LECTURE 3

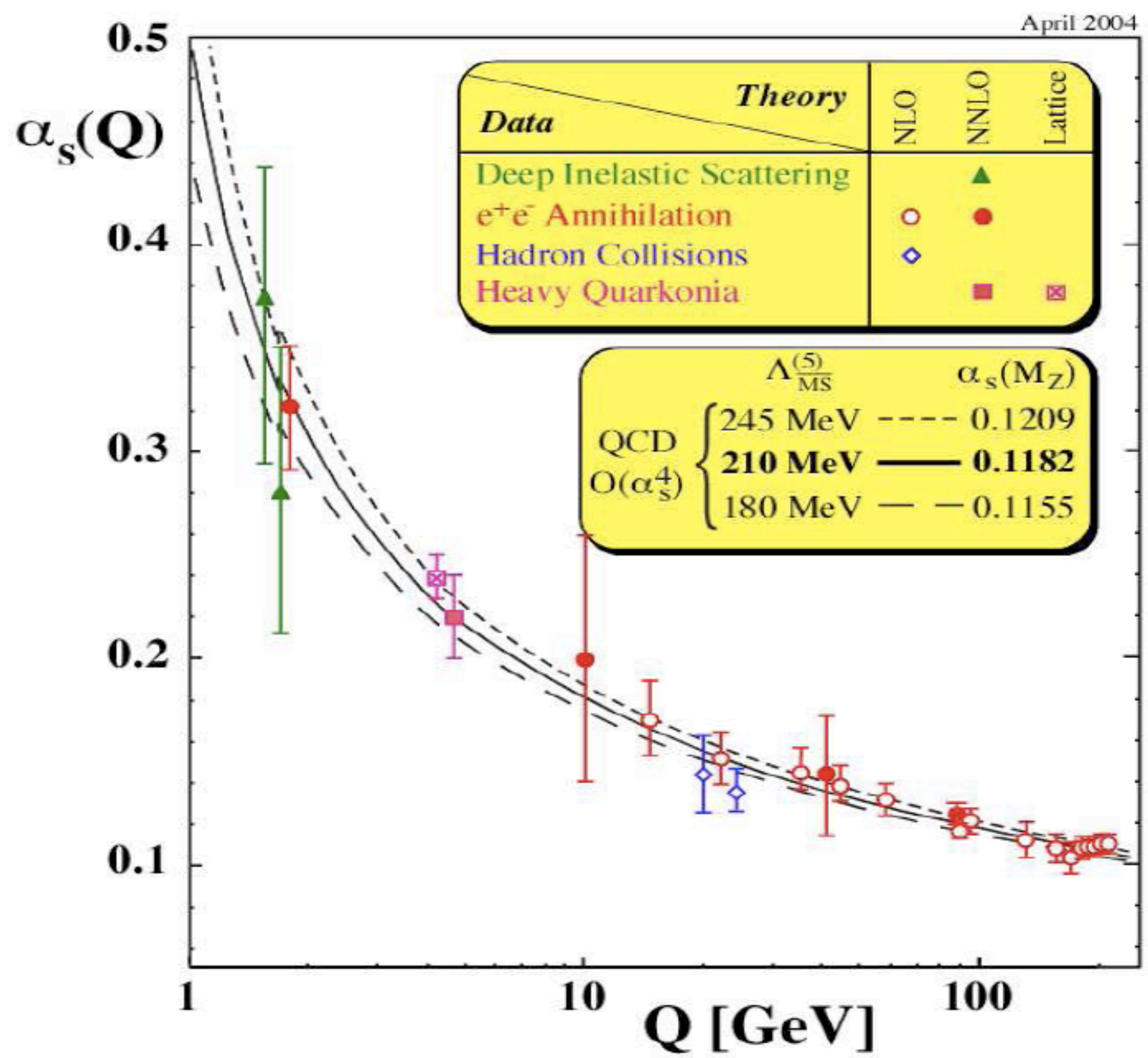
The idea of adiabatic continuity
Its connection to large-N volume independence
Semi-classical solution of Deformed YM, mass gap
IR-renormalons vs. bions?

Mithat Ünsal

North Carolina State University

Asymptotic Freedom

- asymptotic freedom
- Short distance: Weakly coupled, calculable...
- Long distance, strongly coupled. (Lattice works, analytical methods gloomy)



from: S. Bethke, hep-ex/0407021

- Can we find a regime of asymptotically free gauge theories where the NP dynamics become calculable?

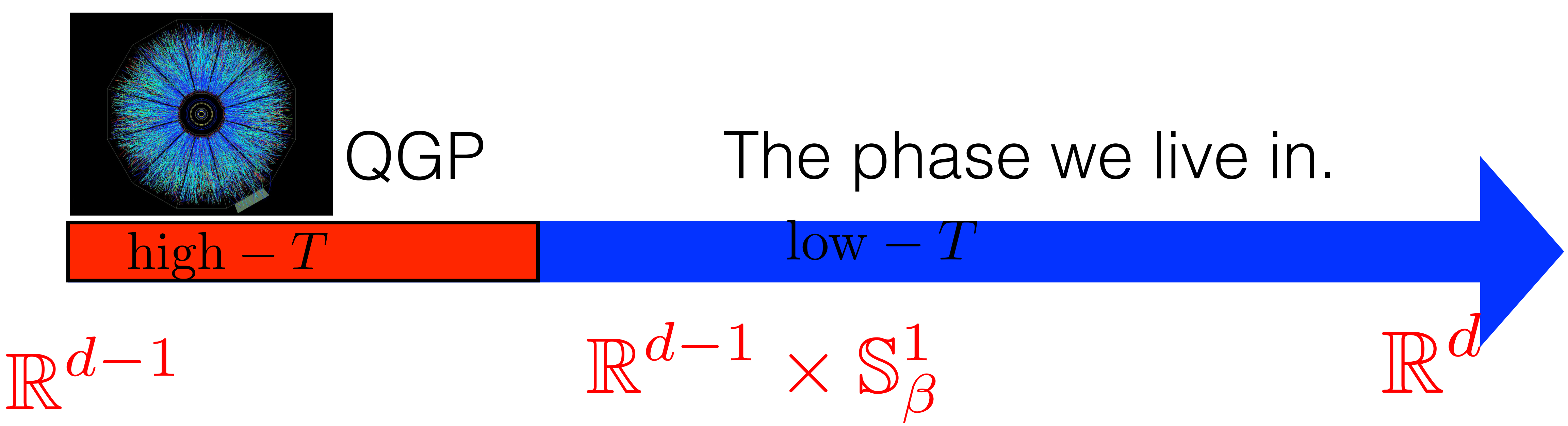
Adiabatic continuity and analyticity for YM? Is it possible?

