_{2+})^2 - x p_1^2 - i0$$

$$= 2x\lambda\, p_1\cdot p_{2+} - x(1-x)\,p_1^2 - i0$$

We now use that:

$$2\,p_1\cdot p_{2+} \approx -q^2 = Q^2 \qquad \text{up to } \mathcal{O}(\lambda^2) \text{ corrections}$$

$$-p_1^2 - i0 = P_1^2$$

$$\hookrightarrow \quad \Delta = \lambda x Q^2 + x(1-x)P_1^2$$

Changing variables from k to $\ell = k+xp_1+\lambda p_{2+}$ yields:

$$I_c = i\,\pi^{-D/2}\mu^{2\epsilon}\cdot(2\pi)^D\cdot 2\int_0^1 dx\int_0^\infty d\lambda\, \underbrace{\int\frac{d^D\ell}{(2\pi)^D}\,\frac{1}{(\ell^2-\Delta)^3}}_{\frac{-i}{(4\pi)^{D/2}}\,\frac{1}{2}\,\Gamma(1+\epsilon)\,\Delta^{-1-\epsilon}}$$

$$= \Gamma(1+\epsilon)\,\mu^{2\epsilon}\int_0^1 dx\,\underbrace{\int_0^\infty d\lambda\,\left(\lambda x Q^2+x(1-x)P_1^2\right)^{-1-\epsilon}}_{-\frac{1}{\epsilon}\,\frac{1}{xQ^2}\left(\lambda x Q^2+x(1-x)P_1^2\right)^{-\epsilon}\Big|_{\lambda=0}^{\infty}}$$

$$= \frac{\Gamma(1+\epsilon)}{Q^2}\,\frac{1}{\epsilon}\left(\frac{\mu^2}{P_1^2}\right)^{\epsilon}\int_0^1 dx\, x^{-1-\epsilon}(1-x)^{-\epsilon}$$

Using the identity

$$\int_0^1 dx\, x^a (1-x)^b = \frac{\Gamma(a+1)\,\Gamma(b+1)}{\Gamma(a+b+2)}$$

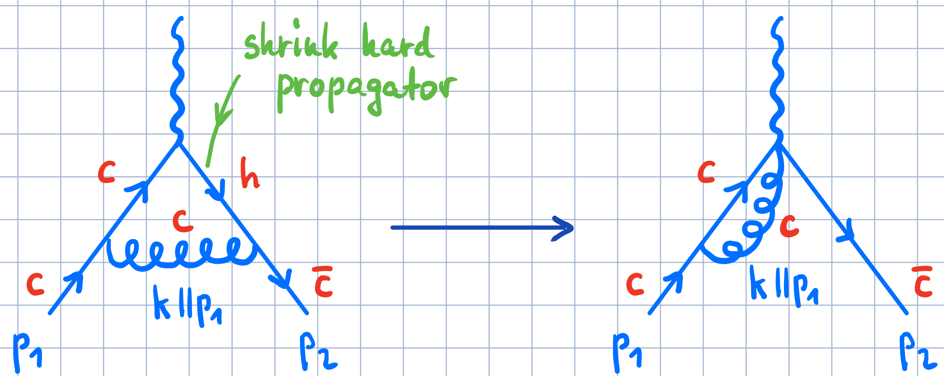
we finally obtain:

$$I_c = \frac{\Gamma(1+\epsilon)}{Q^2}\;\frac{\Gamma(-\epsilon)\,\Gamma(1-\epsilon)}{\epsilon\,\Gamma(1-2\epsilon)}\left(\frac{\mu^2}{p_1^2}\right)^{\epsilon}$$

$$= \frac{\Gamma(1+\epsilon)}{Q^2}\left[-\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\ln\frac{\mu^2}{p_1^2}-\frac{1}{2}\ln^2\frac{\mu^2}{p_1^2}+\frac{\pi^2}{6}+\mathcal{O}(\epsilon)\right]$$

This agrees with the expression given in the lectures.

It is interesting to give a "diagrammatic EFT interpretation" of this integral obtained by shrinking the hard propagator to a point:



The SCET Feynman rules must reproduce the collinear loop graph shown on the right. To this end, SCET must contain an unusual $q\bar{q}g\gamma$ - vertex not present in the full theory.

Exercise 3 :

Evaluate the integral

$$I = \int_1^\infty dx \, \frac{x^{-\epsilon}}{x + \mu x^3} \quad ; \qquad 0 < \mu \ll 1 \qquad (\epsilon > -2)$$

using the method of regions, dropping terms of $O(\mu^2)$.

