Problem Set 2 (12 April 2024)

Exercise 1:

Construct the leading-order ($\sim \lambda^4$) SCET Lagrangian in the presence of a quark mass with scaling a) $m \sim \lambda$ and b) $m \sim \lambda^2$.

Solution:

The projection operators P_n and $P_{\bar{n}}$ included in the definitions of ξ_n and η_n imply that $\bar{\xi}_n \xi_n = 0$ and $\bar{\eta}_n \eta_n = 0$. Hence, the mass term gives:

$$-m\,\bar{\psi}_c \psi_c = -m\left(\bar{\xi}_n \eta_n + \bar{\eta}_n \xi_n\right)$$

From page 42 we see that $(-m)$ always comes together with $i\not{D}_c^\perp \sim \lambda$. It follows that for $m \sim \lambda$:

$$\begin{aligned}
\mathcal{L}_c(x) = {} & \bar{\xi}_n \frac{\not{\bar{n}}}{2}\, i n\cdot D_c\, \xi_n(x) \\
& + \left[\bar{\xi}_n \left(i\not{D}_c^\perp - m\right) W_c\right](x)\, \frac{\not{\bar{n}}}{2}\, i \int_{-\infty}^{0} dt \left[W_c^\dagger \left(i\not{D}_c^\perp - m\right) \xi_n\right](x+t\bar{n}) \\
& + (\text{pure glue terms})
\end{aligned}$$

For $m \sim \lambda^2$, on the other hand, the mass term acts as a power correction and does not contribute at leading order.

Exercise 2 :

Derive the form of the SCET vector current at position $x \neq 0$ and show that it is gauge invariant.

Solution:

We found that:

$$(W_c^\dagger \xi_n)(x) \xrightarrow{U_c} (W_c^\dagger \xi_n)(x)$$

$$\xrightarrow{U_{us}} U_{us}(x_-)\,(W_c^\dagger \xi_n)(x)$$

In an analogous way:

$$(\bar{\xi}_{\bar{n}} W_{\bar{c}})(x) \xrightarrow{U_{\bar{c}}} (\bar{\xi}_{\bar{n}} W_{\bar{c}})(x)$$

$$\xrightarrow{U_{us}} (\bar{\xi}_{\bar{n}} W_{\bar{c}})(x)\, U_{us}^\dagger(x_+)$$

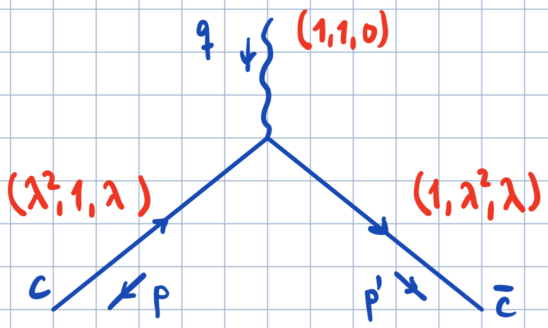
So the current

$$(\bar{\xi}_{\bar{n}} W_{\bar{c}})(x)\ \gamma_\perp^\mu\ (W_c^\dagger \xi_n)(x)$$

is not invariant under ultra-soft gauge transformations, since:

$$U_{us}^\dagger(x_+)\, U_{us}(x_-) \neq 1 \quad \text{for } x_\pm \neq 0$$

However, when collinear and anti-collinear fields are combined in a <u>hard</u> interaction, the (anti-) collinear fields themselves must be <u>multipole expanded</u>! To see this, consider the following vertex:



$n\cdot q = \bar{n}\cdot q = q^0 = \sqrt{s}$ (CMS: $\vec{q} = \vec{0}$)

$$p'^\mu = \underset{\lambda^0}{(n\cdot q} - \underset{\lambda^2}{n\cdot p)}\,\frac{\bar{n}^\mu}{2} + \underset{\lambda^0}{(\bar{n}\cdot q} - \underset{\lambda^0}{\bar{n}\cdot p)}\,\frac{n^\mu}{2} - \underset{\lambda}{p'^\mu_\perp}$$

must be expanded!