- We first want a (semi-classically) calculable regime of field theory, say of Yang-Mills or QCD. Of course, everyone want this. **But is this possible at all?**

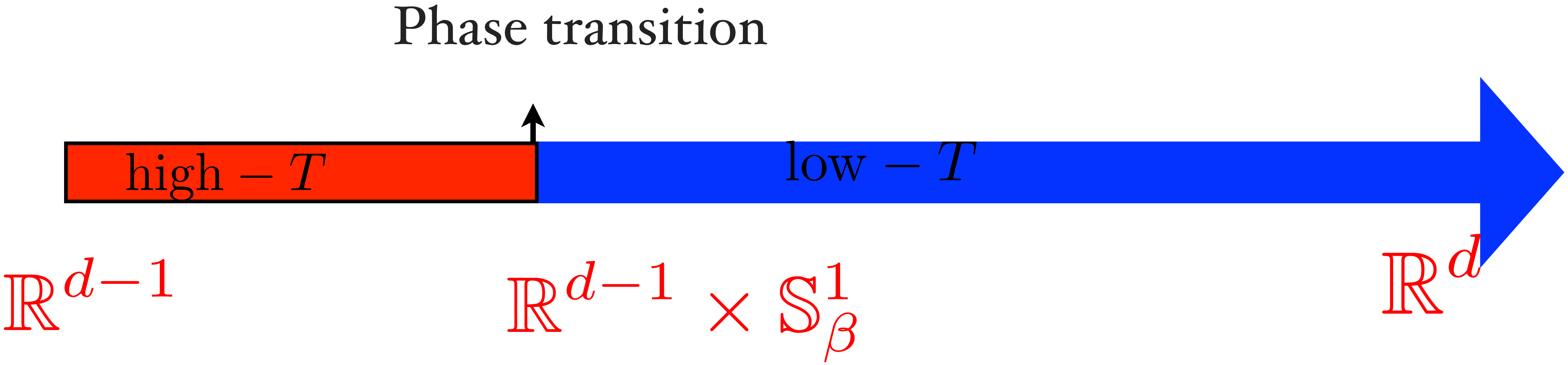
- It is NOT known if such a framework exits on R4. In fact, theory becomes strongly coupled at longer distances.

- Supersymmetry does not help!!

- Consider these theories on four manifold R3 x S1, and study their dynamics as a function of radius. At small-radius, the theory is weakly coupled (thanks to asymptotic freedom) at the scale of the radius. But the theory is non-analytic as a function of radius, there is a phase transition.

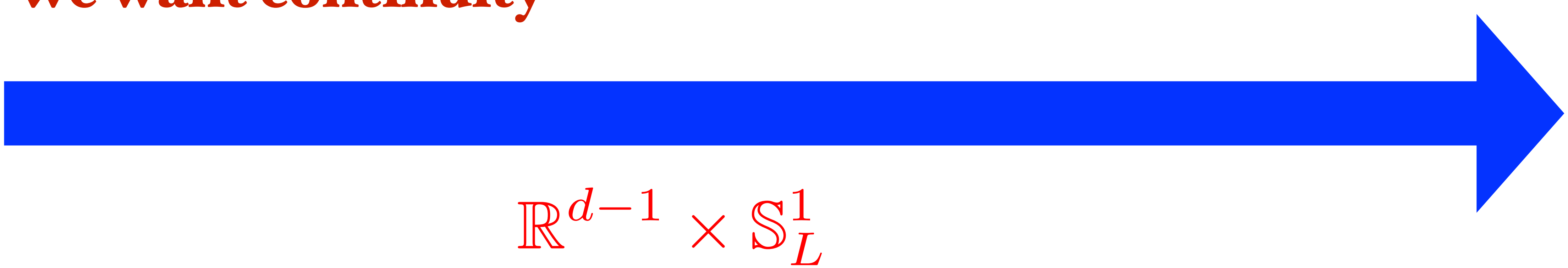


The idea of adiabatic continuity



Thermal: Rapid crossover/phase transition at strong scale

We want continuity



Adiabatic continuity and analyticity

Adiabatic continuity in non-susy theories is a spin-off of a brilliant idea by Eguchi and Kawai (82), called large-N reduction or volume independence.

What does EK say? It says something far more stronger than continuity, it implies volume independence, observable being independent of compactification radius at large-N.

But it is tricky to achieve EK.

Large N volume independence or
"Eguchi-Kawai reduction" or "large-N reduction"

Theorem: $SU(N)$ gauge theory on toroidal compactifications of $\mathbb{R}^4$ to four-manifold $\mathbb{R}^{4-d} \times (S^1)^d$

No volume dependence in leading large N behavior of topologically trivial single-trace observables (or their connected correlators)

provided

there are **no phase transitions as the volume of the space is shrunk.** More technically, **no spontaneous breaking of center symmetry** or translation invariance

Proof: Comparison of large N loop equaions (Eguchi-Kawai 82) in lattice gauge theory or \ N=∞ classical dynamics (Yaffe 82)

The **only** problem was that no-one was able to find **any** example of gauge theory in which **"provided"** holds. (and perhaps violating causality, an example already existed at the time EK was written. This is understood only 25 years later.)

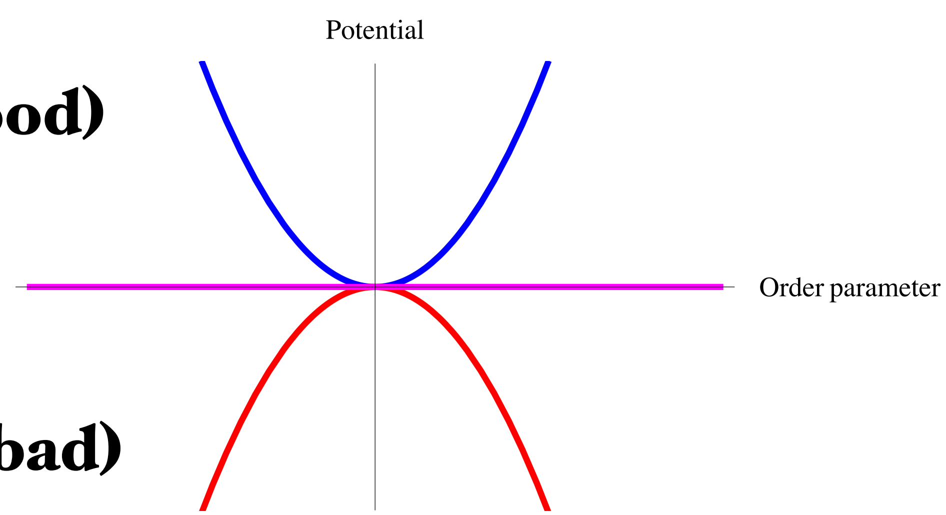
Stumbling block

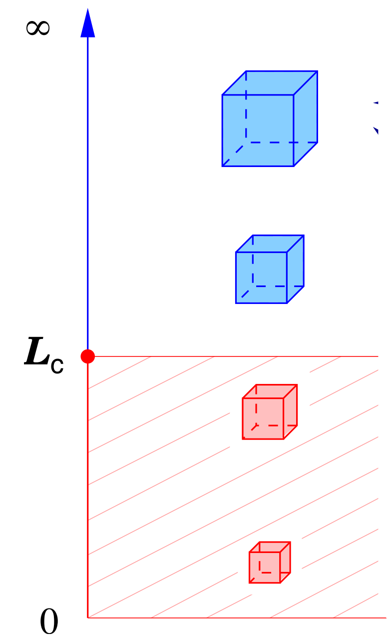
- Because of the attractiveness of the idea, much effort has been devoted. It was one of the hot subjects in mid-80's.
- However, there was always a phase transition when the space shrunk to small volume.
- Technically, an effective potential calculation in terms of Wilson lines (used to determine the phase of the small volume theory) gave a **negative** sign for **all** gauge theories. And we needed a positive sign! People gave up.

Stability (good)

Marginal

Instability (bad)





Instability

80's: EK, QEK, TEK.
Eguchi, Kawai, EK, Briilliantt, but fails

Gonzalez-Arroyo, Okawa, TEK, Failed, and REVIVED. (Many deep connections to non-commutative QFT, and recent works on TQFT coupling to QFT.)

Bhanot, Heller, Neuberger, QEK, Fails

Gross, Kitazawa, (YM Beta function from matrix model assuming working reduction. Clever.)

