# The S-matrix bootstrap in 2d and 4d, primal and dual problem


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Based on

- Work in progress… w/ Yifei He.
- e-Print: 2103.11484, *JHEP* 08 (2021) 125, w/ Yifei He.
- e-Print: 2012.15576, *JHEP* 04 (2021) 097, w/ Harish Murali.
- e-Print: 1909.06495, *JHEP* 04 (2020) 142, w/ Lucia Cordova, Yifei He, Pedro Vieira.
- e-Print: 1805.02812, *JHEP* 11 (2018) 093, w/ Yifei He, Andy Irrgang.


NONPERTURBATIVE AND NUMERICAL APPROACHES TO QUANTUM GRAVITY, STRING THEORY AND HOLOGRAPHY , *ICTS-Bengaluru, Aug 2022*

# Summary

- Introduction and Motivation

- Mapping out the space of allowed S-matrices we find distinguished points (*e.g.* vertices) where interesting theories sit.

- The 2d O(N) model from S-matrix bootstrap and R-matrix bootstrap

- The O(N) model S-matrix at the vertex of the space of allowed S-matrices.
- The O(N) model on a half-line: R-matrix at a vertex after extended analyticity.

- Dual convex maximization problem

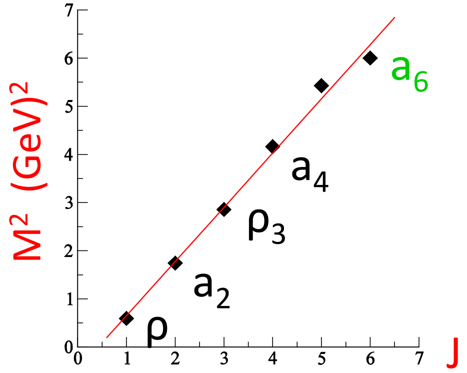
- 2d case: Upper bounds through generalized dispersion relations.
- 4d case: Regularization and dual problem: method and numerical examples.

- Conclusions

Motivation: Before the quark model, many ideas were developed to understand the strong interactions:

Meson physics

Regge trajectories




Veneziano amplitude

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}$$

S-matrix bootstrap

**[Chew, Mandelstam, …]**

The S-matrix satisfies constraints from symmetries, analyticity, unitarity and crossing that can be used to completely determine it in a self-consistent way.

## Recent idea in the S-matrix bootstrap

In the allowed space of S-matrices one can consider a functional and define a theory by the S-matrix that maximizes such functional.

A standard example is the coupling between a particle and its bound states (whose spectrum is assumed fixed). There is a maximum coupling because increasing the coupling further adds more bound states.
Paulos, Penedones, Toledo, van Rees, Vieira

2d O(N) model has no bound states. We studied this model and found that many functionals lead to the same model. We argued that the space of allowed S-matrices is convex and has a vertex where the O(N) model sits. We expect this to be a somewhat generic picture. w/ He and Irrgang

# Unitarity condition for the S matrix in a subspace of states

Unitarity $SS^\dagger = \mathbb{I}$

Considering a subspace D

$$S|\psi_D\rangle = S_D|\psi_D\rangle + |\phi\rangle_\perp \Rightarrow \|S_D|\psi_D\rangle\| \leq 1$$

Take two matrices satisfying

$$\|S_1|\psi\rangle\| \leq 1, \qquad \|S_2|\psi\rangle\| \leq 1,$$

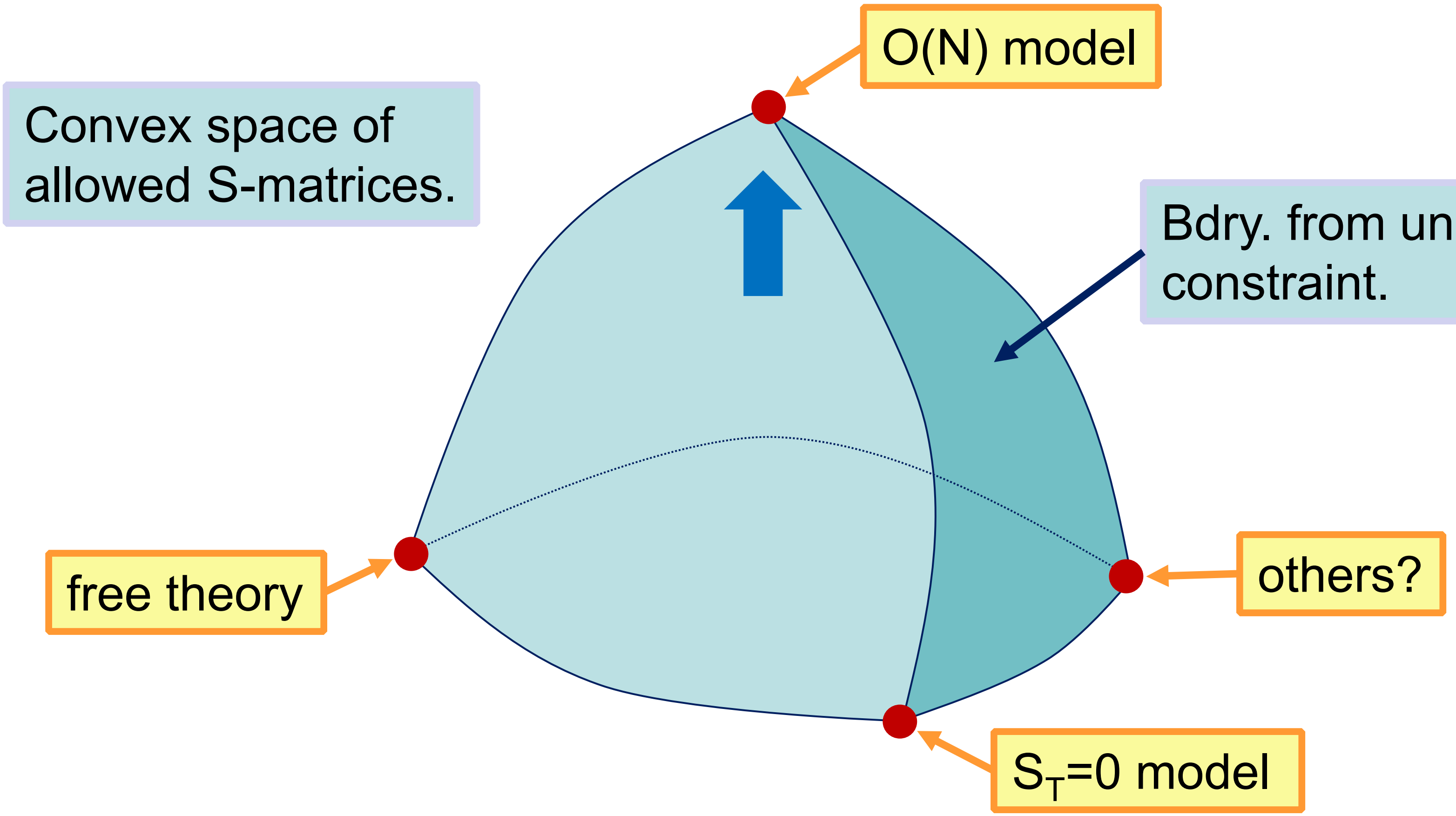
Then $$\tilde{S} = \alpha S_1 + (1-\alpha)S_2, \qquad 0 < \alpha < 1$$

will also satisfy it (convex space)

$$\|\tilde{S}|\psi\rangle\| = \|\alpha S_1|\psi\rangle + (1-\alpha)S_2|\psi\rangle\| \leq 1$$

## The O(N)-model. 1805.02812, *JHEP* 11 (2018) 093, w/ Yifei He, Andy Irrgang

The S-matrix in a subspace D (e.g. two particle states) satisfies unitarity, crossing, and symmetry properties. The allowed space is convex with interesting theories at its vertices. Maximizing linear functionals (typically) finds vertices.



# The 2d O(N) non-linear sigma-model, basic definition.

It is sometimes thought as a toy model for QCD. The variable is a N-dimensional unit vector $n^2=1$ with action

$$S = \frac{1}{2g_0^2}\int d^2x\ (\partial_a \vec{n})^2, \qquad a = 0, 1$$

It is asymptotically free (Polyakov) and develops a mass gap around an O(N) symmetric vacuum. Particle content: N species of bosons w/ mass m. Scattering:

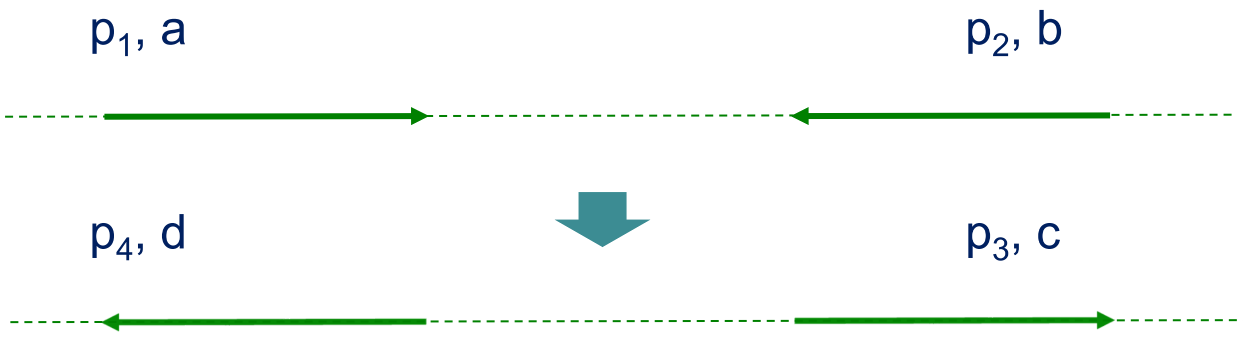
In the large-N limit

$$m = \mu e^{-\frac{2\pi}{g_0^2 N}}$$

$$S_{ab\to cd} = S_{ab}^{cd}(s)\ \delta(p_1 - p_3)\delta(p_2 - p_4) + (p_3 \leftrightarrow p_4)(c \leftrightarrow d)$$

$$S_{ab}^{cd}(s) = \delta_{ab}\delta_{cd}\ S_A(s) + \delta_{ac}\delta_{bd}\ S_T(s) + \delta_{ad}\delta_{bc}\ S_R(s)$$

$$S_{ab}^{cd}(s) = \frac{1}{N}\delta_{ab}\delta_{cd}S_I(s) + \frac{1}{2}(\delta_{ac}\delta_{bd} + \delta_{bc}\delta_{ad} - \frac{2}{N}\delta_{ab}\delta_{cd})S_+(s) + \frac{1}{2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})S_-(s)$$



