Lattice studies of three-dimensional super-Yang–Mills

David Schaich (University of Liverpool)

Nonperturbative and Numerical Approaches
to Quantum Gravity, String Theory and Holography

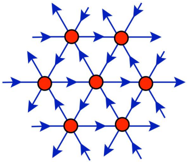
International Centre for Theoretical Sciences, Bangalore, 29 August 2022

arXiv:2010.00026 arXiv:2201.08626 arXiv:2208.03580

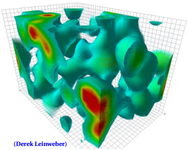
and more to come with R. G. Jha, A. Joseph, A. Sherletov & T. Wiseman

Overview and plan

Three dimensions is a promising frontier
for practical lattice studies of supersymmetric QFTs



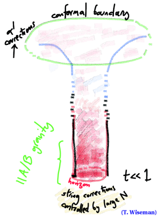
Twisted lattice super-Yang–Mills (SYM) brief review



Recent work: $Q = 16$ SYM and dual D2-branes

Ongoing work: $Q = 16$ SYM phase diagram

$Q = 8$ SYM & 2d quiver super-QCD

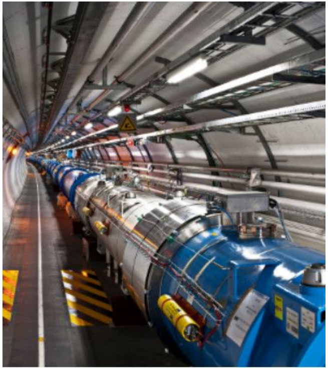


Interaction encouraged — complete coverage unnecessary

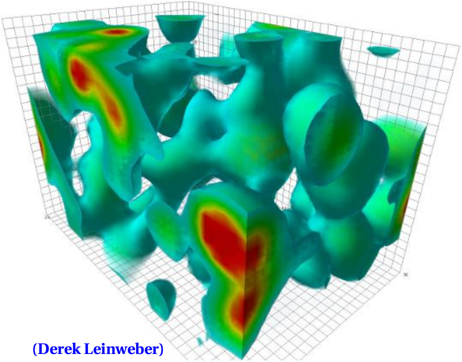
Motivations

Lattice field theory promises first-principles predictions
for strongly coupled supersymmetric QFTs

BSM

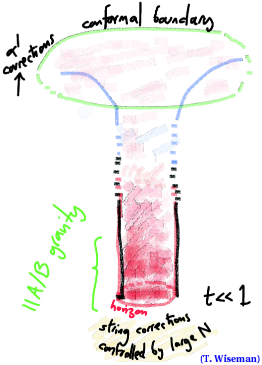


QFT



(Derek Leinweber)

Holography



(T. Wiseman)

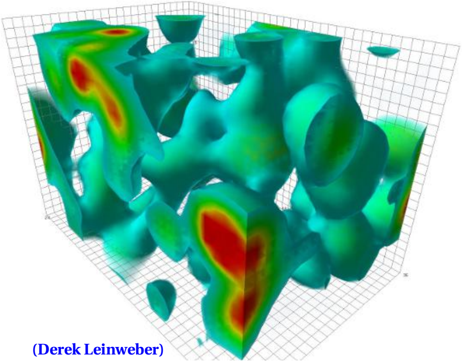
Motivations

Lattice field theory promises first-principles predictions
for strongly coupled supersymmetric QFTs

Three dimensions

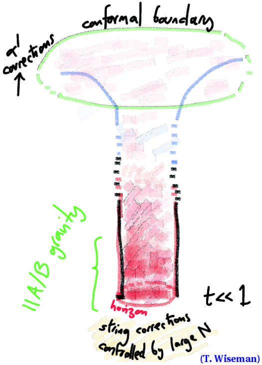
Rich field theory
and holographic
dynamics

QFT



(Derek Leinweber)

Holography



(T. Wiseman)

More manageable computational costs

Results to be shown, and work in progress

use importance sampling to evaluate up to $\sim$crore-dimensional integrals

(Dirac operator as $\sim 10^7 \times 10^7$ matrix)

Significant computational resources required

Many thanks to USQCD–DOE, DiRAC–STFC–UKRI, and computing centres!

USQCD @Fermilab

DiRAC @Cambridge

Barkla @Liverpool

Supersymmetry must be broken on the lattice

Supersymmetry is a space-time symmetry, $(\mathrm{I} = 1, \cdots, \mathcal{N})$

adding spinor generators Q_α^I and $\overline{Q}_{\dot\alpha}^\mathrm{I}$ to translations, rotations, boosts

$\left\{Q_\alpha^\mathrm{I}, \overline{Q}_{\dot\alpha}^\mathrm{J}\right\} = 2\delta^{\mathrm{IJ}}\sigma^\mu_{\alpha\dot\alpha}P_\mu$ broken in discrete space-time

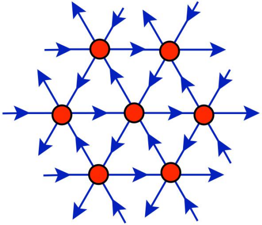
$\longrightarrow$ relevant susy-violating operators

Supersymmetry need not be *completely* broken on the lattice

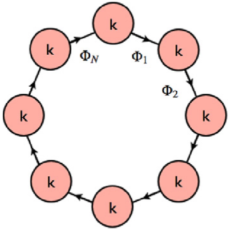
Preserve susy sub-algebra in discrete lattice space-time