Yaffe,

Migdal, Kazakov,
Parisi et.al.
Das, Wadia, Kogut,
+ 500 papers ..... but no single working example!

Stumbling block

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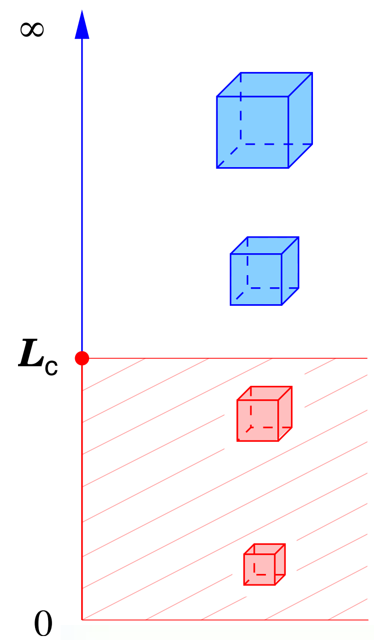

Instability

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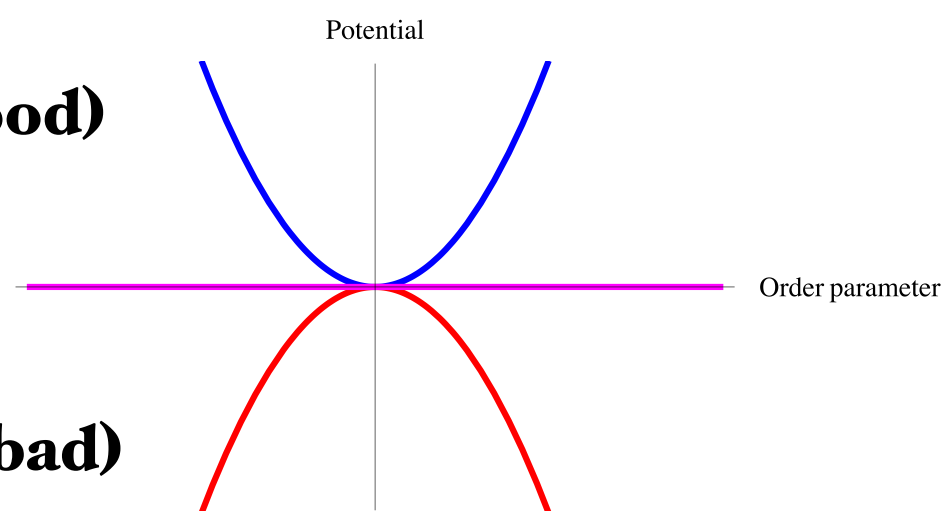


More technically,

$\mathrm{Yang-Mills\ on\ }\mathbb{R}^3 \times S^1$

circumference L

- Z_N center symmetry, order parameter = Wilson line Ω

$g(x+L) = hg(x), \qquad h^N = 1$ Aperiodic gauge rotations, $h \in Z_N$ ‘t Hooft

$$\mathrm{tr}\Omega(x, x+L) \to h\ \mathrm{tr}\Omega(x, x+L)$$

- $L > L_c$: unbroken center symmetry

$$\langle \mathrm{tr}\ \Omega^n \rangle = 0$$

confined phase

- $L < L_c$: broken center symmetry

$$\langle \mathrm{tr}\ \Omega^n \rangle \neq 0$$

deconfined plasma phase failure of EK reduction

Gauge holonomy potential

$$V[\Omega] = -\frac{2}{\pi^2\beta^4}\sum_{n=1}^{\lfloor N/2 \rfloor}\frac{1}{n^4}\left|\mathrm{tr}\left(\Omega^n\right)\right|^2$$

Gross, Pisarski, Yaffe, 1981

In Tutorial, Aleksey will provide a very intuitive and elementary derivation of this formula by using methods from stat. mech.

Minimum at center-broken configuration. The value at min is the Stefan-Boltzmann law for gluons.

$$F = -\frac{\pi^2}{45}T^4(N^2-1)$$

At high-temperature YM theory, **this is inevitable and there is no room for negotiation.** This is also true in any QCD-like theory, and there is no hope here.

Evading the stumbling block(s)

In 2006, I realized that the analog of the effective potential calculation in supersymmetric gauge theory always gave zero.

But that requires using periodic boundary conditions for fermions. I was perfectly happy with it, and interpret it as **non-thermal compactification,** and realize that what you are calculating is not thermal partition function, but

$$\widetilde{Z}(L) = \mathrm{tr}[e^{-LH}(-1)^F]$$

What I did not know then: It was considered as a big "sin" to use periodic b.c. at least in a large-portion of non-supersymmetric QCD community.

At the heart of the super-symmetric cancelation was following identity:

Evading the stumbling block(s)

In 2006, I realized that the analog of the effective potential calculation in a supersymmetric gauge theory gave zero. At the heart of the cancelation was following identity:

$$-1+1=0$$

More precisely,

$$-1\times(\text{stuff})+1\times(\text{same stuff})=0$$

The crucial point: +1 appears due to the boundary conditions, and not supersymmetry!

Immediately, we deduce:

$$-1+N_f>0 \qquad \text{for } N_f>1$$

Our simple calculation was the first positive sign in such a calculation.
All earlier calculations were done for a specific (thermal) boundary condition.

Gauge holonomy potential QCD(adj) Nf-flavor

$$V[\Omega] = (N_f - 1)\frac{2}{\pi^2\beta^4}\sum_{n=1}^{\lfloor N/2 \rfloor}\frac{1}{n^4}\left|\mathrm{tr}\left(\Omega^n\right)\right|^2$$

Kovtun, MU, Yaffe, 2007. Showed that QCD(adj) satisfies volume-independence, Eguchi-Kawai dream naturally.

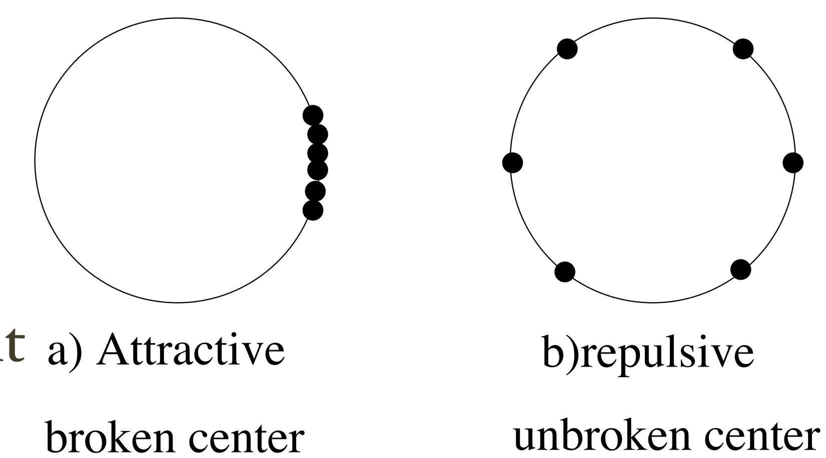
This sign flip probably gave birth to one of the most promising windows to non-perturbative QCD. This is what I thought in 2007, and I will describe later in this talk. I believe it endures the test of time. And in the longer run, we will appreciate this sign flip even more.

Dimensional Reduction ? No!