To expand away $p_+^\mu = n\cdot p\,\frac{\bar{n}^\mu}{2}$, we must expand the collinear fields in x_-^μ:

$$(W_c^\dagger \xi_n)(x) = (W_c^\dagger \xi_n)(x_+ + x_\perp) + x_-\cdot\partial_+ (W_c^\dagger \xi_n)(x_+ + x_\perp) + \ldots$$

power suppressed

Likewise, we must expand the anti-collinear fields in x_+. The correct leading-order SCET current operator at $x \neq 0$ is therefore:

$$\boxed{(\bar{\xi}_{\bar{n}} W_{\bar{c}})(x_- + x_\perp)\,\gamma_\perp^\mu\,(W_c^\dagger \xi_n)(x_+ + x_\perp)}$$

This is invariant under ultra-soft gauge transformations:

$$\ldots \xrightarrow{U_{us}} (\bar{\xi}_{\bar{n}} W_{\bar{c}})(x_- + x_\perp)\,\gamma_\perp^\mu\,\underbrace{U_{us}^\dagger(x_+ = 0)\,U_{us}(x_- = 0)}_{=1}\,(W_c^\dagger \xi_n)(x_+ + x_\perp)$$

Exercise 3 :

Work out the analytic form of the solution of the RG evolution equation for the Wilson coefficient $C_V(\mu)$ at leading order in RG-improved perturbation theory.

Solution:

The general solution to the RGE is:

initial condition

$$C_V(Q^2,\mu) = C_V(Q^2,\mu_h) \times \underbrace{\exp\left[\int_{\mu_h}^{\mu} \frac{d\mu'}{\mu'}\left(\Gamma_{cusp}(\alpha_s(\mu'))\ln\frac{Q^2}{\mu'^2} + \gamma_V(\alpha_s(\mu'))\right)\right]}_{U_V(\mu_h,\mu)}$$

We use the definition of the β-function

$$\beta(\alpha_s) = \mu\frac{d\alpha_s(\mu)}{d\mu}$$

to change variables from μ' to $\alpha_s(\mu')$. We find

$$\ln\frac{Q^2}{\mu'^2} = \ln\frac{Q^2}{\mu_h^2} - 2\int_{\mu_h}^{\mu'}\frac{d\tilde{\mu}}{\tilde{\mu}} = \underbrace{\ln\frac{Q^2}{\mu_h^2}}_{\mathcal{O}(1)} - 2\int_{\alpha_s(\mu_h)}^{\alpha_s(\mu')}\frac{d\alpha}{\beta(\alpha)}$$

and therefore:

$$\ln U_V(\mu_h,\mu) = \int_{\alpha_s(\mu_h)}^{\alpha_s(\mu)}\frac{d\alpha}{\beta(\alpha)}\left[\Gamma_{cusp}(\alpha)\left(\ln\frac{Q^2}{\mu_h^2} - 2\int_{\alpha_s(\mu_h)}^{\alpha}\frac{d\alpha'}{\beta(\alpha')}\right) + \gamma_V(\alpha)\right]$$

We now define the functions:

$$S_\Gamma(\nu,\mu) = -\int_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha \, \frac{\Gamma_{cusp}(\alpha)}{\beta(\alpha)} \int_{\alpha_s(\nu)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')}$$

$$a_\Gamma(\nu,\mu) = -\int_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha \, \frac{\Gamma_{cusp}(\alpha)}{\beta(\alpha)}$$

$$a_{\gamma_V}(\nu,\mu) = -\int_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha \, \frac{\gamma_V(\alpha)}{\beta(\alpha)}$$