$\Longrightarrow$ correct continuum limit with little or no fine tuning

Equivalent constructions from ‘topological’ twisting and dim’l deconstruction



Review:
Catterall–Kaplan–Ünsal,
arXiv:0903.4881



Need $Q = 2^d$ supersymmetries in d dimensions

$d = 3 \longrightarrow$ super-Yang–Mills (SYM) with $Q = 8$ or (maximal) $Q = 16$

3d maximal SYM in a nutshell

May be easiest to grok as dimensional reduction of 4d $\mathcal{N} = 4$ SYM

All fields massless and in adjoint rep. of SU(N) gauge group

4d: Gauge field A_μ plus 6 scalars Φ^{IJ}

$\mathcal{N} = 4$ four-component fermions $\Psi^{\mathrm{I}} \longleftrightarrow$ 16 supersymmetries Q^{I}_α and $\overline{Q}^{\mathrm{I}}_{\dot{\alpha}}$

Global SU(4) $\sim$ SO(6) R symmetry

3d: Gauge field A_μ plus 7 scalars Φ

$\mathcal{N} = 8$ two-component fermions $\Psi \longleftrightarrow$ 16 supersymmetries

Global Spin$(7, \mathbb{C}) \sim$ SO(8) $\supset$ SO(4) $\sim$ SU(2) $\times$ SU(2) R symmetry

Symmetries relate kinetic, Yukawa and Φ^4 terms $\longrightarrow$ single coupling $\lambda = g^2 N$

Twisting 3d maximal SYM

May be easiest to grok as dim'l reduction of 4d twisted $\mathcal{N} = 4$ SYM [2002.10517]

$$\{\mathcal{Q}, \mathcal{Q}_a, \mathcal{Q}_{ab}\} \longrightarrow \{\mathcal{Q}, \mathcal{Q}_0, \mathcal{Q}_\mu, \mathcal{Q}_{0\mu}, \mathcal{Q}_{\mu\nu}\}$$

$$\{\eta, \psi_a, \chi_{ab}\} \longrightarrow \{\eta, \eta_0, \psi_\mu, \psi_{0\mu}, \chi_{\mu\nu}\}$$

$$\{\mathcal{U}_a, \overline{\mathcal{U}}_a\} \longrightarrow \{\phi, \overline{\phi}, \mathcal{U}_\mu, \overline{\mathcal{U}}_\mu\}$$

with $\mu, \nu = 1, \cdots, 4$

Twisted rotation group now

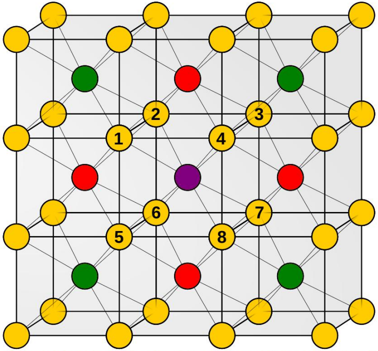
$$\text{SO}(3)_{\text{tw}} \equiv \text{diag}\left[\text{SO}(3)_{\text{euc}} \otimes \text{SO}(3)_R\right] \qquad \text{SO}(3)_R \subset \text{SO}(4)_R$$

Two closed supersymmetry sub-algebras in discrete space-time

$$\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0 \qquad \{\mathcal{Q}_0, \mathcal{Q}_0\} = 2\mathcal{Q}_0^2 = 0$$

Four links in three dimensions $\longrightarrow$ A_3^* lattice

A_3^* (body-centered cubic) lattice
from dimensional reduction of 4d A_4^* lattice



msestudent.com/body-centered-cubic-bcc-unit-cell

Basis vectors linearly dependent and non-orthogonal

Large S_4 point group symmetry
$\longrightarrow$ continuum limit without fine-tuning

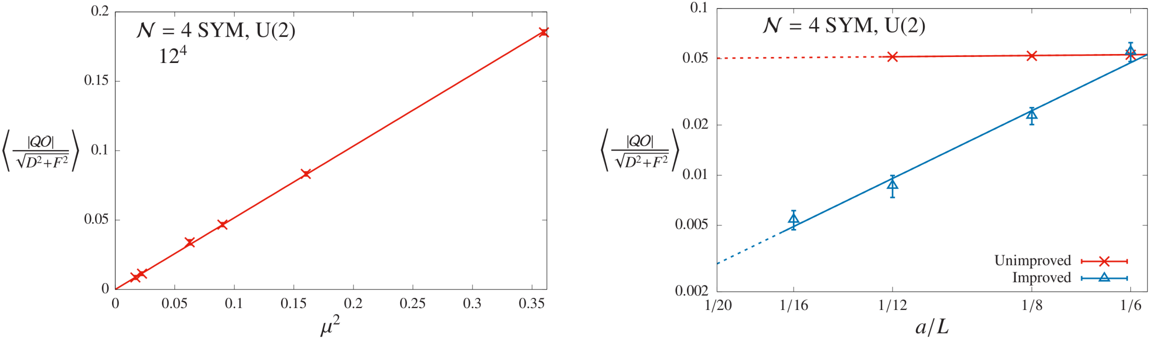
Numerical calculations require regulating zero modes and flat directions
and stabilizing dimensional reduction

Deformations to stabilize lattice calculations

Recall soft $\mathcal{Q}$-breaking SU(N) scalar potential $\propto \mu^2 \sum_a \text{Tr}\left[\left(\mathcal{U}_a\overline{\mathcal{U}}_a - \mathbb{I}_N\right)^2\right]$

and supersymmetric U(1) constraint $\sim G \sum_{a<b} \left(\det \mathcal{P}_{ab} - 1\right)$