- small L, asymptotic freedom, heavy, weakly coupled KK modes

- usual case: broken center symmetry
  $\langle \mathrm{tr}\, \Omega \rangle \neq 0 \Leftrightarrow$ eigenvalues clump

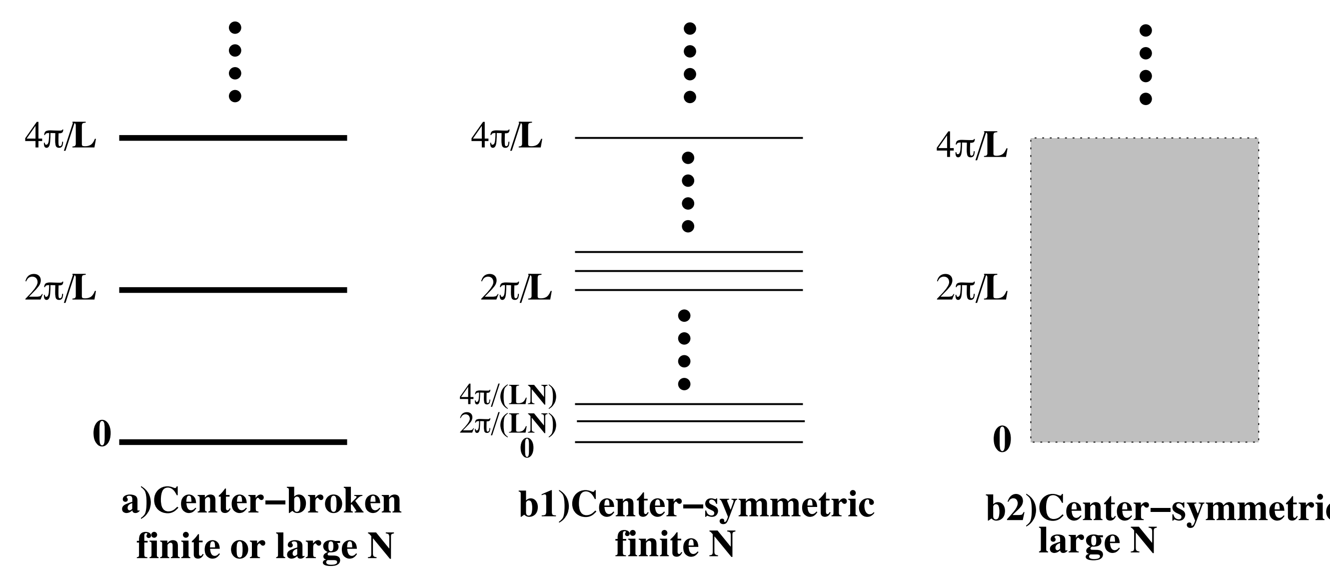
  $m_{KK} = 1/L, 2/L, \ldots,$
  perturbative control when $L\Lambda << 1$
  integrate out $\Rightarrow$ $3d$ effective theory, L-dependent



- center-symmetric case:
  $\langle \mathrm{tr}\, \Omega \rangle = 0 \Leftrightarrow$ eigenvalues repel

  $m_{KK} = 1/NL, 2/NL, \ldots,$
  perturbative control when $NL\Lambda << 1$

topological defects (instantons),
mass gap, confinement, later......

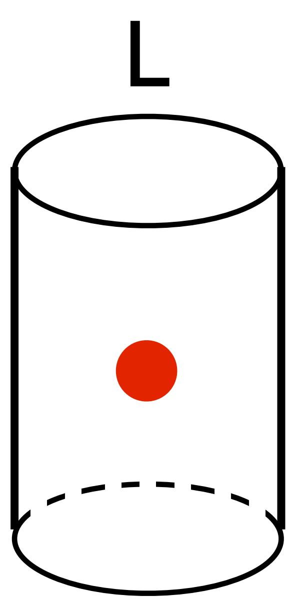


What is volume (in)dependence?

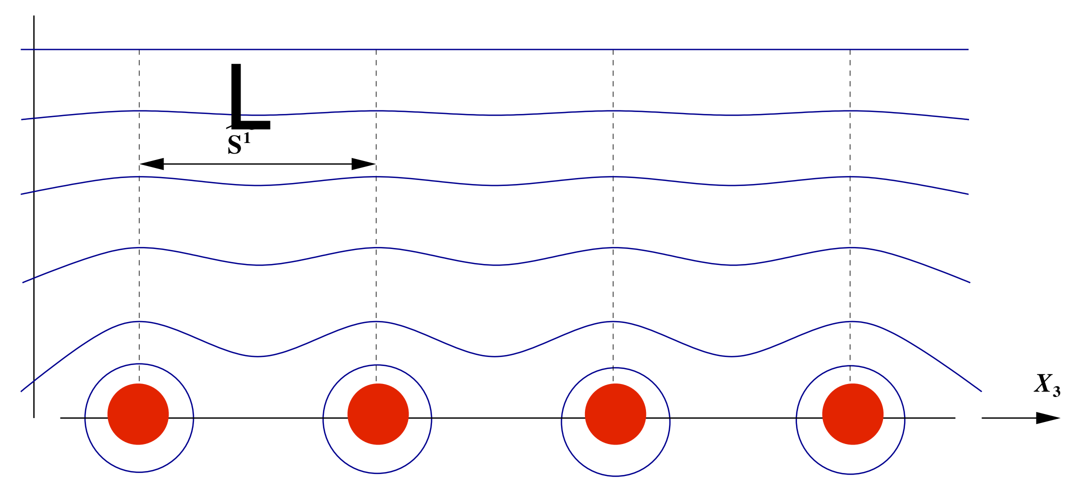
with super-simple electrostatic analogy

- Consider a point charge in R3. Its potential is 1/r.
- Now, compactify one of the dimensions to a circle with size L. Space is R2 x S1.

R2 x S1



by method of images



$r >> L \rightarrow \ln r.$

$r < L \rightarrow 1/r$

- The characteristic length at which the potential (interaction between charges) changes from 3d behavior to 2d behavior is L. Intuitive!

- By compactify more dimensions down to a space with size L, and using method of images, we obtain

The potential of a point charge in d-dimension: Gauss' law

$$3d: \qquad \frac{1}{r} = \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$2d: \qquad \log r = \log \sqrt{x_1^2 + x_2^2}$$

$$1d: \qquad |r| = |x_1|$$

Whereas volume independence demands

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$$2d: \qquad \frac{1}{r} = \frac{1}{\sqrt{x_1^2 + x_2^2}}$$

$$1d: \qquad \frac{1}{r} = \frac{1}{\sqrt{x_1^2}}$$

Sounds outrageous.

Certainly wrong in electrodynamics, (or U(1) gauge theory), where our intuition is based on.

QCD(adj) on $\mathbb{R}^3 \times S^1$

$N_f \geq 1$ massless adjoint rep. fermions
periodic boundary conditions ➡ stabilized center symmetry

$$\widetilde{Z}(L) = \mathrm{tr}[e^{-LH}(-1)^F]$$

$$Z = Z_B + Z_F$$
$$\widetilde{Z} = Z_B - Z_F$$

Susy-theory: **Supersymmetric Witten Index, useful.**
Non-susy theory: **Twisted partition function, probably as useful!**

$$V_{1-\text{loop}}[\Omega] = \frac{2}{\pi^2 L^4} \sum_{n=1}^{\infty} \underbrace{\frac{1}{n^4}(-1+N_f)}_{m_n^2} |\mathrm{tr}\,\Omega^n|^2$$

$m_n^2 < 0$ instability, “calculations between 1980-2007”

$m_n^2 = 0$ Supersymmetric case, $N_f = 1$, marginal,

$m_n^2 > 0$ QCD(adj), $N_f > 1$, stability Kovtun, Unsal, Yaffe,07

This sign flip probably gave birth to one of the most promising windows to non-perturbative QCD. Still ongoing work.