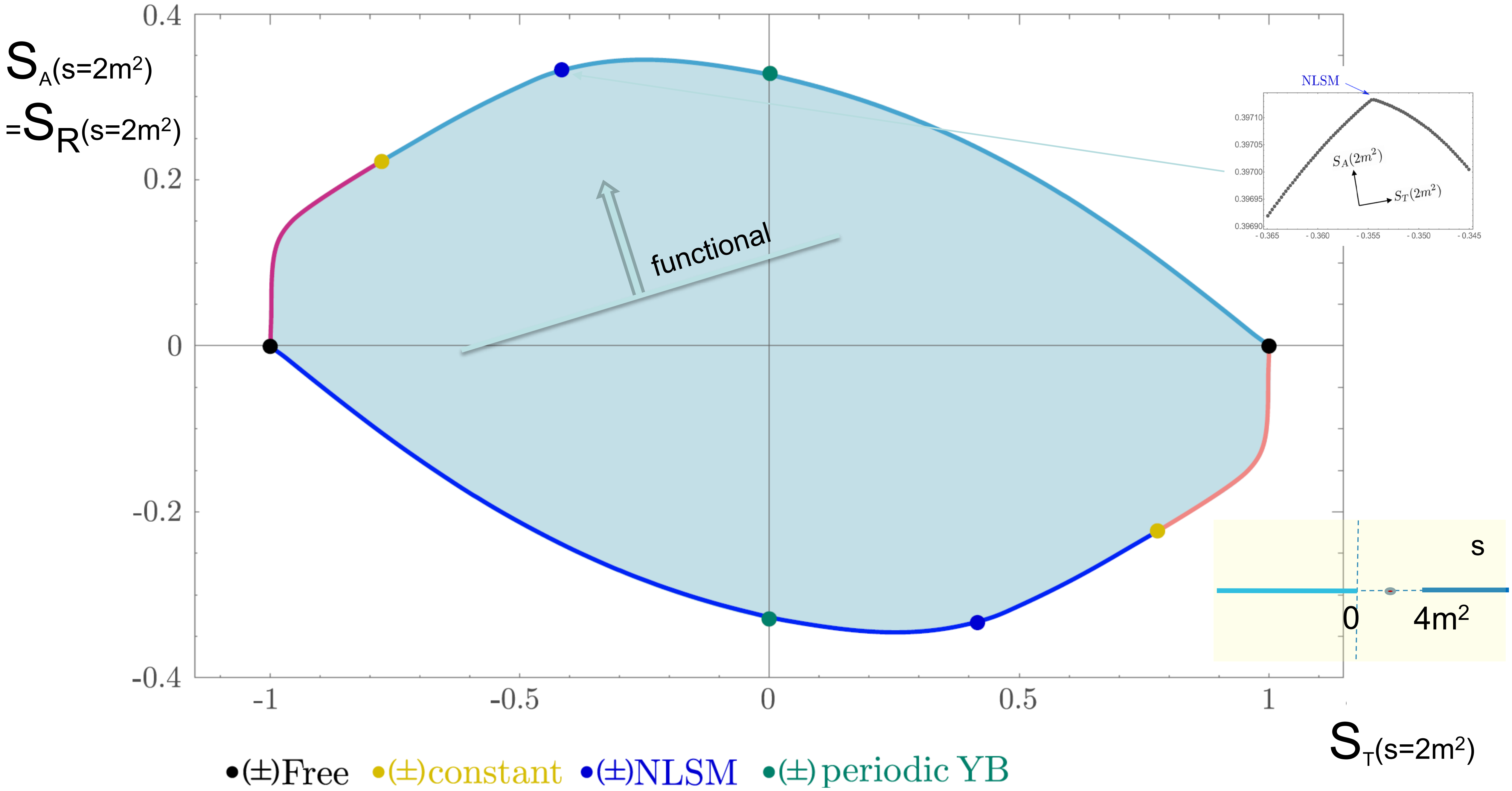

$$s = (p_1 + p_2)^2$$

Crossing

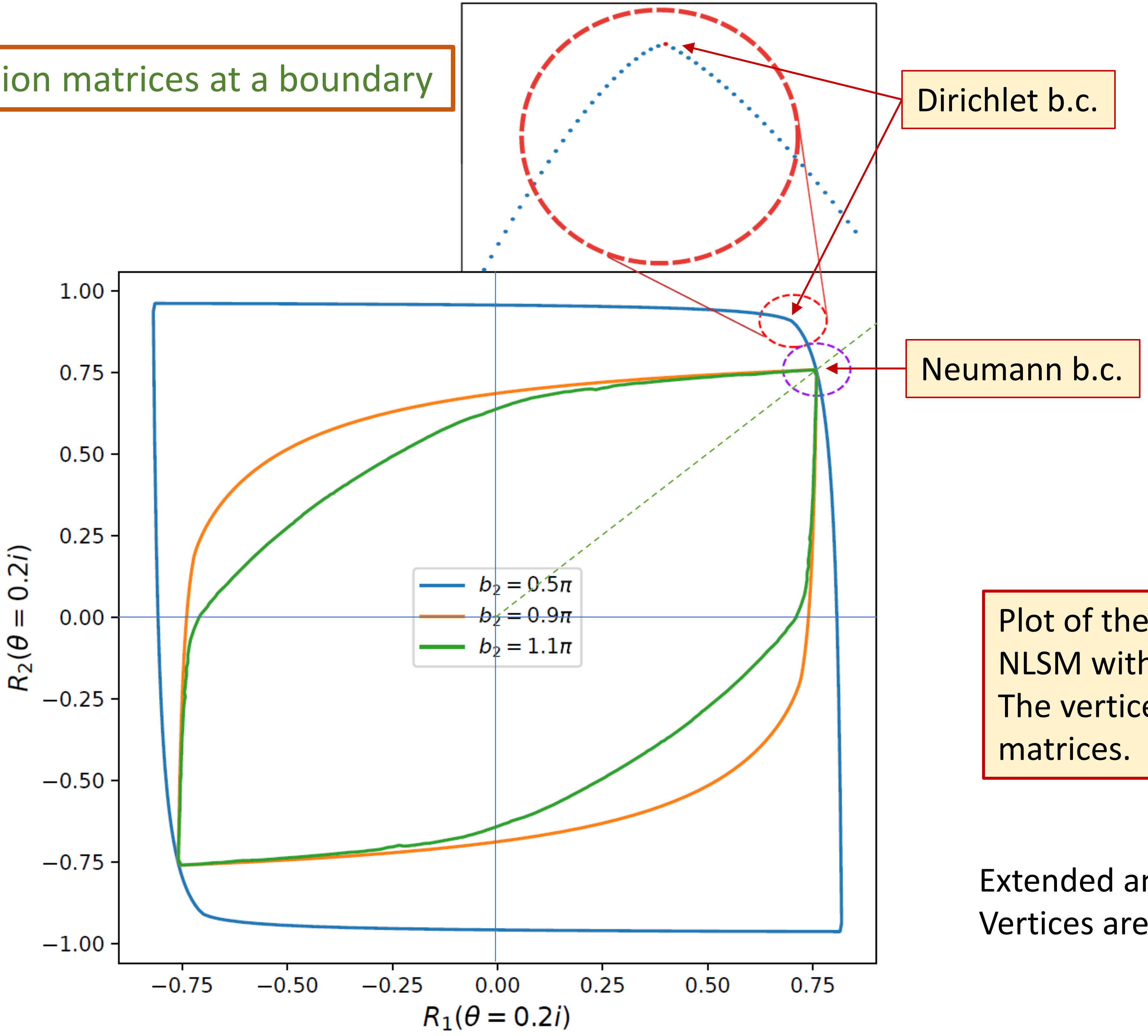
$$S_T(s) = S_T(4 - s)$$

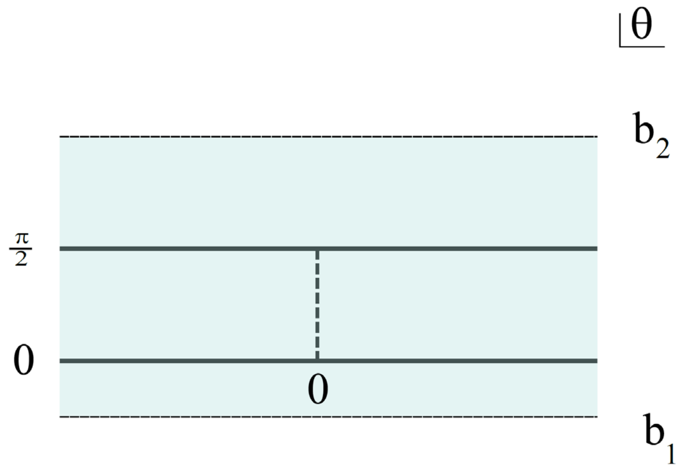
$$S_R(s) = S_A(4 - s)$$

Allowed space of values for S 1909.06495, *JHEP* 04 (2020) 142, w/ Lucia Cordova, Yifei He, Pedro Vieira.