Monitor $\mathcal{Q}$ restoration via Ward identity violations $\left\langle \text{Tr}\ \mathcal{Q}\left[\eta\,\mathcal{U}_a\overline{\mathcal{U}}_a\right]\right\rangle \neq 0$

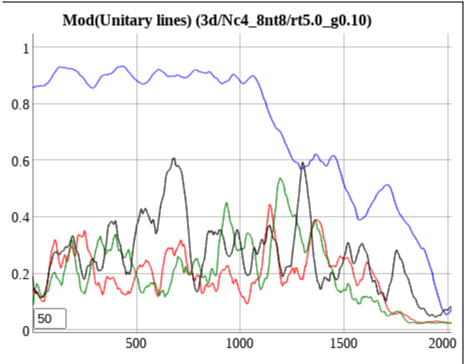


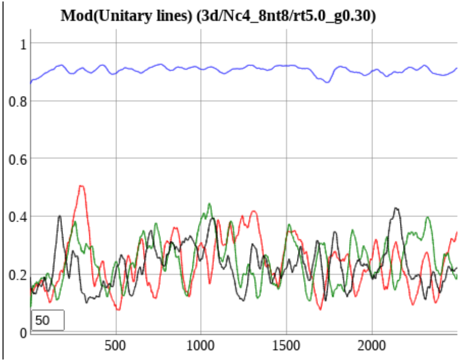
Deformations to stabilize lattice calculations

Enable naive dimensional reduction (4d code with $N_x = 1$)

Potential $\propto \mu^2 \text{Tr}\left[\left(\varphi - \mathbb{I}_N\right)^\dagger \left(\varphi - \mathbb{I}_N\right)\right]$ to break center symmetry in reduced dir

$\longrightarrow$ Kaluza–Klein reduction rather than Eguchi–Kawai





Public code for supersymmetric lattice field theories

so that the full improved action becomes

$$S_{\rm imp} = S'_{\rm exact} + S_{\rm closed} + S'_{\rm soft} \tag{18}$$

$$S'_{\rm exact} = \frac{N}{4\lambda_{\rm lat}} \sum_n \mathrm{Tr}\Bigg[-\overline{\mathcal{F}}_{ab}(n)\mathcal{F}_{ab}(n) - \chi_{ab}(n)\mathcal{D}^{(+)}_{[a}\psi_{b]}(n) - \eta(n)\overline{\mathcal{D}}^{(-)}_a\psi_a(n)$$
$$+ \frac{1}{2}\left(\overline{\mathcal{D}}^{(-)}_a\mathcal{U}_a(n) + G\sum_{a\neq b}\left(\det \mathcal{P}_{ab}(n) - 1\right)\mathbb{I}_N\right)^2\Bigg] - S_{\rm det}$$

$$S_{\rm det} = \frac{N}{4\lambda_{\rm lat}} G \sum_n \mathrm{Tr}\left[\eta(n)\right] \sum_{a\neq b}\left[\det \mathcal{P}_{ab}(n)\right] \mathrm{Tr}\left[\mathcal{U}_b^{-1}(n)\psi_b(n) + \mathcal{U}_a^{-1}(n+\widehat{\mu}_b)\psi_a(n+\widehat{\mu}_b)\right]$$

$$S_{\rm closed} = -\frac{N}{16\lambda_{\rm lat}} \sum_n \mathrm{Tr}\left[\epsilon_{abcde}\ \chi_{de}(n+\widehat{\mu}_a+\widehat{\mu}_b+\widehat{\mu}_c)\overline{\mathcal{D}}^{(-)}_c\chi_{ab}(n)\right],$$

$$S'_{\rm soft} = \frac{N}{4\lambda_{\rm lat}}\mu^2 \sum_n \sum_a \left(\frac{1}{N}\mathrm{Tr}\left[\mathcal{U}_a(n)\overline{\mathcal{U}}_a(n)\right] - 1\right)^2$$

$\gtrsim$100 inter-node data transfers in the fermion operator — non-trivial

Public parallel code github.com/daschaich/susy [arXiv:1410.6971]
actively developed for improved performance and new applications

3d maximal SYM thermodynamics

arXiv:2010.00026

Formulate on $r_1 \times r_2 \times r_\beta$ (skewed) 3-torus

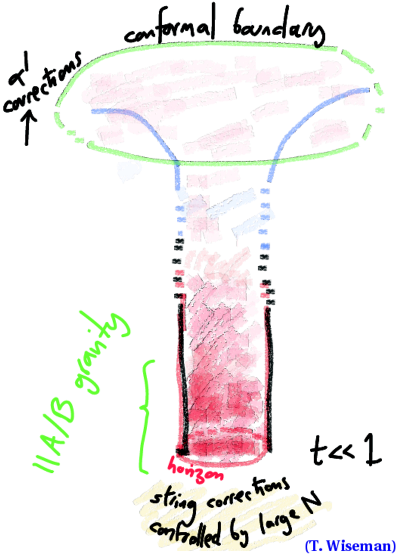
Thermal boundary conditions

$\longrightarrow$ dimensionless temperature $t = \dfrac{T}{\lambda} = \dfrac{1}{r_\beta}$