In terms of these objects, the exact solution reads:

$$\boxed{U_V(\mu_h,\mu) = \exp\left[2\, S_\Gamma(\mu_h,\mu) - \underbrace{\ln\frac{Q^2}{\mu_h^2}}_{O(1),\ \text{since } \mu_h \approx Q} \cdot a_\Gamma(\mu_h,\mu) - a_{\gamma_V}(\mu_h,\mu) \right]}$$

(see e.g. Section 3.1 in hep-ph/0607228)

We now work out a perturbative approximation to this result, using the expansions:

$$\beta(\alpha_s) = -2\alpha_s \left[\beta_0 \frac{\alpha_s}{4\pi} + \beta_1 \left(\frac{\alpha_s}{4\pi}\right)^2 + \dots \right]$$

$$\Gamma_{cusp}(\alpha_s) = \Gamma_0 \frac{\alpha_s}{4\pi} + \Gamma_1 \left(\frac{\alpha_s}{4\pi}\right)^2 + \dots$$

$$\gamma_V(\alpha_s) = \gamma_0 \frac{\alpha_s}{4\pi} + \gamma_1 \left(\frac{\alpha_s}{4\pi}\right)^2 + \dots$$

The one-loop coefficients are:

$$\beta_0 = \frac{11}{3} \underbrace{C_A}_{=3} - \frac{4}{3} \underbrace{T_F}_{=1/2} n_f \,, \qquad \Gamma_0 = 4 \underbrace{C_F}_{=4/3} \,, \qquad \gamma_0 = -6 C_F$$

n_f: # light quark flavors

Using these expansion to evaluate the integrals, one obtains:

$$a_\Gamma(\nu,\mu) = \frac{\Gamma_0}{2\beta_0} \left[\ln \frac{\alpha_s(\mu)}{\alpha_s(\nu)} + \left(\frac{\Gamma_1}{\Gamma_0} - \frac{\beta_1}{\beta_0} \right) \frac{\alpha_s(\mu) - \alpha_s(\nu)}{4\pi} + \dots \right]$$

$$a_{\gamma_V}(\nu,\mu) = \frac{\gamma_0}{2\beta_0} \left[\ln \frac{\alpha_s(\mu)}{\alpha_s(\nu)} + \left(\frac{\gamma_1}{\gamma_0} - \frac{\beta_1}{\beta_0} \right) \frac{\alpha_s(\mu) - \alpha_s(\nu)}{4\pi} + \dots \right]$$

$O(1)$ "LO" $\quad$ $O(\alpha_s)$ "NLO" $\quad$ $O(\alpha_s^2)$ "NNLO"

in "RG-improved perturbation theory"

The solution for the "Sudakov exponent" S_Γ is more involved. One finds:

$$S_\Gamma(\nu,\mu) = \frac{\Gamma_0}{4\beta_0^2} \left\{ \frac{4\pi}{\alpha_s(\nu)} \left(1 - \frac{1}{r} - \ln r \right) \right.$$

larger than $O(1)$

$$+ \left(\frac{\Gamma_1}{\Gamma_0} - \frac{\beta_1}{\beta_0} \right) (1 - r + \ln r) + \frac{\beta_1}{2\beta_0} \ln^2 r$$

"LO"

$$\left. + O\left(\frac{\alpha_s(\mu) - \alpha_s(\nu)}{4\pi} \right) \right\}$$

"NLO"

with:

$$r = \frac{\alpha_s(\mu)}{\alpha_s(\nu)}$$

The presence of a "super-leading" term $\sim 1/\alpha_s$ is characteristic of Sudakov problems.

At leading order in RG-improved perturbation theory, we finally obtain:

$$U_V(\mu_h, \mu) = e^{2 S_\Gamma^{LO}(\mu_h, \mu)} \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_h)} \right)^{-\frac{\gamma_0}{2\beta_0} - \frac{\Gamma_0}{2\beta_0} \ln \frac{Q^2}{\mu_h^2}}$$

$$\times \left\{ 1 + \mathcal{O}\left(\frac{\alpha_s(\mu) - \alpha_s(v)}{4\pi} \right) \right\}$$