# Reflection matrices at a boundary

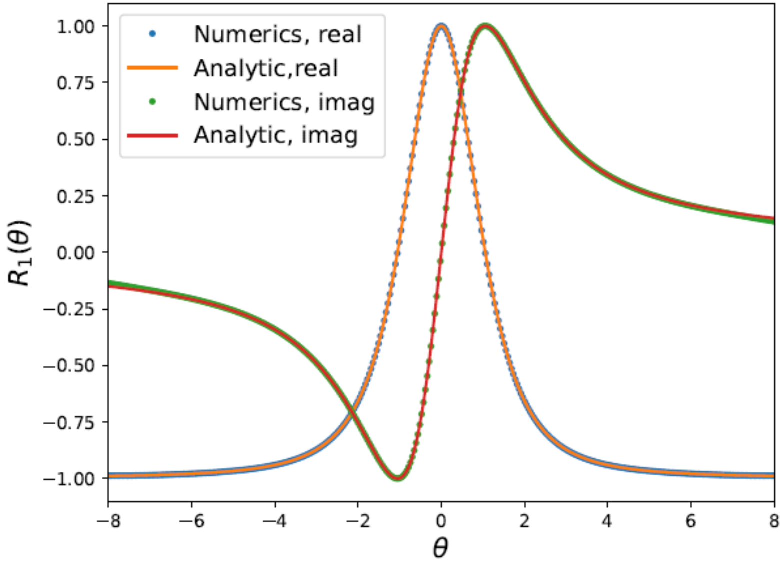


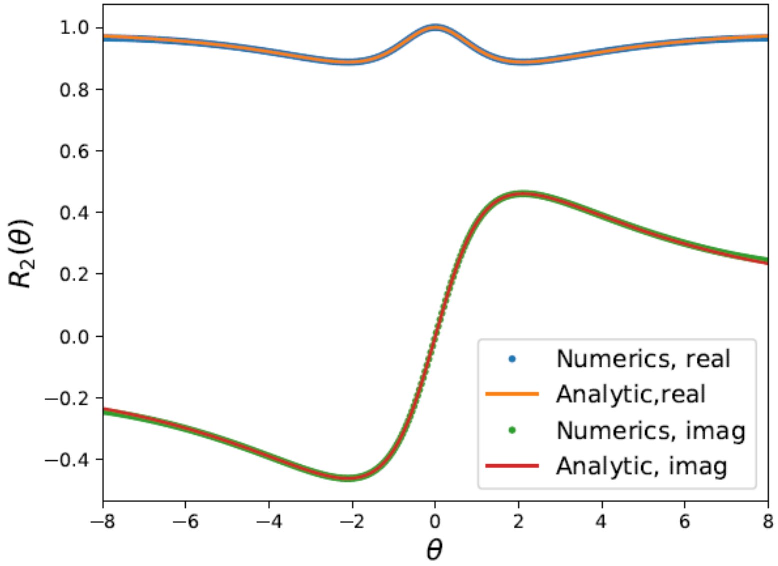

$$s = 4\cosh^2\left(\frac{\theta}{2}\right)$$

Plot of the allowed region ($R_1$,$R_2$) for the NLSM with N=6, k=1.
The vertices correspond to integrable R-matrices.

Extended analyticity idea.
Vertices are due to poles reaching the physical line.

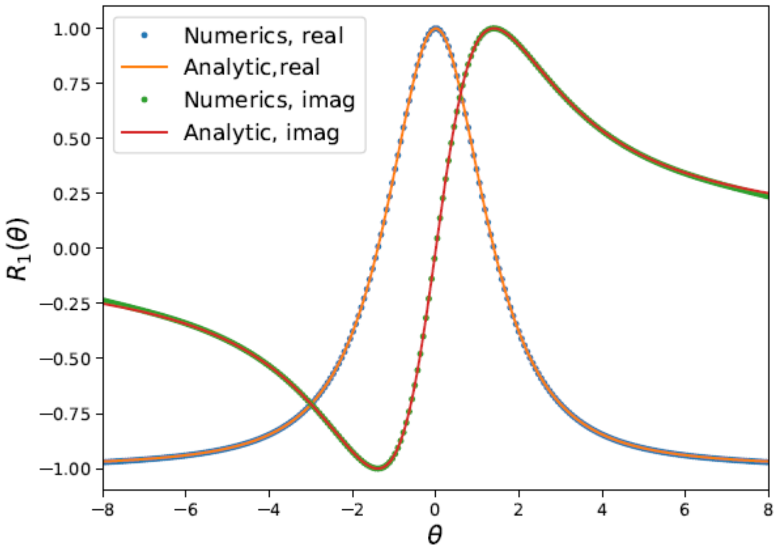



(a) $R_1(\theta)$, Dirichlet




(b) $R_2(\theta)$, Dirichlet

$$s = 4\cosh^2\left(\frac{\theta}{2}\right)$$

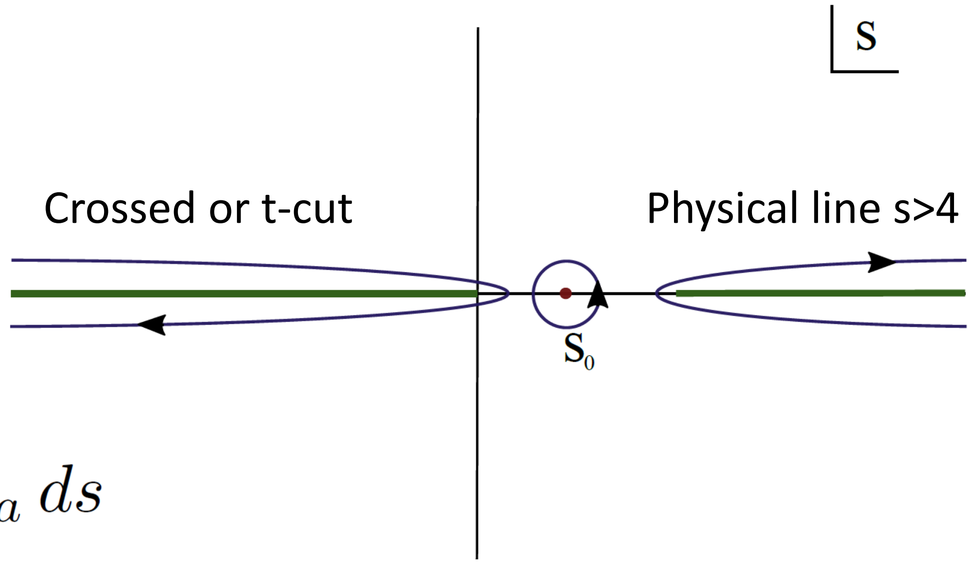



(c) $R_1(\theta)$, Neumann

Plot of the functions $R_1$,$R_2$ on the real axis (physical region) for the two vertices of the previous figure.

# Dual problem from generalized dispersion relations

We can rewrite the functional by using dispersion relations.




$$\mathcal{F} = \sum_{a=I,-,+} n_a S_a(s_0)$$

$$S_a(s_0) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{S_a(s)}{s - s_0} = \frac{1}{2\pi i} \left[ \int_{4m^2}^{\infty} + \int_{-\infty}^{0} \right] \Delta S_a \, ds$$

However, there is no need to use just a pole, we can use any function that has a pole and is analytic in the region outside the cuts so that we can deform the contours (namely it can have the same cuts). We can then write ($K_a$ pole at $s_0$ residue $n_a$):

$$\mathcal{F} = \frac{1}{2\pi i} \oint_{\mathcal{C}} K_a(s) S_a(s) = \frac{1}{2\pi i} \left[ \int_{4m^2}^{\infty} + \int_{-\infty}^{0} \right] \Delta(K_a S_a) \, ds$$

With extra assumptions (K is real analytic and satisfies anti-crossing, the it also has pole at $4\text{-}s_0$) we can integrate only on the physical cut and the jump is given by the imaginary part of $K_a S_a$.

We can now use a simple inequality to find a bound on the functional.

$$\mathcal{F} = \frac{2}{\pi}\int_{4m^2}^{\infty} \mathrm{Im}[K_a(s^+)S_a(s^+)]ds \leq \frac{2}{\pi}\int_{4m^2}^{\infty} |K_a(s^+)S_a(s^+)|ds \leq \frac{2}{\pi}\int_{4m^2}^{\infty} |K_a(s^+)|ds$$

Sum over *a*

Since the functions K are arbitrary we find the best bound by minimizing over K. This defines the dual problem:

$$\max_{\{S_a\}}\left\{\mathcal{F}_P = \sum_a \mathrm{Re}\left[S_a(s_0)\right]\right\} \leq \min_{\{K_a\}}\left\{\mathcal{F}_D = \frac{2}{\pi}\int_{4m^2}^{\infty}\sum_a |K_a(s^+)|ds\right\}$$

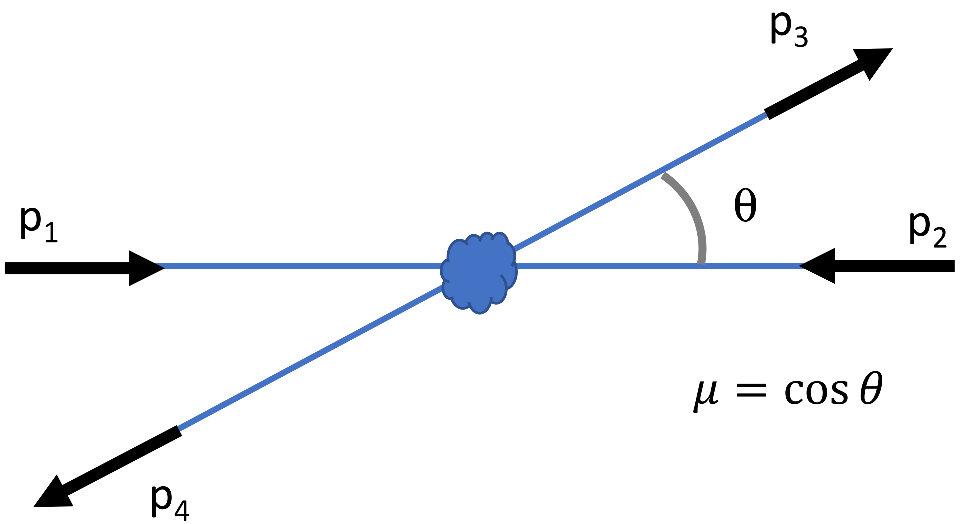
Increase space of $S_a$

Increase space of $K_a$

The functions $K_a$ are constrained by crossing and real analyticity

S-matrix bootstrap in 4d, scattering of spin 0 particles (pions)
2103.11484, *JHEP* 08 (2021) 125, w/ Yifei He.