Low temperatures t at large N

$\updownarrow$

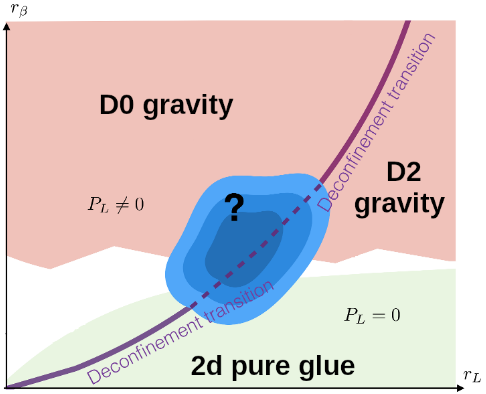
Black branes in dual supergravity

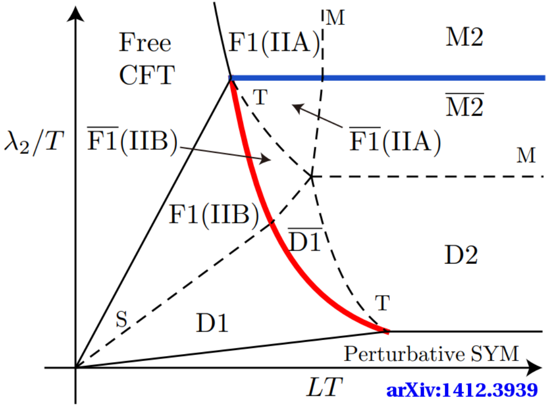


Holographic expectations for 3d maximal SYM

Rich holographic phase diagram, especially when $r_1 \neq r_2$

Left: $r_1 = r_2$ **Right:** $r_1 = \infty, \quad r_2 = L\lambda$

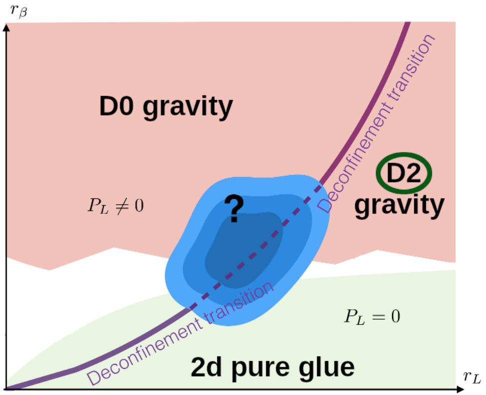


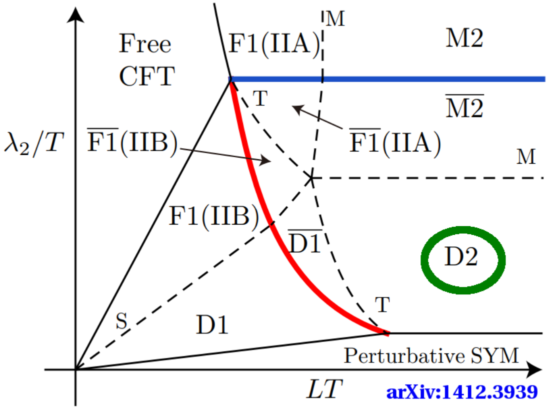


Holographic expectations for 3d maximal SYM

Rich holographic phase diagram, especially when $r_1 \neq r_2$

First consider simplest homogeneous black D2-branes $\longrightarrow$ $r_1 = r_2 = r_\beta$

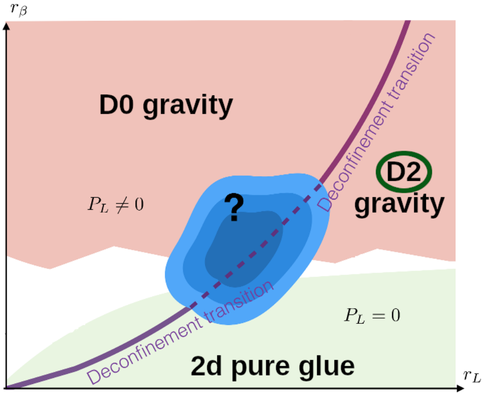


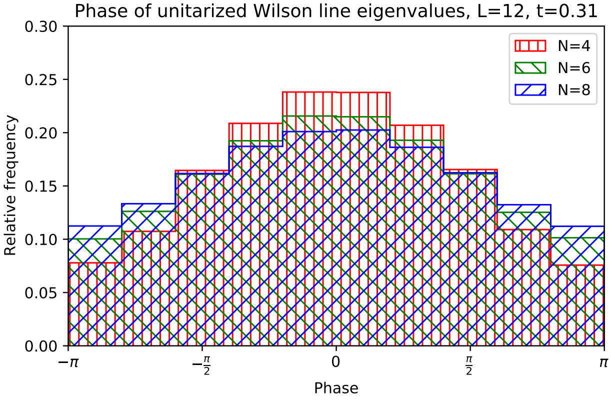


Homogeneous D2 phase

arXiv:2010.00026

Homogeneous D2-branes $\longleftrightarrow$ uniform Wilson line eigenvalue phases at large N



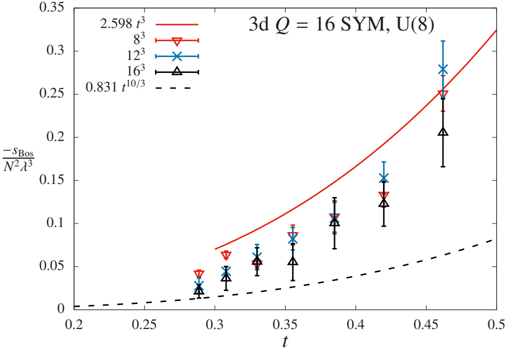


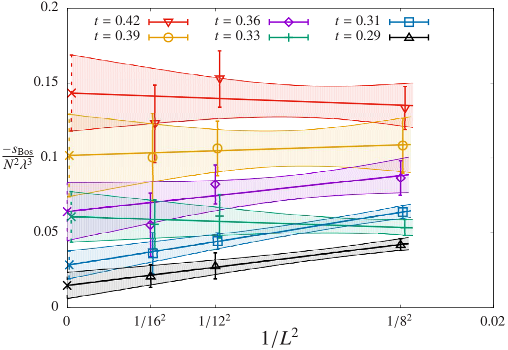
Holographic black brane energies and continuum extrapolation