Can we achieve center-stability in YM in small-L?

$$S^{\mathrm{YM}^*} = S^{\mathrm{YM}} + \int_{R^3 \times S^1} P[\Omega(\mathbf{x})] \qquad P[\Omega] = A \frac{2}{\pi^2 L^4} \sum_{n=1}^{\lfloor N/2 \rfloor} \frac{1}{n^4} \left| \mathrm{tr}\,(\Omega^n) \right|^2$$

- Motivated by QCD(adj), it is reasonable to proposed a double-trace deformation that prevents center-breaking. (Yaffe, MU, 2008).

- This is a large deformation of the action, not a small perturbation in the sense of large-N counting. We are changing the action with something as large as action itself, and expecting Hilbert space to remain invariant!

- This of course sounds absurd.

- But there is something deeper here!

- Indeed, deformation is O(N^2). But after it does its job of stabilization, its effect on the dynamics is N-suppressed. How is this possible?

Loop equations for Lattice YM theory

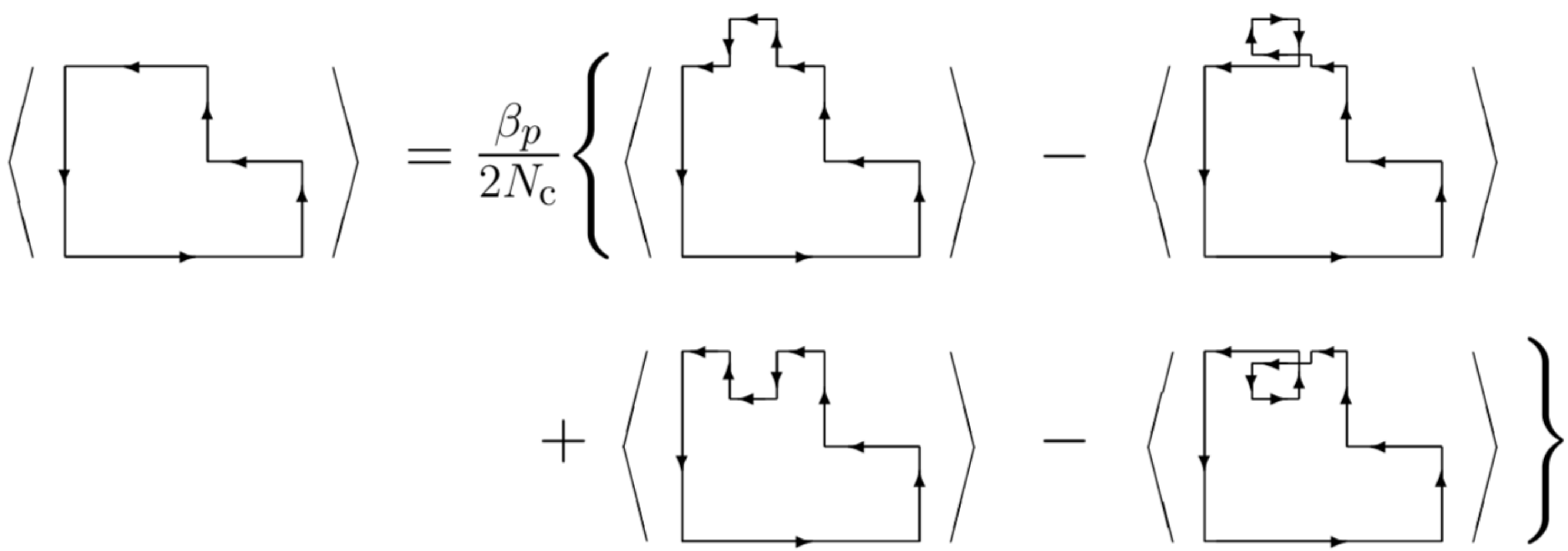
$$\langle \delta S \cdot \delta W[C] \rangle + \langle \delta^2 W[C] \rangle = 0\,.$$

Schwinger-Dyson equation for Wilson loops

$$\tfrac{1}{2}|C|\langle W[C]\rangle = \sum_{\ell \subset C} \sum_{p | l \subset \partial p} \frac{\widetilde{\beta}}{4} \Big[\langle W[(\overline{\partial p})C]\rangle - \langle W[(\partial p)C]\rangle \Big] + \sum_{\mathrm{self-intersections}} \mp \langle W[C']\rangle \, \langle W[C'']\rangle\,.$$

Loop equations

Makeenko-Migdal equation for W(C)



Loop equations, effects of deformation

$$\langle \delta(\Delta S) \cdot \delta W[C] \rangle = \sum_{k \neq 0} b_k[C] \, \langle W[\Omega^k C] \, W[\Omega^{-k}] \rangle \, ,$$

Deformation changes action and loop equations.

$$\langle W[\Omega^k C] W[\Omega^{-k}] \rangle = \langle W[\Omega^k C] \rangle \, \langle W[\Omega^{-k}] \rangle + O(1/N^2) \, .$$

Deformation operator guarantees center stability, hence, ensures large-N factorization.

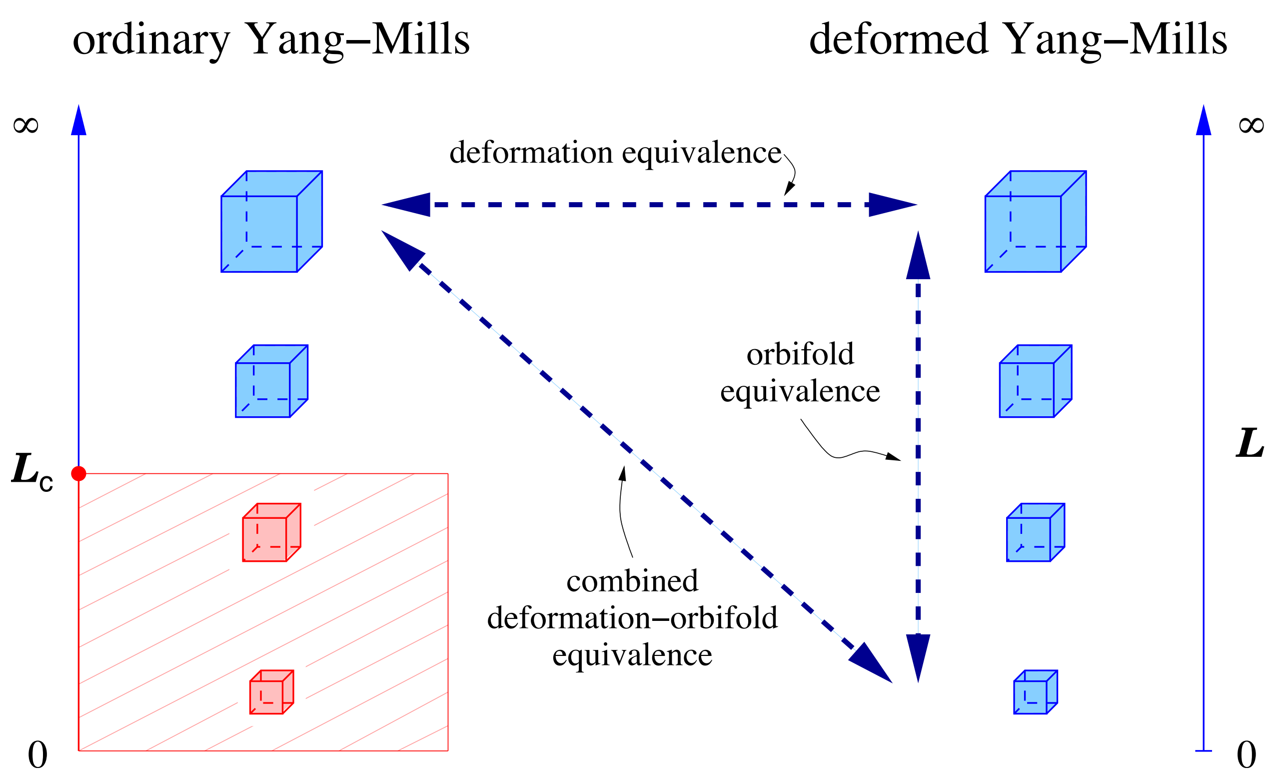
Each factorized piece is charged under center. Hence, by center stability, its vev is zero.