Single pion scattering $\pi(p_1) + \pi(p_2) \to \pi(p_3) + \pi(p_4)$ in the Mandelstam representation




$s = (p_1 + p_2)^2$

$t = (p_1 - p_3)^2$

$u = (p_1 - p_4)^2$

$s + t + u = 4$

$$F(s,t,u) = f_0 + \int_4^\infty dx\, \mathcal{K}(s,t,u;x)\sigma(x) + \int_4^\infty dx \int_4^\infty dy\, \mathcal{K}(s,t,u;x,y)\, \rho(x,y)$$

$$\mathcal{K}^{(1)}(s,t,u;x) = \frac{1}{\pi}\left[\frac{1}{x-s} + \frac{1}{x-t} + \frac{1}{x-u}\right]$$

$$\mathcal{K}^{(2)}(s,t,u;x,y) = \frac{1}{2\pi^2}\left[\frac{1}{(x-s)(y-t)} + \frac{1}{(x-s)(y-u)} + \frac{1}{(x-u)(y-t)}\right]$$

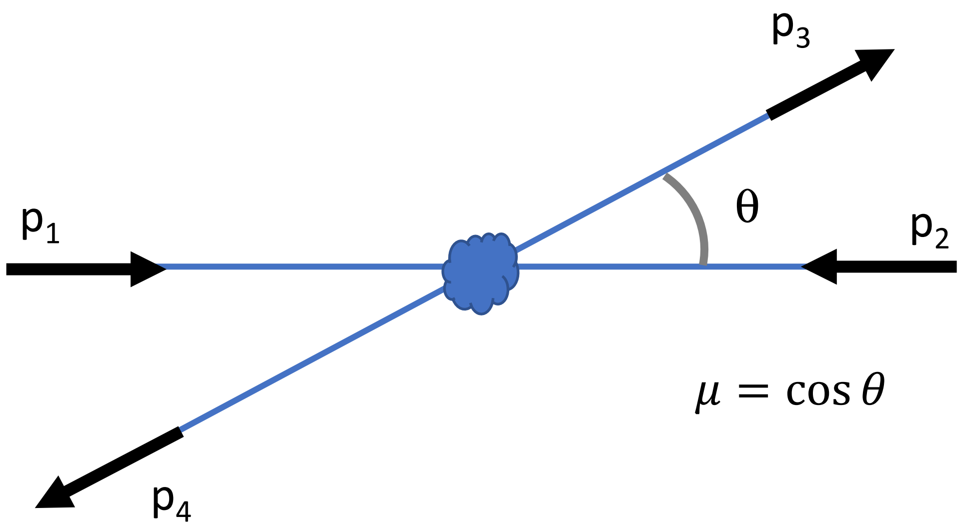
Now we can compute the usual partial waves (scattering with fixed angular momentum)

$$f_\ell(s) = \frac{1}{4}\int_{-1}^{+1} d\mu\, P_\ell(\mu)\, F\left(s^+, t(\mu,s), u(\mu,s)\right)$$

$$\begin{aligned} t(\mu,s) &= \frac{(s-4)(\mu-1)}{2} \\ u(\mu,s) &= 4 - s - t(\mu,s) = t(-\mu,s). \end{aligned}$$

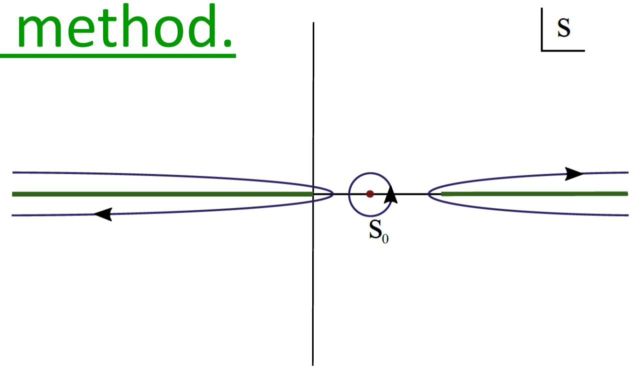
$$\begin{aligned} h_\ell(s) &= \pi\sqrt{\frac{s-4}{s}}\, f_\ell(s) \\ S_\ell(s) &= 1 + i\, h_\ell(s) \end{aligned}$$

$$|S_\ell(s)| \leq 1 \quad \Longleftrightarrow \quad |h_\ell(s)|^2 \leq 2\mathrm{Im} h_\ell(s)$$




$$F(s^+, t) = \sum_\ell \int_4^\infty ds\, \frac{2}{\pi}\sqrt{\frac{s}{4-s}}(2\ell+1) h_\ell(s) P_\ell\left(1 + \frac{2t}{s-4}\right)$$

# Aside: Generalized dispersion relations for forward amplitudes using 2d method.




$$F(s) = F(s, t=0, u=4-s), \quad \Rightarrow \quad F(s) = F(4-s)$$

It is like 2d!

$$F(s_0) \quad = \quad \frac{2}{\pi} \int_4^\infty \mathrm{Im}\left(F(s^+) K(s^+, s_0)\right)$$

Unitarity?

$$|F(s_0)| \quad \leq \quad \sum_\ell \int_4^\infty ds \, \underbrace{\frac{2}{\pi} \sqrt{\frac{s}{4-s}} (2\ell+1) \sqrt{2\mathrm{Im} h_\ell(s)} \, |K(s^+, s_0)|}_{u_i v_i}$$

Use scattering length $a_2$

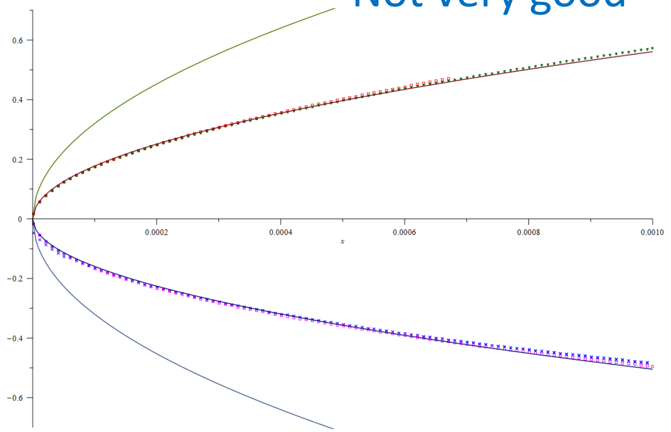
$$a_2 = \sum_\ell \int_4^\infty ds \, \underbrace{\frac{4}{15\pi s^3} \sqrt{\frac{s}{4-s}} (2\ell+1) \, 2\mathrm{Im} h_\ell(s) \, P_\ell\left(1 + \frac{8}{s-4}\right)}_{u_i^2}$$

$$\sum_i u_i v_i \leq \sqrt{\sum_i u_i^2} \sqrt{\sum_i v_i^2}$$

**[Chung, Mau, 1976]**

$$|F(s_0)| \leq \sqrt{a_2} \, \min_{\{K\}} \left( \sum_\ell \int_4^\infty ds \, \underbrace{\frac{15}{\pi} \sqrt{\frac{s}{4-s}} \frac{(2\ell+1)}{P_\ell\left(1+\frac{8}{s-4}\right)} s^3 \, |K(s^+, s_0)|^2}_{v_i^2} \right)^{\frac{1}{2}}$$

Not very good



# Primal problem

Maximize

$$\mathcal{F}_P = F(s_0, t_0, u_0)$$

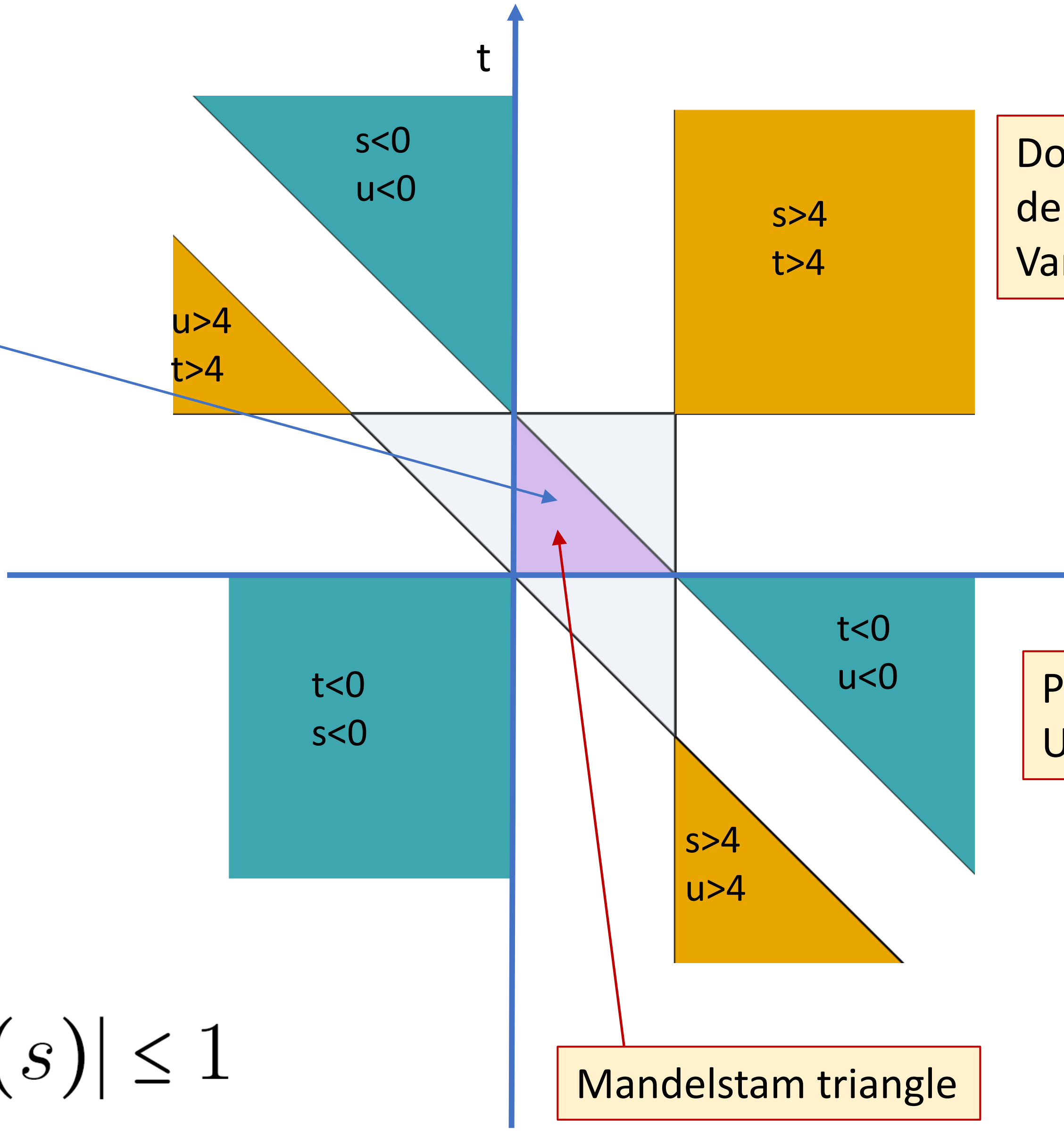
Under the previous conditions.