Lattice volume L^3 with gauge group U(8)

$\longrightarrow$ results approach leading holographic expectation $\propto t^{10/3}$ for low $t \lesssim 0.4$

Carry out first 3d continuum extrapolations, $L \to \infty$ with fixed $t = 1/(L\lambda_{\text{lat}})$

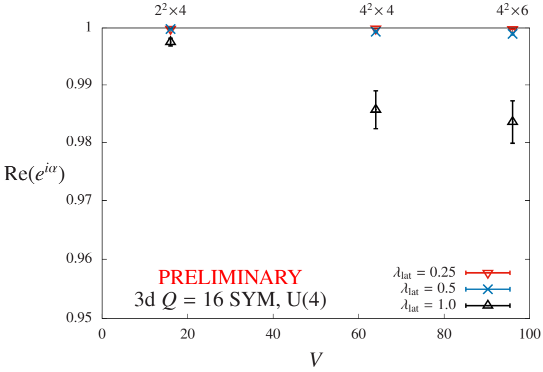


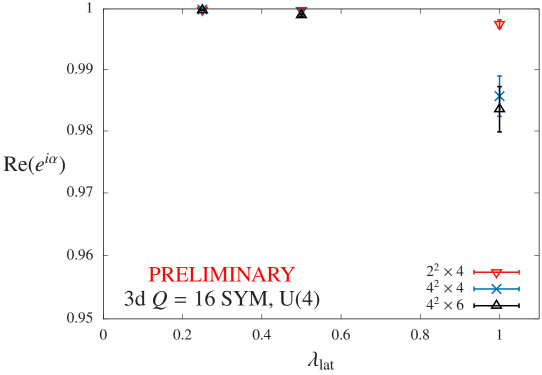


Aside: No sign problem for 3d maximal SYM

Continuum limit $L \to \infty$ with fixed $t = 1/(L\lambda_{\text{lat}}) \implies \lambda_{\text{lat}} \to 0$

Pfaffian nearly real positive for $\lambda_{\text{lat}} \leq 1$ on small volumes $\longrightarrow$ no sign problem





Next step: Exploring the 3d $Q = 16$ SYM phase diagram

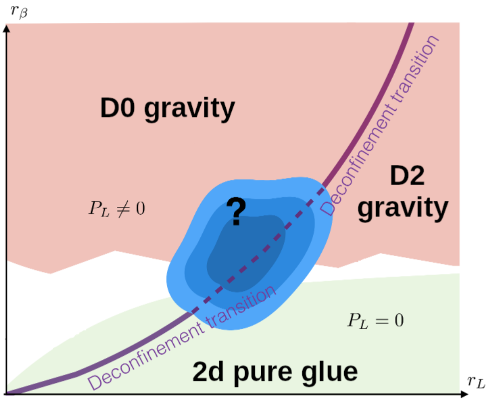
Work in progress to investigate D2–D0 transition with $r_L = r_1 = r_2$
$\longrightarrow$ scan in r_β fixing aspect ratio $\alpha = r_L / r_\beta$

For decreasing r_L at large N

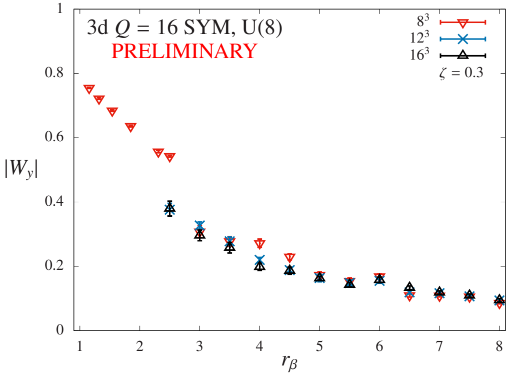
homogeneous black D2 brane
$\longrightarrow$ localized D0 black hole

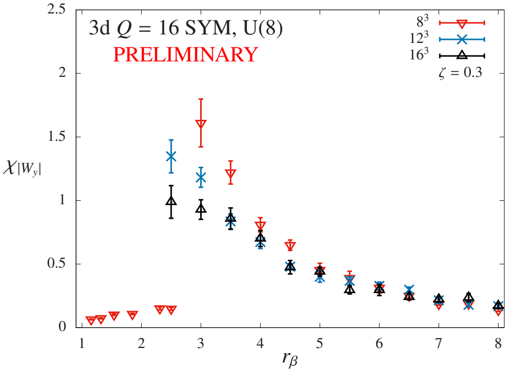
$\Updownarrow$

"spatial deconfinement"
signalled by Wilson line P_L



3d $Q = 16$ SYM spatial deconfinement transition signals





Preliminary U(8) results for 8^3 vs. 12^3 vs. 16^3 lattices (aspect ratio $\alpha = 1$)

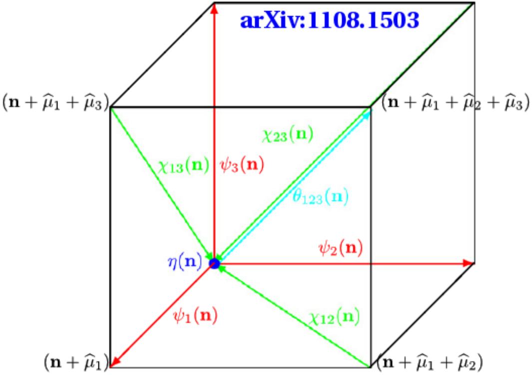
Still locating peaks in Wilson line susceptibility and checking hysteresis