$$\langle \delta(\Delta S) \cdot \delta W[C] \rangle = O(1/N^2) \, .$$

Its effect on observables is $1/N^2$

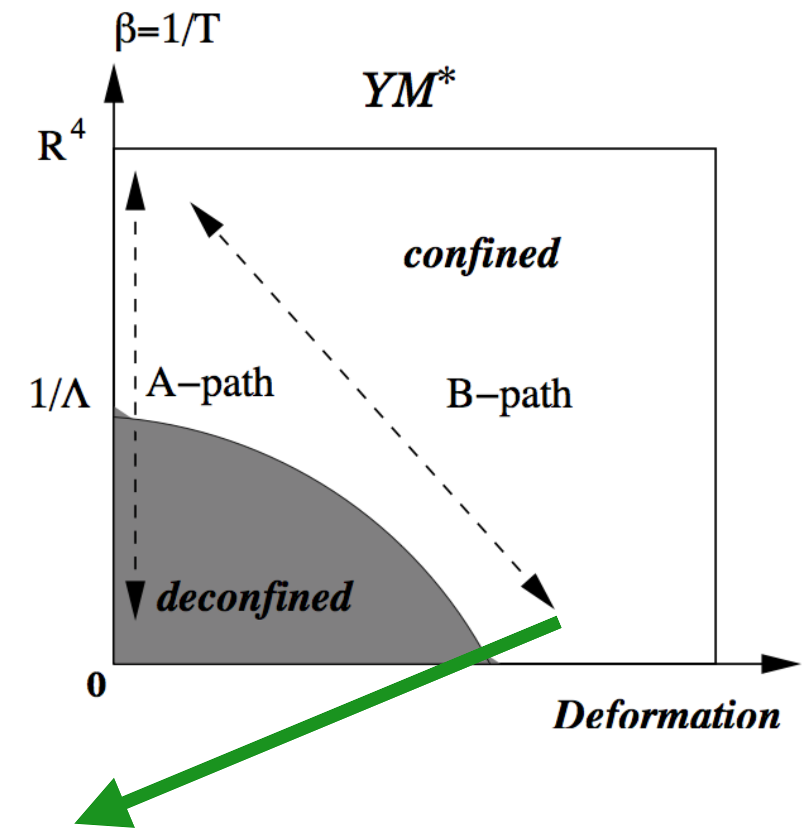
- Like a good Samaritan, it does the good deed, and you do not even know it existed (as Gabriele Veneziano insightfully put it in 2009.)

Large-N:exact volume independence



- (w/ Yaffe 2008)

Finite-N: analiticity or continuity



We can now do reliable semi-classics here, and it is continuously cconnected to YM on R4.

This is the reason why large-N deformed YM is not a model, but YM itself. One just removes the deconfined phase.

Interestingly, later on, I learned that Joe Polchinski has one paper on EK. And there, he proposes an analytic continuation of confined phase. The above construction turns out to be an explicit realization of that idea.
(1991) High temperature limit of the confining phase

Dynamics of Deformed Yang-Mills on

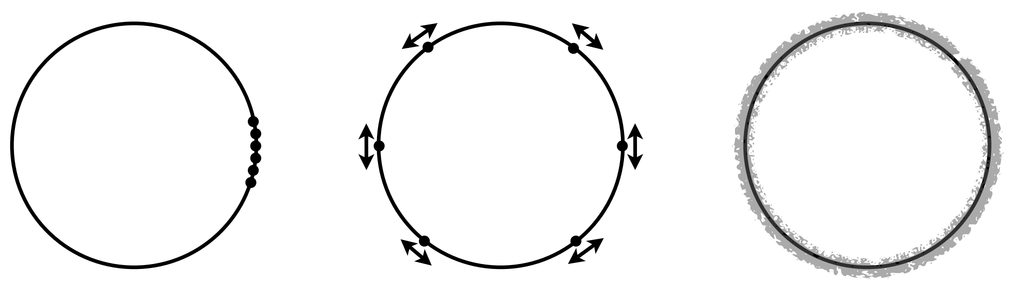
$$\mathbb{R}^3 \times S^1$$

Abelianization and abelian duality

$$SU(N) \to U(1)^{N-1}$$

Similar to Polyakov model in 3d (1974) and Seiberg-Witten in 4d (1994), dynamics abelianize, but via a compact group valued field

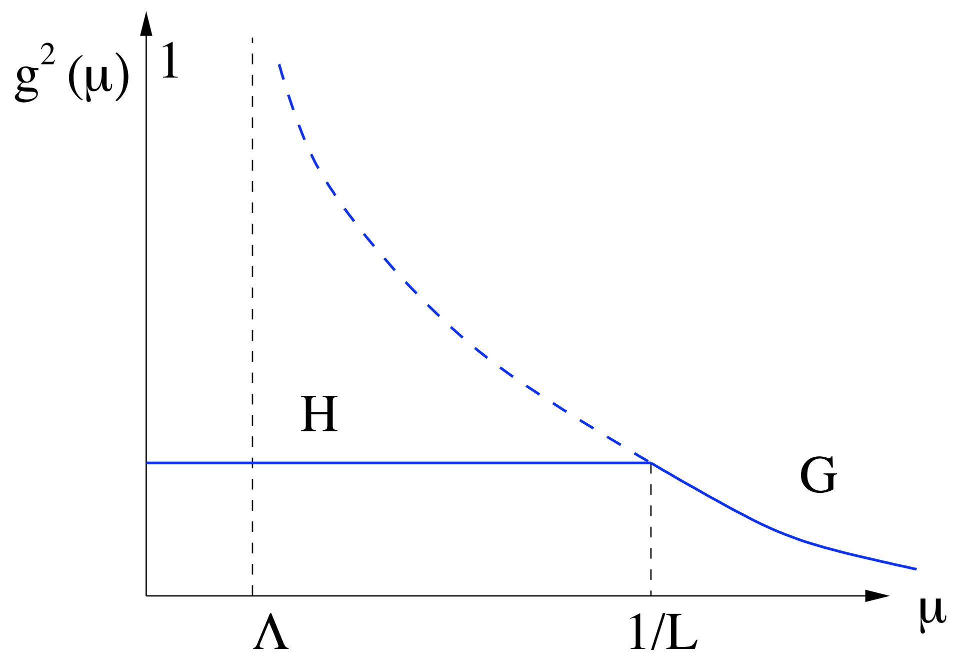
Three types of holonomy



(a) center broken

(b) center-stable weak coupling

(c) center-stable strong coupling



$$L = \frac{1}{4} F_{\mu\nu}^2 \longleftrightarrow \frac{1}{2}(\partial_\mu \sigma)^2$$

Gapless to all orders in perturbation theory.
How about NP-effects?