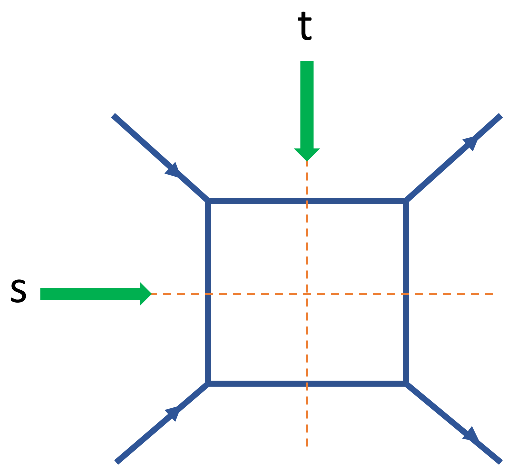
Formally we can write:

$$\max_{\alpha_n} \ \mathcal{F}_P = a_n \alpha_n$$

$$s.t. \ \ h_\ell^I(s) = \alpha_n h_{\ell n}^I(s)$$

$$|S_\ell^I(s)| = |1 + i h_\ell^I(s)| \leq 1$$

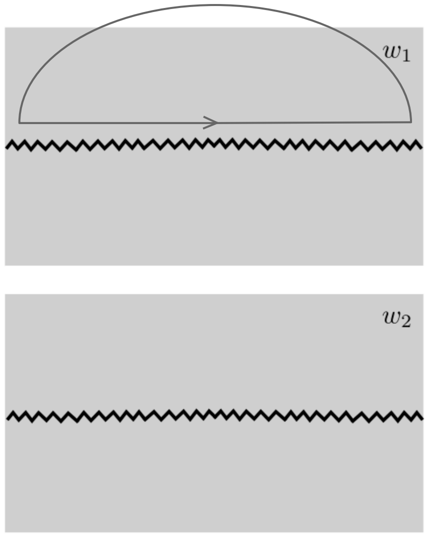


# Generalized dispersion relation, 4d case

2103.11484, *JHEP* 08 (2021) 125, w/ Yifei He.

analytic function of two variables $$G(w_1,w_2)=\frac{1}{\pi^2}\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\,\frac{g(x,y)}{(x-w_1)(y-w_2)}$$




double spectral density $$\Delta_{12}G(x,y)=G(x^+,y^+)-G(x^-,y^+)-G(x^+,y^-)+G(x^-,y^-)=-4g(x,y)$$

$$\int_{-\infty}^{+\infty}dx\;G(x^\pm,w_2)\;=\;0\qquad\int_{-\infty}^{+\infty}dy\;G(w_1,y^\pm)\;=\;0$$

*close contour and drop infinity*

---

product of two such analytic functions: $$G(w_1,w_2)=H(w_1,w_2)K(w_1,w_2)$$

**amplitude support in s>4, t>4**

**dual amplitude Support in physical region (s>4,4-s<t<0) plus poles.**

*generalized dispersion relation*

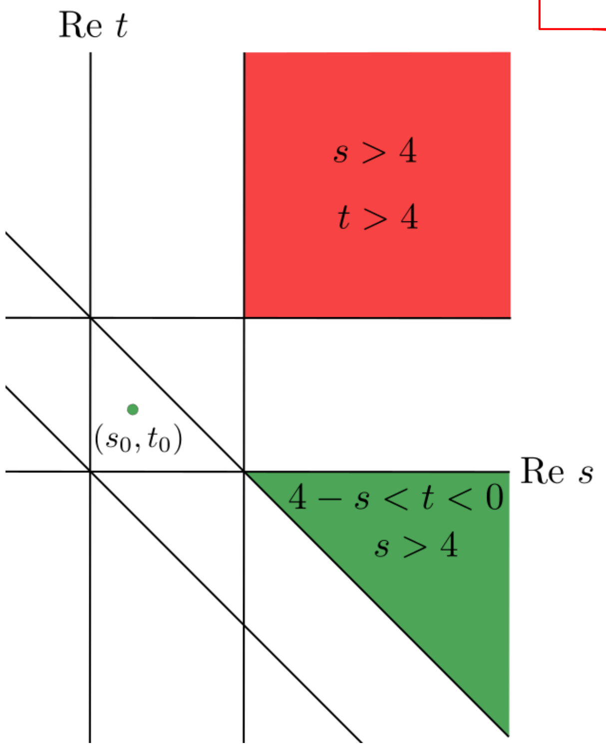
$$\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\,\left[\Delta_{12}H(x,y)K(x^-,y^+)-H(x^+,y^-)\Delta_{12}K(x,y)\right]=0$$

*relate amplitude to physical region*

# Analytic amplitudes: primal and dual

$$H(s,t) = \frac{1}{\pi^2}\int_4^\infty dx \int_4^\infty dy \frac{\rho(x,y)}{(x-s)(y-t)}$$

*regularization* $\|\rho\| \leq M_{\text{reg}}$




Mandelstam region

$$H(s_0,t_0) = \frac{i}{\pi^2}\int_4^\infty dx \int_{4-x}^0 dy H(x^+,y)\bar{k}(x,y) - \frac{1}{\pi^2}\int_4^\infty dx \int_4^\infty dy \rho(x,y) K(x^-,y^+)$$

physical region ➡ can use partial waves

$$H(s^+,t) = \frac{2}{\pi}\sqrt{\frac{s}{s-4}}\sum_\ell (2\ell+1) h_\ell(s) P_\ell\left(1+\frac{2t}{s-4}\right)$$

*unitarity*

$$K(s,t) = -\frac{1}{(s-s_0)(t-t_0)} + \frac{i}{\pi^2}\int_4^\infty dx \int_{4-x}^0 dy \frac{\bar{k}(x,y)}{(s-x)(t-y)}$$

# Dual S-matrix bootstrap for single pion

**define dual partial waves:**

$$k_\ell(s) = \frac{(2\ell+1)}{\pi^3}\sqrt{s(s-4)}\int_{-1}^{1} d\cos\theta P_\ell(\cos\theta)k(s,t)$$

**bound the double spectral function ρ(x,y)**

$$\langle \rho\cdot\Theta\rangle = \int_4^\infty dx \int_4^\infty dy\, \rho(x,y)\Theta(x,y) \le ||\rho||\,||\Theta||_* \le M_{\text{reg}}||\Theta||_*$$

$$H(s_0,t_0) = -\sum_\ell \int_4^\infty ds\, \text{Im}[h_\ell(s)\bar{k}_\ell(s)] - \frac{1}{\pi^2}\int_4^\infty dx \int_4^\infty dy \rho(x,y)\underbrace{\text{Re}K(x^-,y^+)}_{\Theta(x,y)}$$

$$h_l = -iS_l + i$$

$$-\text{Im}[h_\ell(s)\bar{k}_\ell(s)] \le |S_\ell k_\ell| - \text{Re}k_\ell \le |k_\ell| - \text{Re}k_\ell$$

*take into account crossed regions*

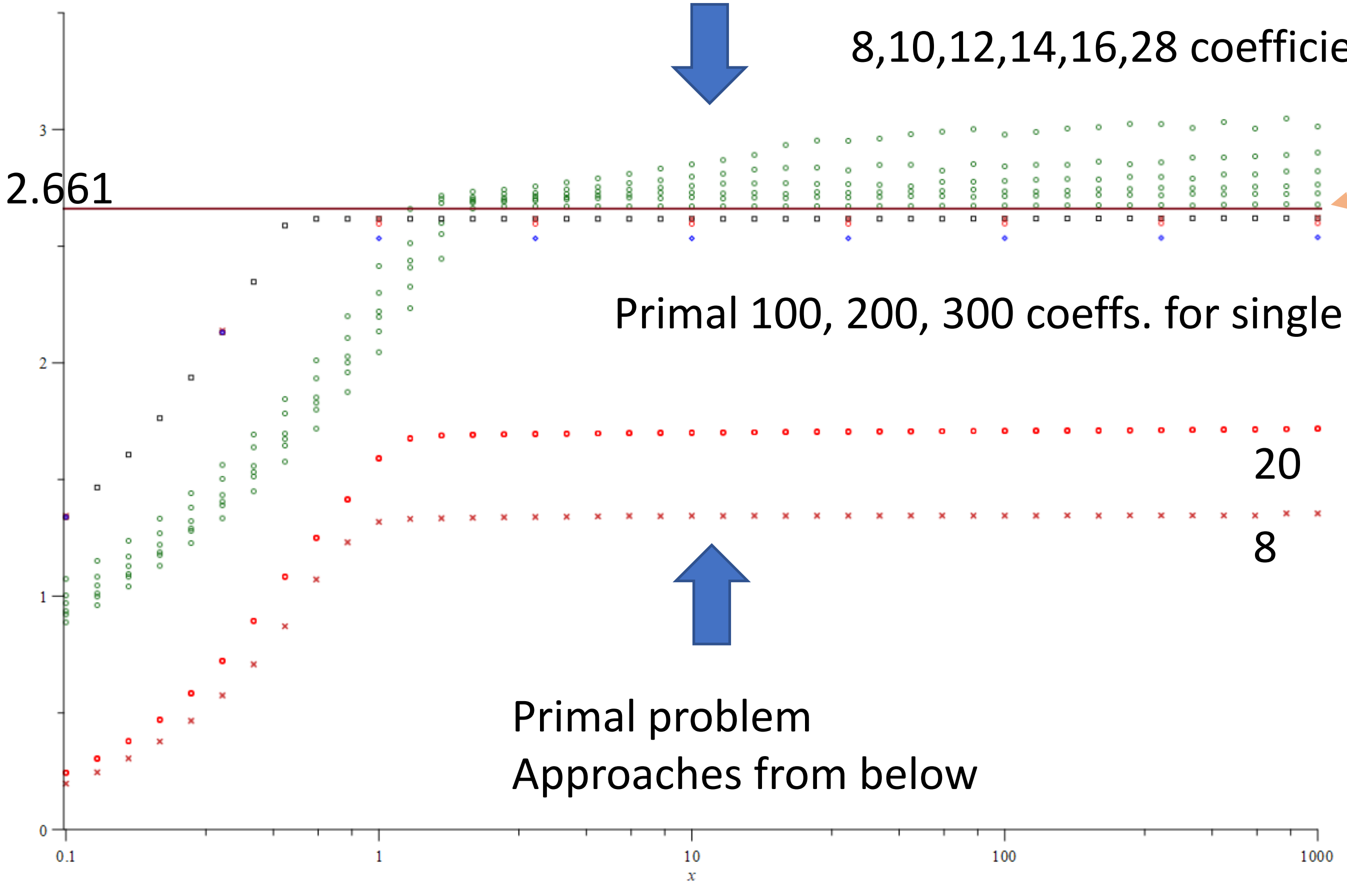
$$\min_{\{k_\ell(s)\}} \mathcal{F}_D = \sum_{\ell\text{ even}} \int_4^\infty ds \Big(|k_\ell(s)| - \text{Re}k_\ell(s)\Big) + M_{\text{reg}}||\text{Re}K||_*$$