Work in progress: 3d $Q = 8$ SYM

Simpler [Blau–Thompson] twisted formulation

$Q = 8$ supercharges $\{\mathcal{Q}, \mathcal{Q}_a, \mathcal{Q}_{ab}, \mathcal{Q}_{abc}\}$ with $a, b = 1, \cdots, 3$

$\longrightarrow$ site / link / plaquette / cube fermions $\{\eta, \psi_a, \chi_{ab}, \theta_{abc}\}$ on simple cubic lattice



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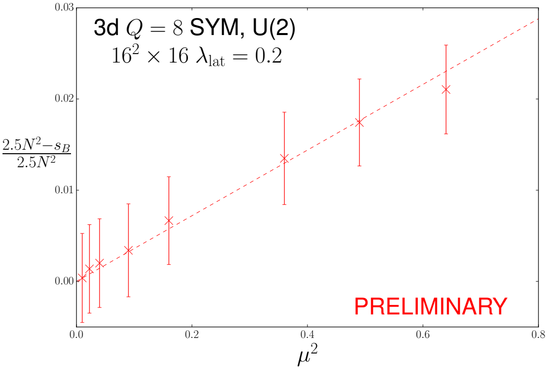
Parallel code developed
(Angel Sherletov)

Tests passed
$\longrightarrow$ larger-scale calculations
[mirror symmetry?]



Work in progress: Quiver superQCD from twisted SYM

First check 3d SYM $\longrightarrow$ 2d superQCD   then new 4d SYM $\longrightarrow$ 3d superQCD

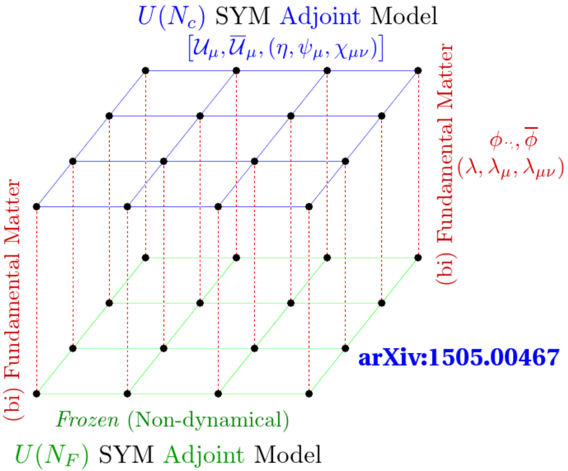
2-slice lattice SYM
with U(N) $\times$ U(F) gauge group

Adj. fields on each slice

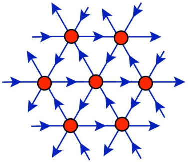
Bi-fundamental in between

Decouple U(F) slice

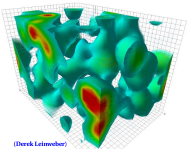
$\longrightarrow$ U(N) SQCD in $d-1$ dims.
with F fund. hypermultiplets



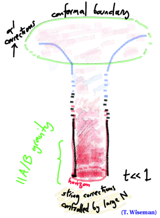
Recap: An exciting time for lattice supersymmetry



Three dimensions is a promising frontier
for practical lattice studies of supersymmetric QFTs



Preserving susy sub-algebra enables lattice calculations,
public code available



3d $Q = 16$ SYM thermodynamics consistent with holography,
work in progress on phase diagram

Work in progress on 3d $Q = 8$ SYM $\longrightarrow$ 2d superQCD
and much more for the future

Thanks for your attention! Any further questions?

Collaborators

Simon Catterall, Joel Giedt, Raghav Jha,
Anosh Joseph, Angel Sherletov, Toby Wiseman

Funding and computing resources

UK Research and Innovation

Backup: Lattice $\mathcal{N} = 4$ SYM

Lattice theory looks nearly the same despite breaking $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Covariant derivatives $\longrightarrow$ finite difference operators

Complexified gauge fields $\{\mathcal{A}_a, \overline{\mathcal{A}}_a\}$ $\longrightarrow$ gauge links $\{\mathcal{U}_a, \overline{\mathcal{U}}_a\} \in \mathfrak{gl}(N, \mathbb{C})$
with gauge-invariant flat measure $D\mathcal{U}\, D\overline{\mathcal{U}}$

Need $\mathcal{U}_a \to \mathbb{I}_N + \mathcal{A}_a$ to recover continuum covariant derivative

✓ $\mathcal{Q}$ interchanges bosonic $\longleftrightarrow$ fermionic d.o.f. with $\mathcal{Q}^2 = 0$

$$\mathcal{Q}\,\mathcal{A}_a \longrightarrow \mathcal{Q}\,\mathcal{U}_a = \psi_a \qquad\qquad \mathcal{Q}\,\psi_a = 0$$

$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad\qquad \mathcal{Q}\,\overline{\mathcal{A}}_a \longrightarrow \mathcal{Q}\,\overline{\mathcal{U}}_a = 0$$

$$\mathcal{Q}\,\eta = d \qquad\qquad \mathcal{Q}\,d = 0$$

Backup: Sign problems

Recall typical algorithms sample field configurations Φ with probability $\frac{1}{\mathcal{Z}} e^{-S[\Phi]}$