Duality

In 2+1 d, photon has just one polarization, one degree of freedom. It is dual to a scalar.

$$B = \partial_0 \sigma$$
$$E_x = \partial_y \sigma$$
$$E_y = -\partial_x \sigma$$

$$L = \frac{1}{4} F_{\mu\nu}^2 \longleftrightarrow \frac{1}{2} (\partial_\mu \sigma)^2$$

Maxwell term

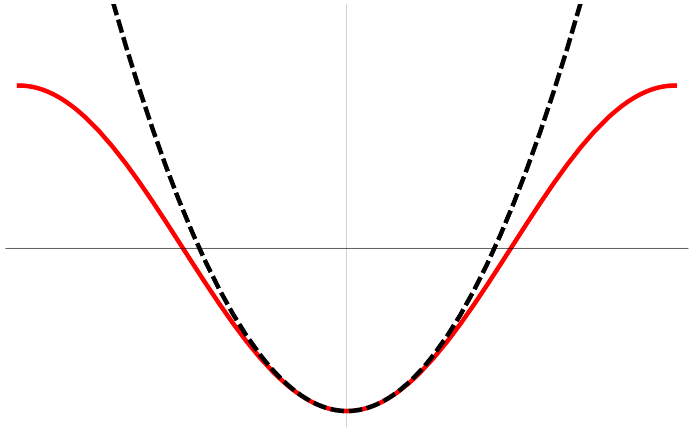
The proliferation of monopoles generate monopole operators in Lagrangian.

$$L = \frac{1}{2}(\partial\sigma)^2 - e^{-S_0}(e^{i\sigma} + e^{-i\sigma})$$

Monopole operators: Contribution of instanton amplitude to the effective lagrangian.

Expanding the cos potential to quadratic order,

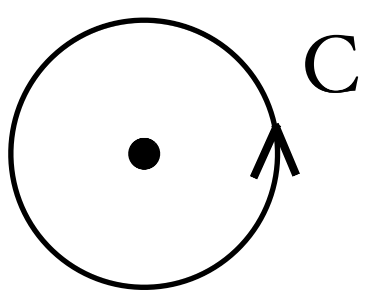
$$L^{\text{small fluc.}} = \frac{1}{2}(\partial\sigma)^2 + e^{-S_0}\sigma^2$$



Inverse Debye length = mass gap

$$B = \partial_0 \sigma$$
$$E_x = \partial_y \sigma$$
$$E_y = -\partial_x \sigma$$

By using this relation, we can show that



$$\int_C d\sigma = Q_e$$

a vortex in the sigma field correspond to an electrically charged particle in the original theory.

Confinement of electric charges mean that the work required to separate two test charges grows linearly in separation as (tension) x L!

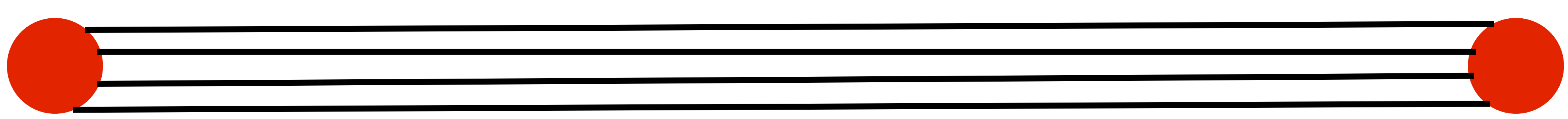
If one ignore monopoles, the interaction between a vortex and anti-vortex is log R.

$$\sigma(\theta) = Q_e \theta \Rightarrow \vec{E} = \frac{Q_e}{r}\hat{e}_r, \qquad V(R) \sim \log R$$

Incorporating monopole operators, and using the above ansatz, interaction becomes $V(R) \sim R^2$, a huge energy coast.

Instead, if we have sigma zero everywhere except for a cut, connecting vortex to anti-vortex, (recall that vorticity need to kept intact), the potential becomes

$$V(R) \sim TR$$



Corresponding to linear confinement of electric charges.

Topological configurations: Monopole-instantons

1-defects, Monopole-instantons: Associated with the N-nodes of the affine Dynkin diagram of SU(N) algebra. The Nth type corresponds to the affine root and is present only because the theory is *locally 4d*! [**van Baal, Kraan, (97/98), Lee-Yi, Lee-Lu (97)**]

$$\mathcal{M}_k \sim e^{-S_k} e^{-\alpha_k \cdot b + i\alpha_k \sigma + i\theta/N}, \qquad k = 1, \ldots, N$$

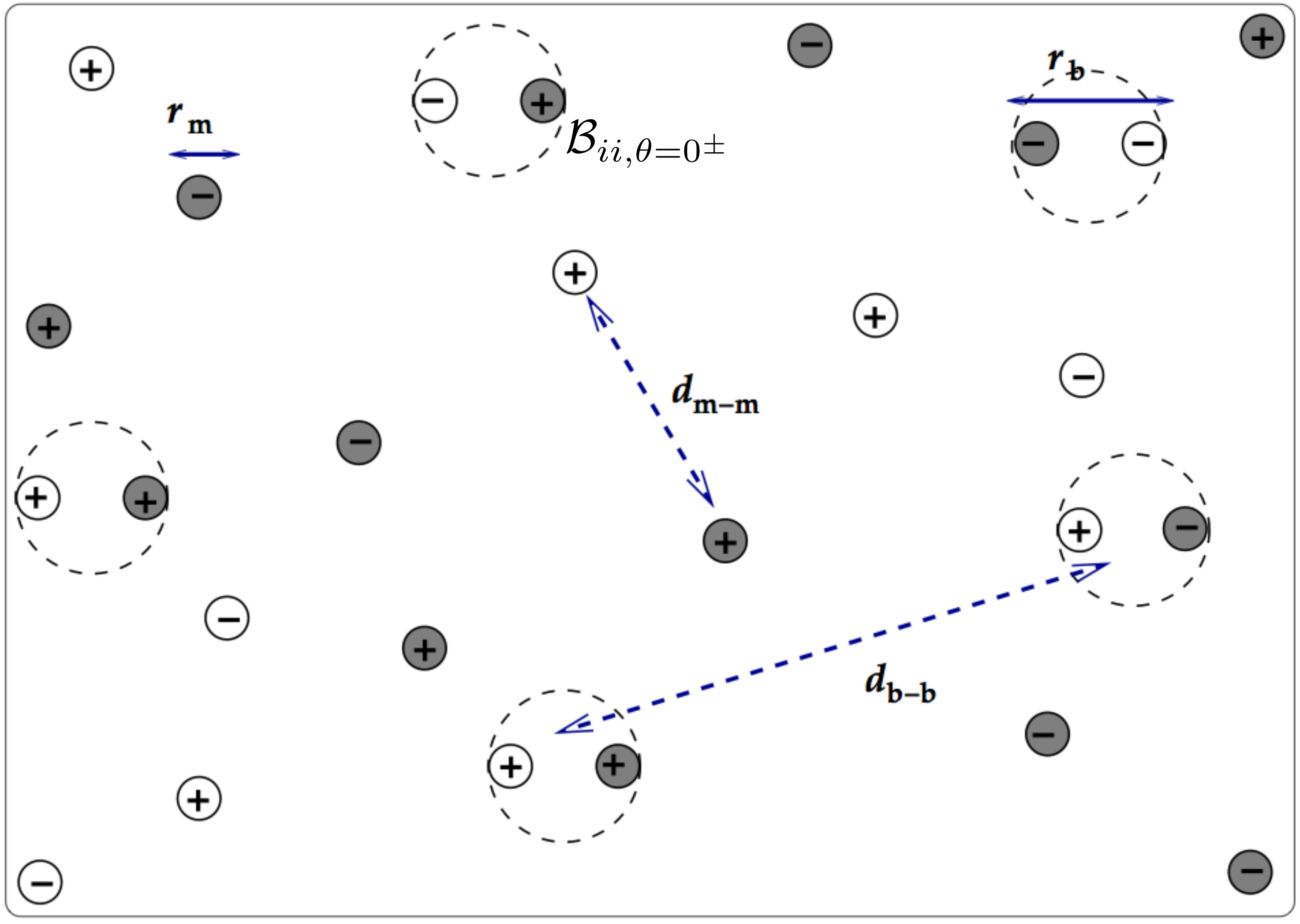
$$S_k = \frac{8\pi^2}{g^2 N} = \frac{S_I}{N}$$

Action 1/N of the 4d instanton, keep this in mind!