Dual problem
Approaches from above

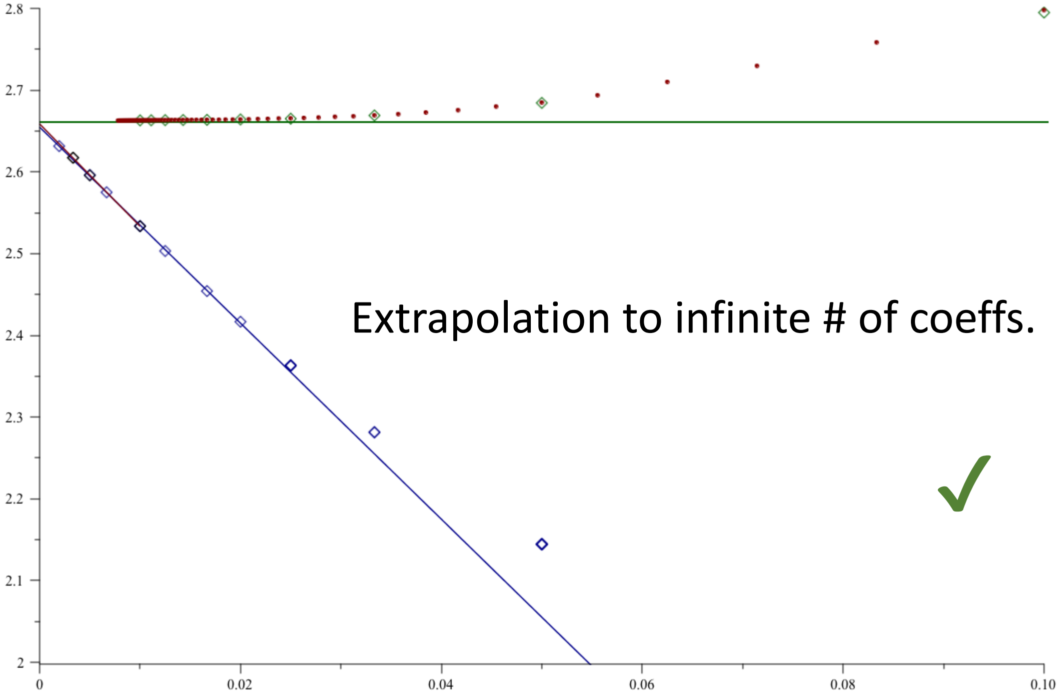
8,10,12,14,16,28 coefficients for K

**Note:** Amplitude at maximum has a pole at threshold (s=4). Here we avoid including that information and try to find the pole from the maximization procedure.

Primal 100, 200, 300 coeffs. for single disp.




Primal problem
Approaches from below



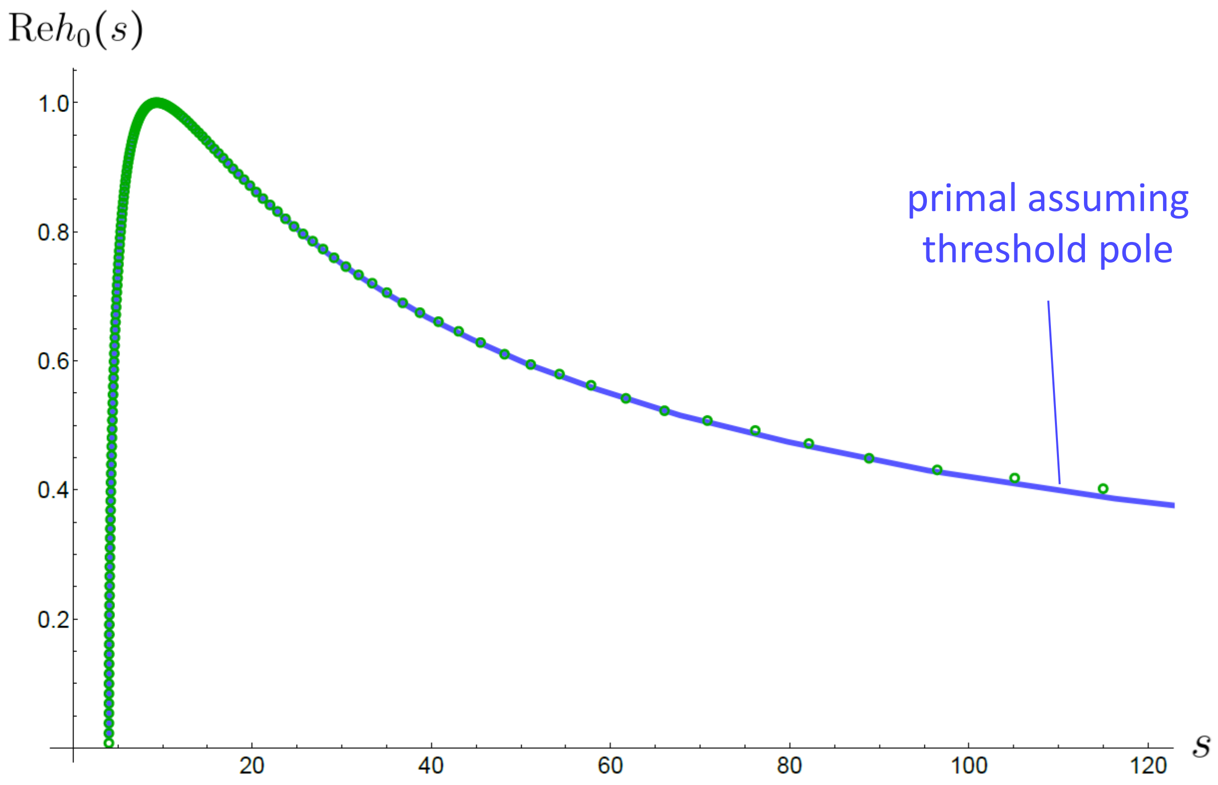

Extrapolation to infinite # of coeffs.