$\longrightarrow$ "sign problem" if action $S[\Phi]$ can be negative or complex

Lattice SYM has complex pfaffian $\text{pf}\,\mathcal{D} = |\text{pf}\,\mathcal{D}| e^{i\alpha}$

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int [d\mathcal{U}][d\overline{\mathcal{U}}] \;\; \mathcal{O} \;\; e^{-S_B[\mathcal{U},\overline{\mathcal{U}}]} \;\; \text{pf}\,\mathcal{D}[\mathcal{U},\overline{\mathcal{U}}]$$

We phase quench $\text{pf}\,\mathcal{D} \longrightarrow |\text{pf}\,\mathcal{D}|$, need to reweight $\langle \mathcal{O} \rangle = \frac{\left\langle \mathcal{O} e^{i\alpha} \right\rangle_{\text{pq}}}{\left\langle e^{i\alpha} \right\rangle_{\text{pq}}}$

$\Longrightarrow \left\langle e^{i\alpha} \right\rangle_{\text{pq}} = \frac{\mathcal{Z}}{\mathcal{Z}_{\text{pq}}}$ quantifies severity of sign problem

Backup: 2d maximal SYM phase diagram

arXiv:1709.07025

Dimensionally reduce to 2d $\mathcal{N} = (8, 8)$ SYM on $(r_L \times r_\beta)$ torus with four scalar $\mathcal{Q}$

Low temperatures $t = 1/r_\beta \longleftrightarrow$ black holes in dual supergravity

For decreasing r_L at large N

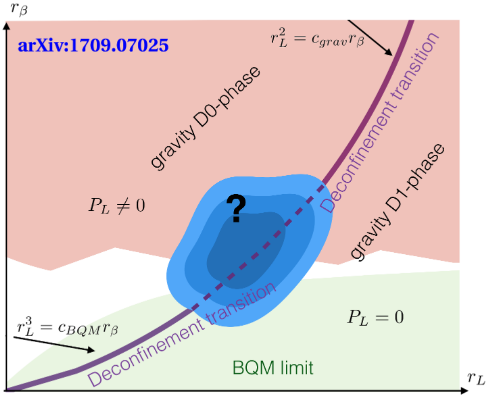
homogeneous black string (D1)

$\longrightarrow$ localized black hole (D0)

$\updownarrow$

"spatial deconfinement"

signalled by Wilson line P_L



Backup: Dimensional reduction to 2d $\mathcal{N} = (8, 8)$ SYM

Naive for now: 4d $\mathcal{N} = 4$ SYM code with $N_x = N_y = 1$

$A_4^* \longrightarrow A_2^*$ (triangular) lattice

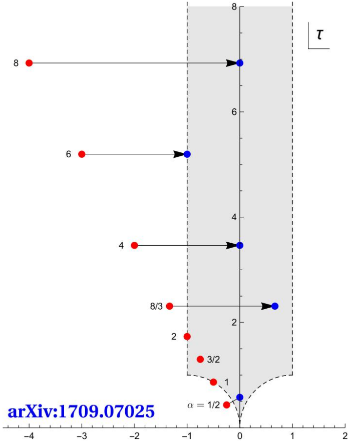
Torus **skewed** depending on $\alpha = L/N_t$

Modular transformation into fundamental domain

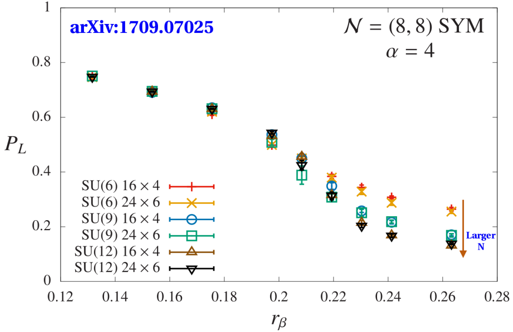
$\longrightarrow$ some skewed tori actually rectangular

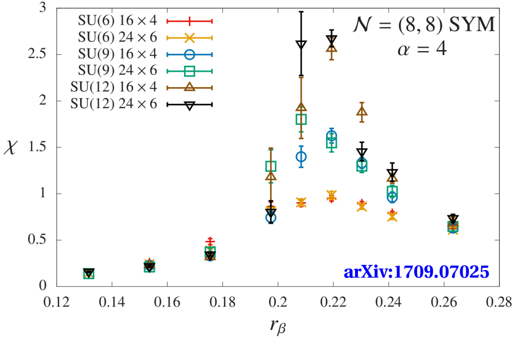
Again need to stabilize compactified links

to ensure broken center symmetries



Backup: 2d spatial deconfinement transition signals





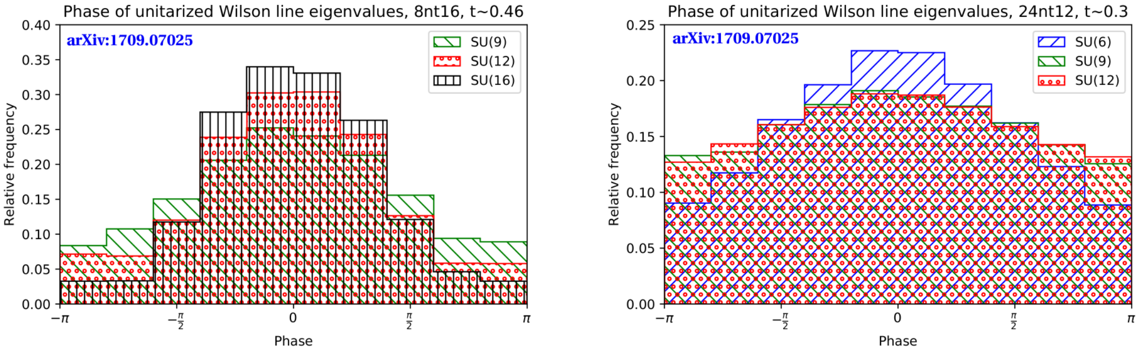
Peaks in Wilson line susceptibility match change in its magnitude |PL|,

grow with size of SU(N) gauge group, comparing $N = 6, 9, 12$

Agreement for 16×4 vs. 24×6 lattices (aspect ratio $\alpha = r_L / r_\beta = 4$)