Solution:

The integral depends on two "scales": the lower integration limit $x_{min} = 1$ and the parameter μ. We can rewrite:

$$I = \int_0^\infty dx \, \frac{x^{-\epsilon}}{x + \mu x^3} \; \Theta(x-1)$$

The exact expression for the integral is rather complicated. One finds:

hypergeometric function

$$I = \frac{1}{\epsilon}\left[1 - {}_2F_1\left(1, \frac{\epsilon}{2}; 1+\frac{\epsilon}{2}; -\frac{1}{\mu}\right)\right]$$

HypExp package

$$= \frac{1}{\epsilon}\left[1 - \Gamma\left(1+\frac{\epsilon}{2}\right)\Gamma\left(1-\frac{\epsilon}{2}\right)\mu^{\epsilon/2}\right] + \frac{\mu}{2-\epsilon} + O(\mu^2)$$

Let us reproduce this result, including the first-order correction in μ, using the method of regions. The two relevant regions are $x \sim 1$ and $x \sim \mu^{-1/2}$.

Region 1: $x \sim 1$

$$I_1 = \int_0^\infty dx \, \frac{x^{-\epsilon}}{x} \left[1 - \mu x^2 + \mathcal{O}(\mu^2) \right] \Theta(x-1)$$

$$= \int_1^\infty dx \, x^{-1-\epsilon} - \mu \int_1^\infty dx \, x^{1-\epsilon} + \mathcal{O}(\mu^2)$$

$$= -\frac{1}{\epsilon} x^{-\epsilon} \Big|_{x=1}^{\infty} - \frac{\mu}{2-\epsilon} x^{2-\epsilon} \Big|_{x=1}^{\infty} + \mathcal{O}(\mu^2)$$

$$= \frac{1}{\epsilon} + \frac{\mu}{2-\epsilon} + \mathcal{O}(\mu^2)$$

Note that the first integral converges for $\epsilon > 0$, while for the second one we need to assume that $\epsilon > 2$. But this is o.k., since at the final result is defined by analytic continuation in ϵ to the region $\epsilon > -2$.

Region 2: $x \sim \mu^{-1/2}$

This region is chosen such that the two terms in the denominator are of the same order. Setting $x = \frac{y}{\sqrt{\mu}}$, we obtain:

must expand, since $y = O(1)$ in this region

$$I_2 = \int_0^\infty dy \, \frac{y^{-\epsilon}}{y(1+y^2)} \mu^{\epsilon/2} \, \underset{\Theta(y^2-\mu)}{\underset{\shortparallel}{\Theta(y-\sqrt{\mu})}}$$

$$= \mu^{\epsilon/2} \int_0^\infty dy \, \frac{y^{-\epsilon}}{y(1+y^2)} \left[\Theta(y^2) - \mu \, \delta(y^2) + \mathcal{O}(\mu^2) \right]$$

Changing variables from y to $z = y^2$, we find:

$$I_2 = \mu^{\epsilon/2} \frac{1}{2} \int_0^\infty dz \, \frac{z^{-\epsilon/2}}{z(1+z)} \left[1 - \mu \, \delta(z) + \mathcal{O}(\mu^2) \right]$$

The contribution from the Dirac δ-function converges if we assume that $\epsilon < -2$, and if that condition is satisfied it evaluates to zero. The integral for the first term converges for $-2 < \epsilon < 0$. Setting $w = \frac{1}{1+z}$, we find:

$$\int_0^\infty dz \, \frac{z^{-\epsilon/2}}{z(1+z)} = \int_0^1 dw \, w^{\epsilon/2} (1-w)^{-1-\epsilon/2} = \Gamma(1+\tfrac{\epsilon}{2}) \, \Gamma(-\tfrac{\epsilon}{2})$$

This gives:

$$\begin{aligned} I_2 &= \frac{\mu^{\epsilon/2}}{2} \Gamma(1+\tfrac{\epsilon}{2}) \, \Gamma(-\tfrac{\epsilon}{2}) + \mathcal{O}(\mu^2) \\ &= -\frac{1}{\epsilon} \frac{\mu^{\epsilon/2}}{2} \Gamma(1+\tfrac{\epsilon}{2}) \, \Gamma(1-\tfrac{\epsilon}{2}) + \mathcal{O}(\mu^2) \end{aligned}$$

The sum $I_1 + I_2$ correctly reproduces the exact result for I given on page A.9, including the power correction of $\mathcal{O}(\mu)$. Any other region of x values leads to scaleless integrals.