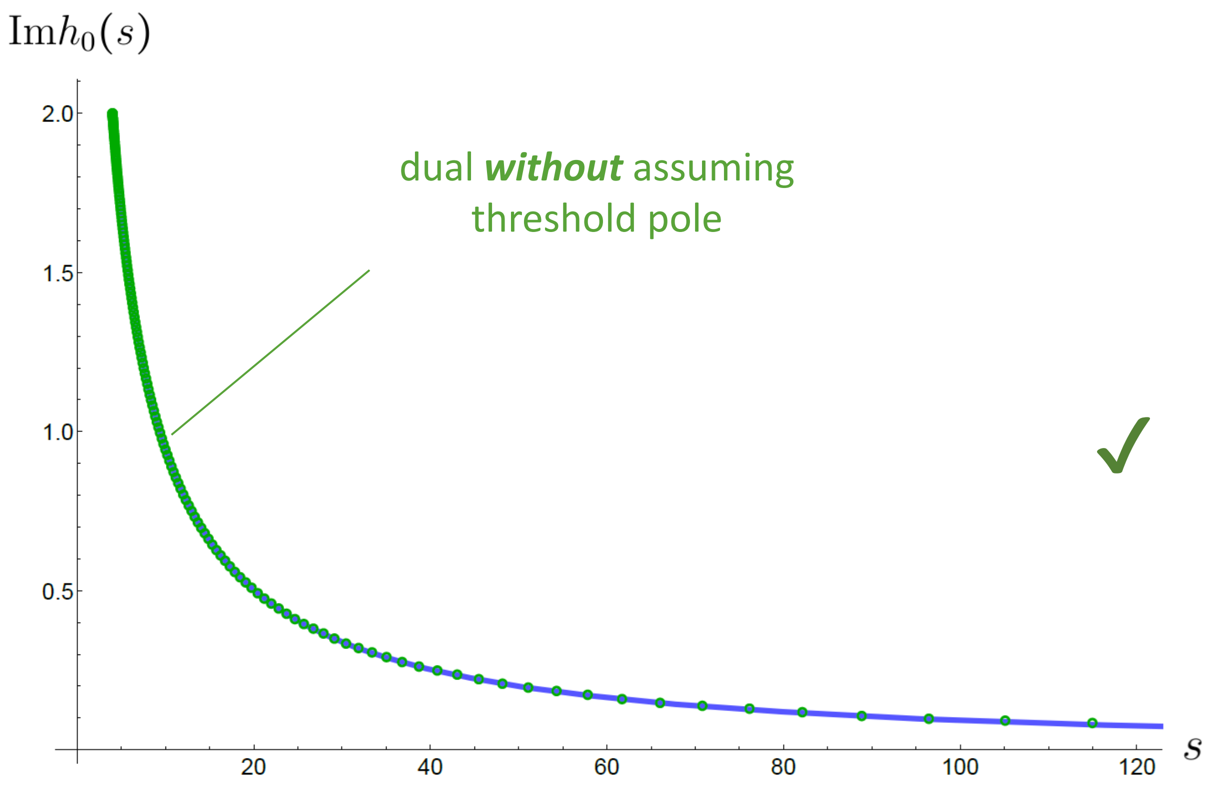
✓

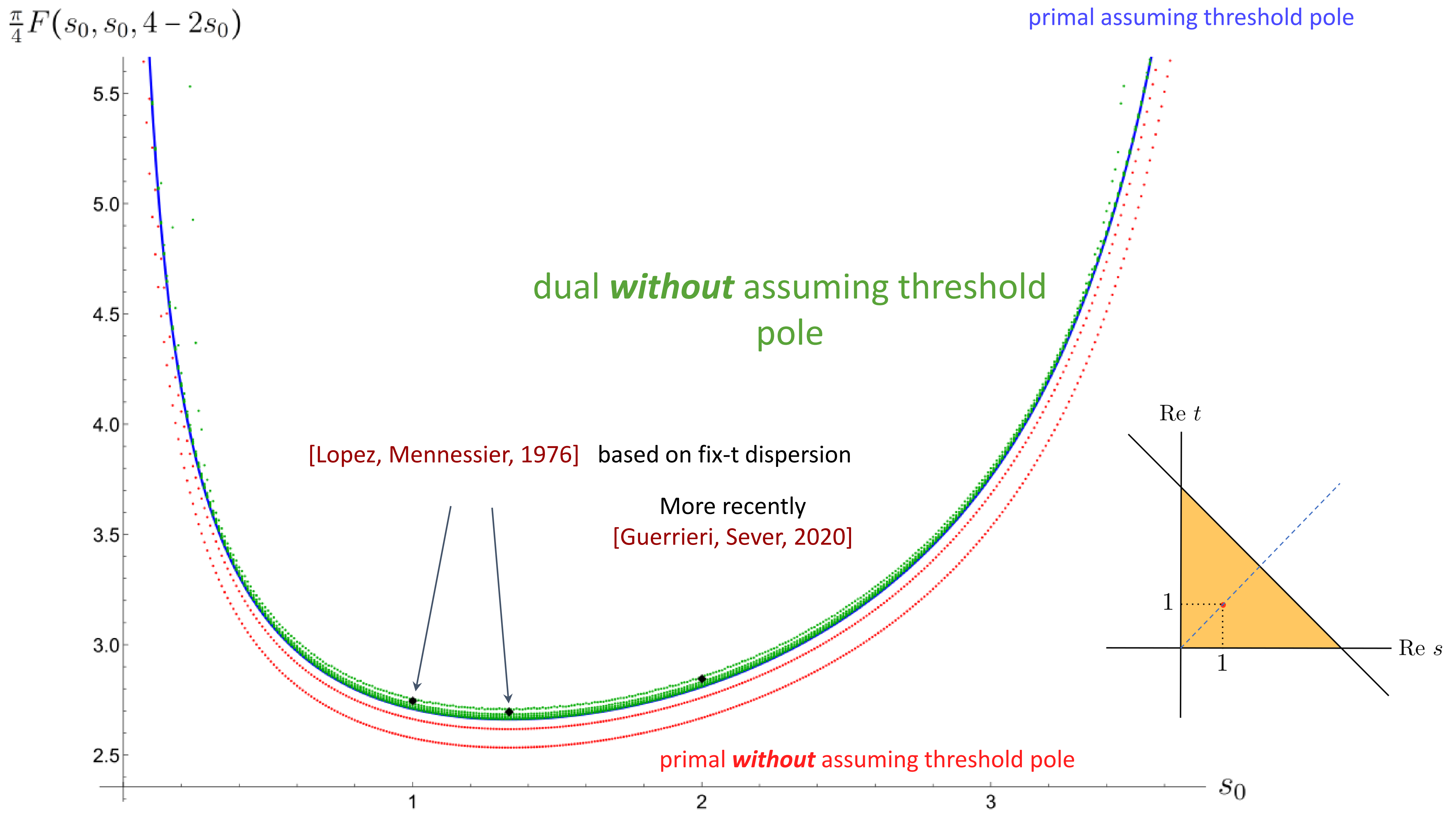
$s_0=t_0=u_0=4/3$

# S-wave: primal and dual

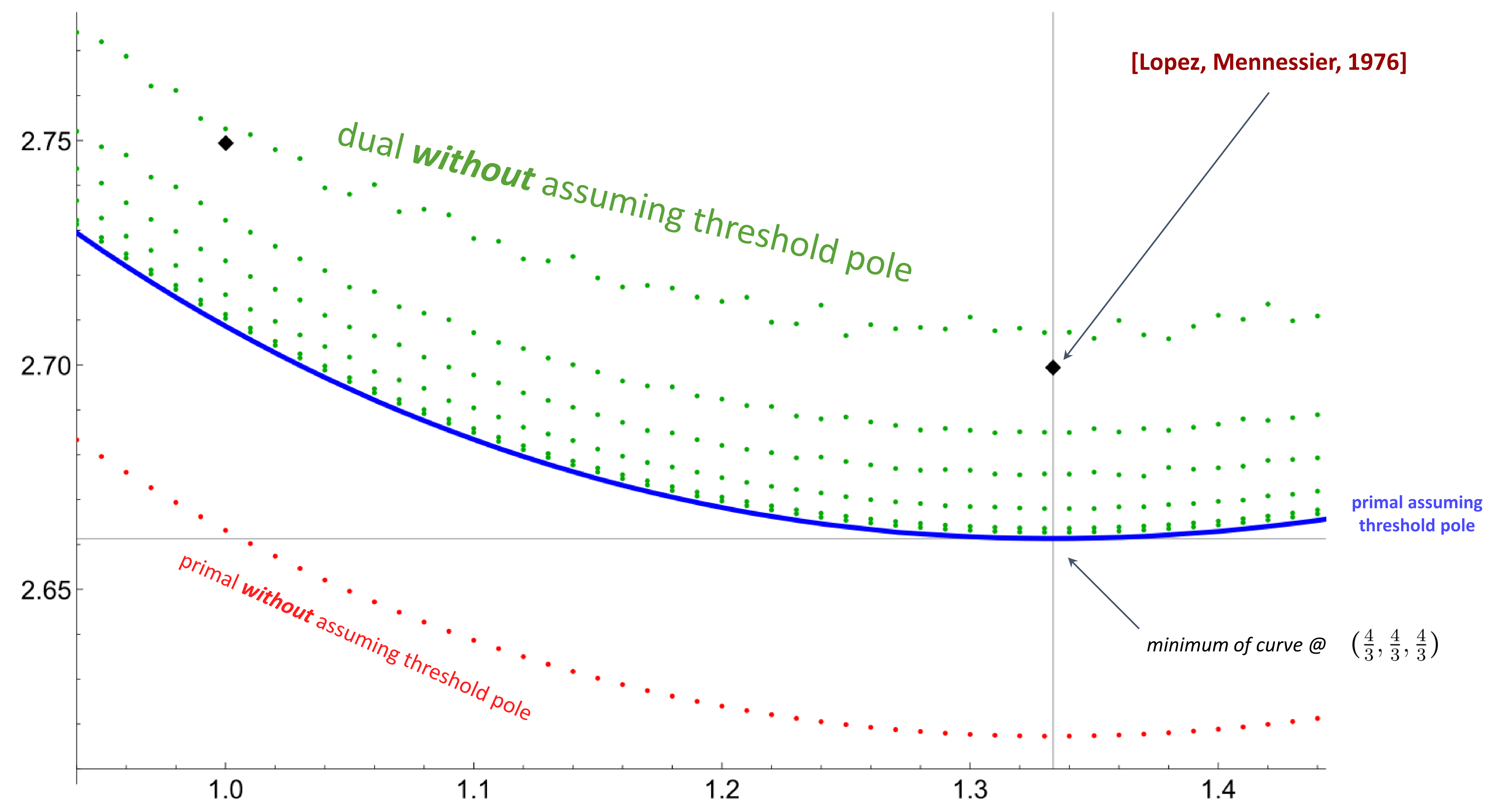
$$h_0(s) = -i\left(\frac{k_0(s)}{|k_0(s)|} - 1\right)$$



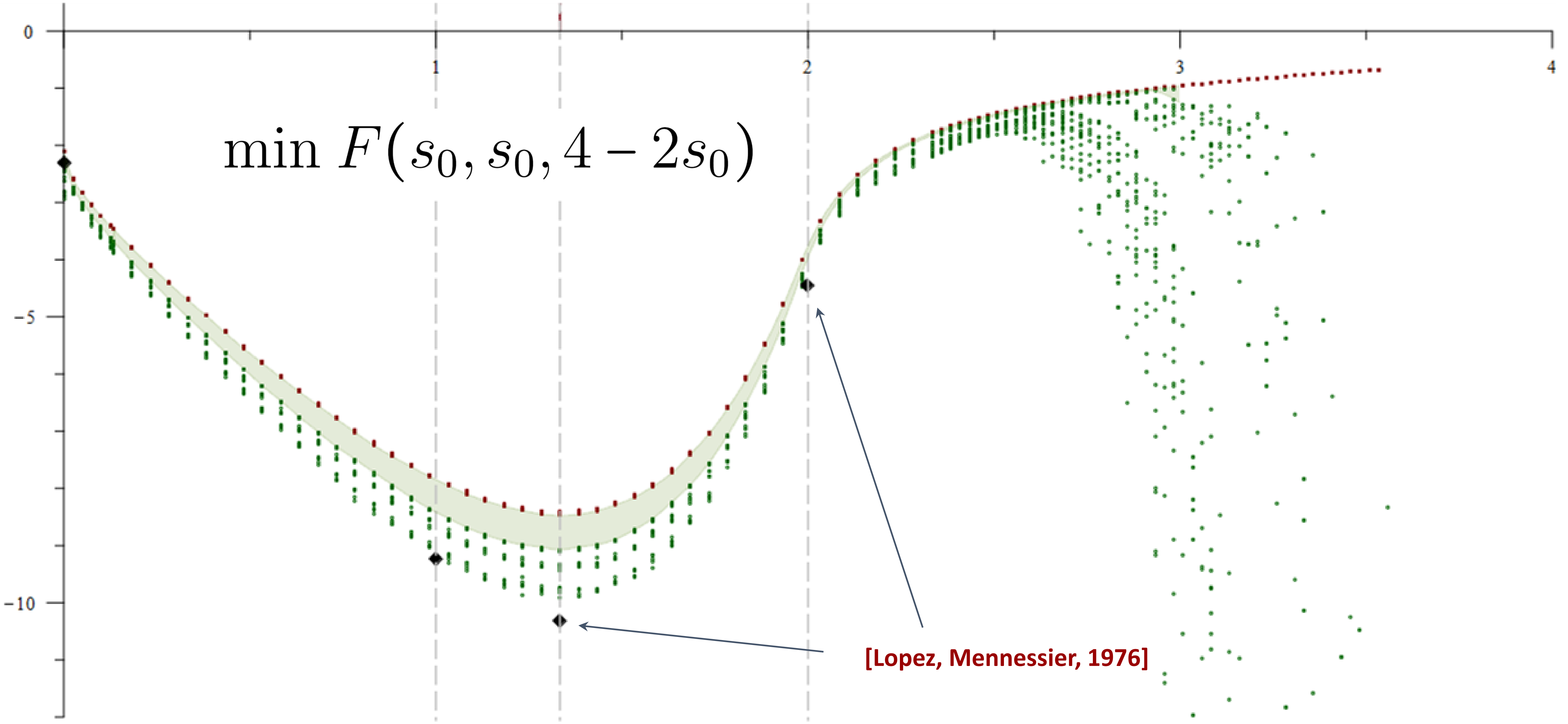





$\frac{\pi}{4}F(s_0, s_0, 4-2s_0)$
primal assuming threshold pole
dual *without* assuming threshold pole
[Lopez, Mennessier, 1976] based on fix-t dispersion
More recently
[Guerrieri, Sever, 2020]
primal *without* assuming threshold pole
Re $t$
Re $s$
1
1
$s_0$
5.5
5.0
4.5
4.0
3.5
3.0
2.5
1
2
3