Backup: 2d Wilson line eigenvalues

Large-N eigenvalue phase distribution also signals spatial deconfinement



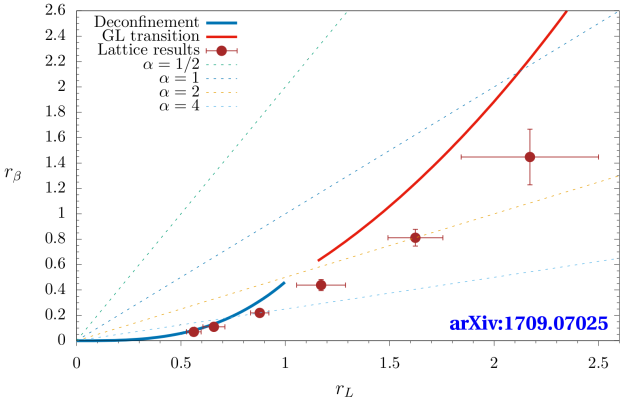
Left: $\alpha = 1/2$ distributions more localized as N increases $\longrightarrow$ D0 black hole

Right: $\alpha = 2$ distributions more uniform as N increases $\longrightarrow$ D1 black string

Backup: Lattice results for 2d $\mathcal{N} = (8, 8)$ SYM phase diagram

Good agreement with bosonic QM at high temperatures $(\alpha \gtrsim 4)$

Harder to control low-temperature uncertainties (larger $N > 16$ should help)



Overall consistent with holography

Comparing multiple lattice sizes and $6 \leq N \leq 16$

Controlled extrapolations are work in progress

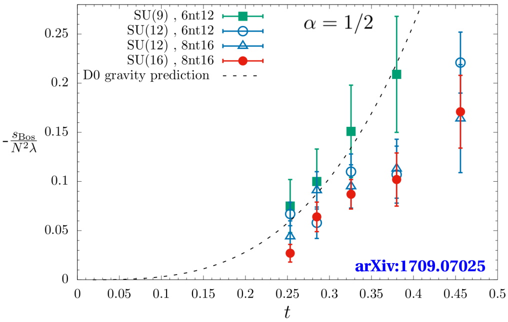
Backup: 2d holographic black hole energies

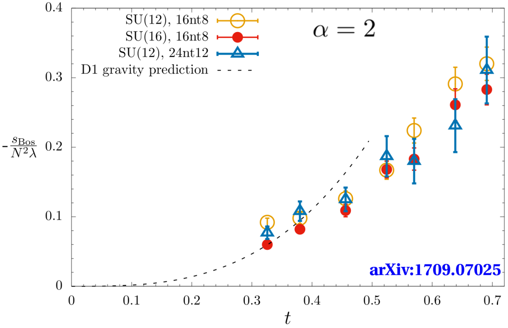
Lattice results consistent with leading expectation for sufficiently low $t \lesssim 0.4$

Similar behavior $\longrightarrow$ difficult to distinguish phases

$\propto t^{3.2}$ for small-r_L D0 phase

$\propto t^3$ for large-r_L D1 phase

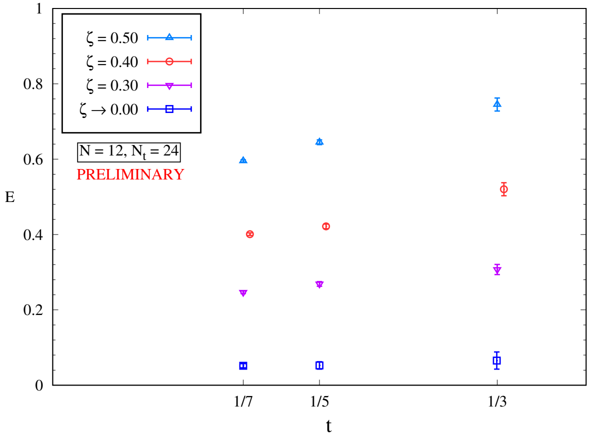




Backup: $\mathcal{N} = (2, 2)$ SYM

arXiv:2109.01001

Much simpler twisted formulation: $Q = 4$ supercharges $\{\mathcal{Q}, \mathcal{Q}_a, \mathcal{Q}_{ab}\}$
$\longrightarrow$ site / link / plaquette fermions $\{\eta, \psi_a, \chi_{ab}\}$ on square lattice $(a, b = 1, 2)$



Work by Navdeep Singh Dhindsa

Prelim. $\mu^2 \to 0$ extrapolations
for $r_L = r_\beta \longleftrightarrow \alpha = 1$

Energy independent of $t \lesssim 0.33$
vs. $\sim t^3$ for $\mathcal{N} = (8, 8)$ SYM

Backup: High-temperature ($t \gtrsim 1$) 3d maximal SYM

Wilson line eigenvalue phases localized rather than uniform (**left**)

Thermodynamics consistent with weak-coupling expectation $\propto t^3$ (**right**)