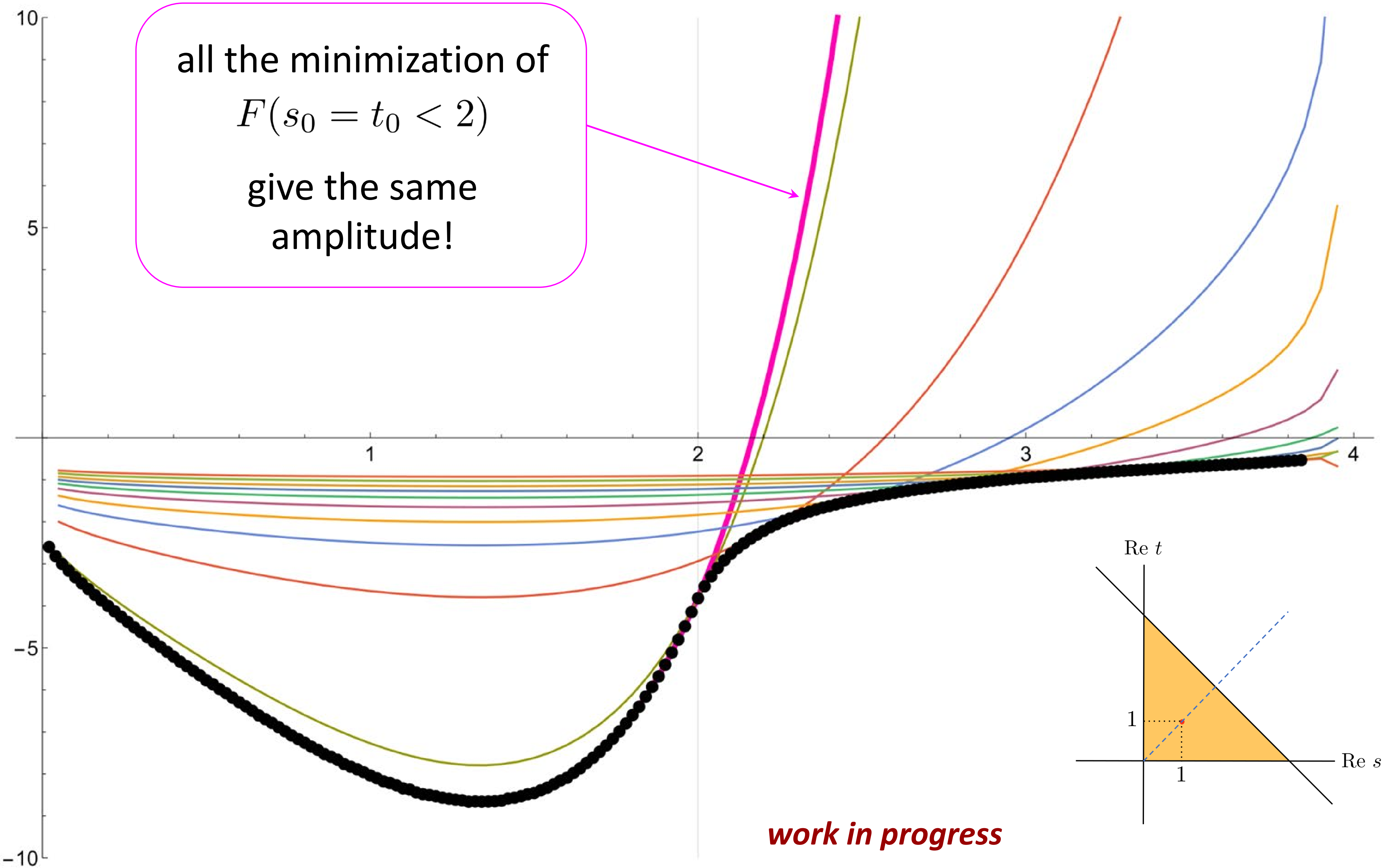


[Lopez, Mennessier, 1976]
dual without assuming threshold pole
primal assuming threshold pole
primal without assuming threshold pole
minimum of curve @ $\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$
2.75
2.70
2.65
1.0
1.1
1.2
1.3
1.4




*What are these extremal amplitudes?*
*What do they tell us about the space of pion physics?*

*work in progress*

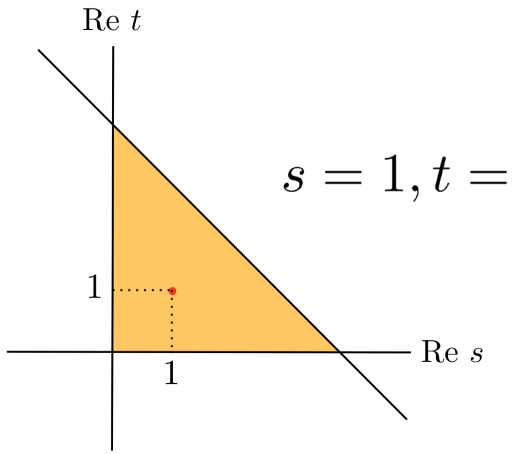


all the minimization of
$F(s_0 = t_0 < 2)$
give the same amplitude!
10
5
−5
−10
1
2
3
4
Re $t$
Re $s$
1
1
work in progress

## space of O(3) amplitudes
## (dual is a simple generalization)

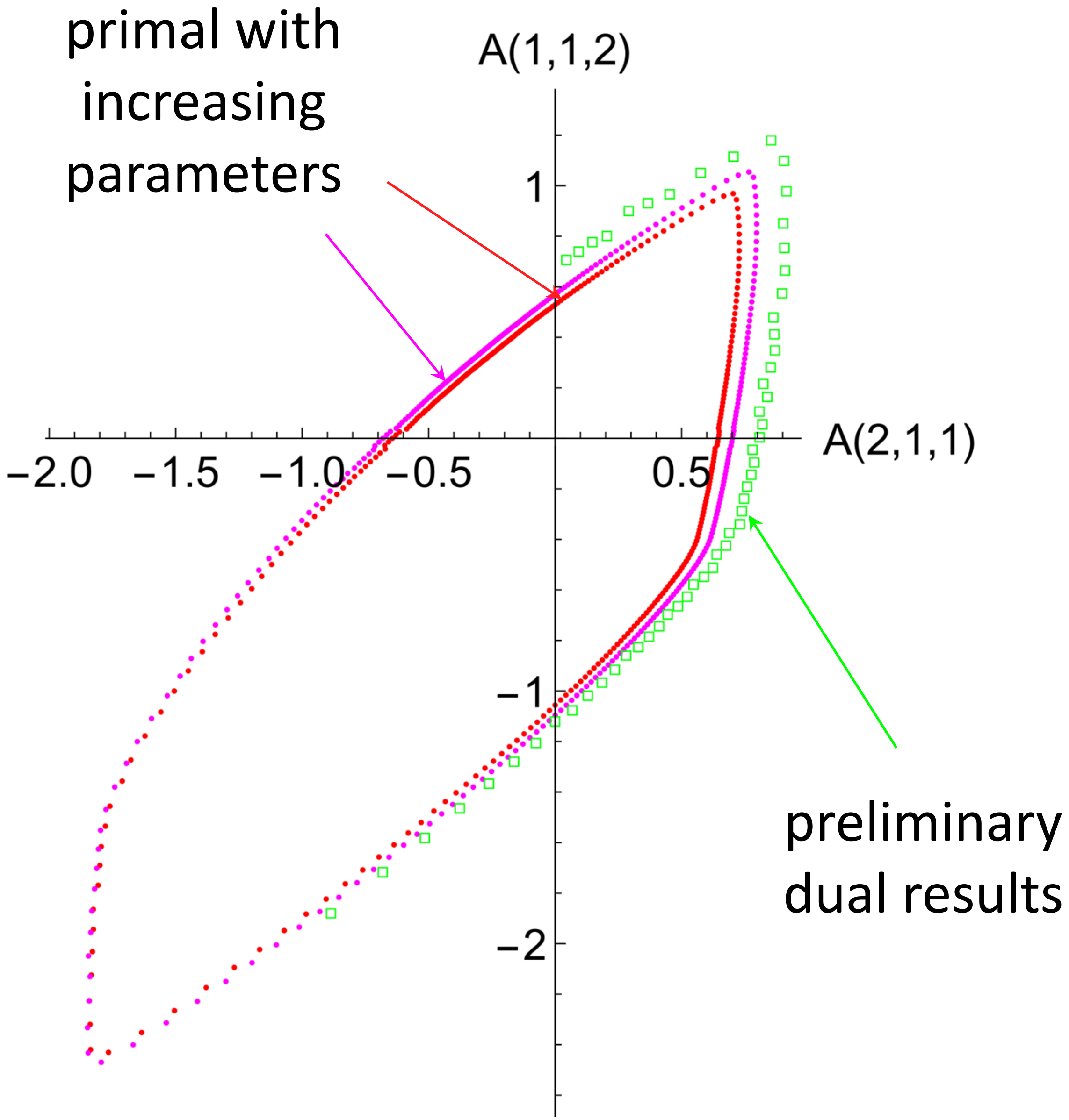
***work in progress***
with He and Murali

$$
\begin{aligned}
T^{I=0}(s,t,u) &= 3A(s,t,u) + A(t,s,u) + A(u,t,s) \\
T^{I=1}(s,t,u) &= A(t,s,u) - A(u,t,s) \\
T^{I=2}(s,t,u) &= A(t,s,u) + A(u,t,s)
\end{aligned}
$$




$$s = 1, t = 1, u = 2$$

$A(1,1,2)$ and $A(2,1,1)$




***detailed structures of the amplitudes to be examined!***

# Conclusions

- The S-matrix bootstrap, new version

- Define a field theory by finding distinguished points in the allowed space of S-matrices
- That space can be found by maximizing linear functionals in convex spaces: vertices are distinguished points.

- In 2d we find integrable models with no particle production.

- The O(N) model S-matrix and R-matrix from a convex maximization problem.

- Dual convex maximization problem

- 2d case: Upper bounds through generalized dispersion relations.
- 4d case: Upper bounds and better numerical methods

# Discussion points/further work.

- Particle creation?
  S-matrices at the boundary tend to saturate unitarity.

- Which theories are at vertices?
  (integrable theories with arbitrary parameters fill edges or faces)

- Incorporate UV properties in the low energy bootstrap.
  E.g. SU(N) + 2 quarks
  ?

- Do we need to introduce an extra scale in pion bootstrap (rho mass, scattering length)?,
  or do we need to consider nucleons?.

- Can we use a more physical way of regularizing the dual problem ($a_2$)?.

- Relation with EFT approach, Lagrangian/Hamiltonian (recent paper by Albert, Rastelli ).