Proliferation of monopole-instantons generates a non-perturbative mass gap for gauge fluctuations, similar to 3d Polyakov model (Polyakov, 74). It is first generalization thereof to local 4d theory!

Deformed YM, Euclidean vacuum

Dilute gas of monopole instantons and correlated bion events



Relation to R4? Will comment on this later...

$$\langle F^2 \rangle_{0^\pm} \propto \mathcal{M}_i + [\mathcal{M}_i \bar{\mathcal{M}}_j] + [\mathcal{M}_i \bar{\mathcal{M}}_i]_{0^\pm} + \ldots$$

Ambiguity in condensate sourced by neutral bion.

The essence of mass gap in Polyakov-mechanism in 3d

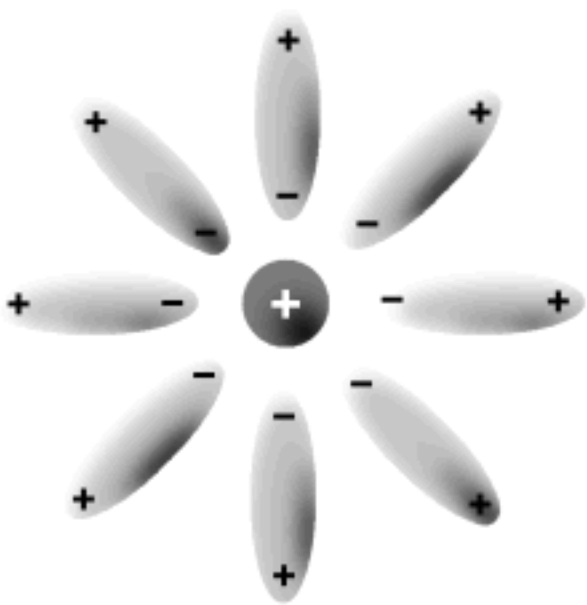
Polyakov 1977

't Hooft-Polyakov monopole solutions (instantons in 3d) in Georgi-Glashow model.

Partition function of gauge theory = The grand canonical ensemble of classical monopole plasma.

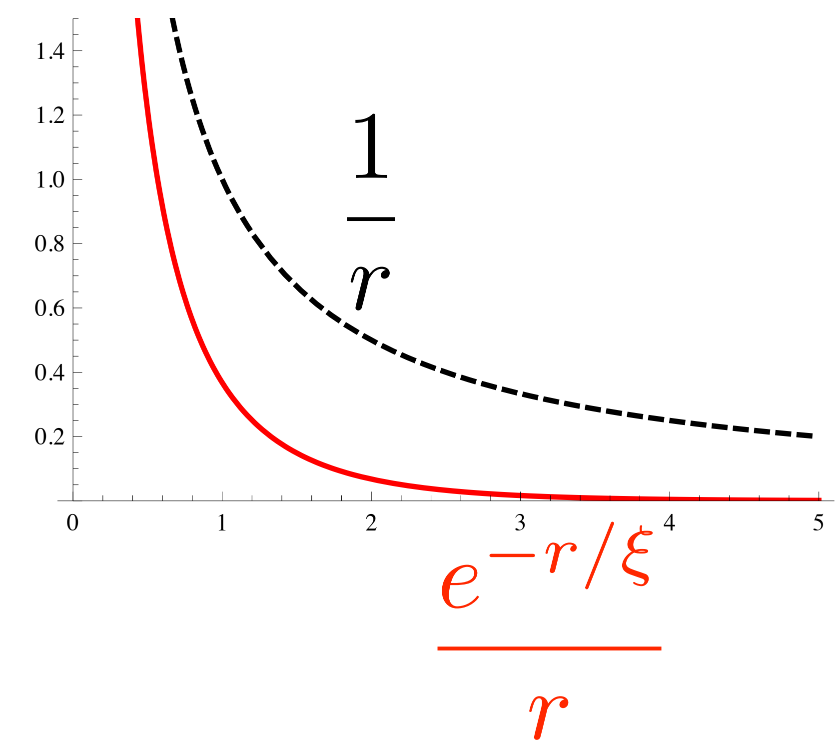
The field of external charge in a classical plasma decay exponentially. Debye-Hückel 1923.

Proliferation of monopole-instantons generates mass gap for gauge fluctuations.



Due to screening

$$\frac{1}{r} \longrightarrow \frac{e^{-r/\xi}}{r}$$



Finite magnetic screening length=mass for gauge fluctuations for U(1) photon=Confinement of electric charge (I will not show this part explicitly since I would like to emphasize mass gap. But the two are intimately related.)

Long-distance 3d dual theory

$$S^{\rm dual} = \int_{\mathbb{R}^3} \left[\frac{1}{2L} \left(\frac{g}{2\pi}\right)^2 (\nabla\sigma)^2 - \zeta \sum_{i=1}^{N} \cos(\alpha_i \cdot \sigma) \right].$$

Maxwell term

Monopole Operator

Abelian duality

$$F^{(j)}_{\mu\nu} = \frac{g^2}{2\pi L} \epsilon_{\mu\nu\rho} \partial_\rho \sigma^j$$

Monopole charges $\Delta^0_{\rm aff} \equiv \{\alpha_1, \alpha_2, \ldots, \alpha_{N-1}, \overline{\alpha_N}\}$.

usual N-1 monopoles

monopole due to compactness of Higgs scalar

Lee, Yi, Kraan, vanBaal

Semi-classical Mass gap on R3 x S1

$$m_g^2 = \Lambda^2 (\Lambda L N)^{5/3} \max_k \cos \frac{\theta + 2\pi k}{N}$$

Mass gap monopole-instanton effect.

Expected non-trivial theta angle dependence (not present in Polyakov model).

For SU(2), mass gap vanishes at theta=pi. An exponentially smaller mass gap appears due to magnetic bion effects. The vacuum is 2-fold degenerate due to CP-breaking, as per magnetic bion induced potential.

Analysis strictly reliable for $(\Lambda L N) \lesssim 1$

Although I have not discussed, there is only one fundamental string tension, just like pure Yang-Mills.

This is a non-trivial issue, and in sharp contrast with Polyakov and Seiberg-Witten. In both of those cases, there are N-1 fundamental strings, very unlike pure YM as pointed by Douglas-Shenker (1995).

Topological susceptibility

Topological susceptibility in SU(4) dYM on small S1 x R3 vs Pure YM on the confined phase approximately R4. The deformation parameters for single winding and double winding loop is denoted by h.

Green curve is roughly the sharp drop associated with the deconfinement phase transition.

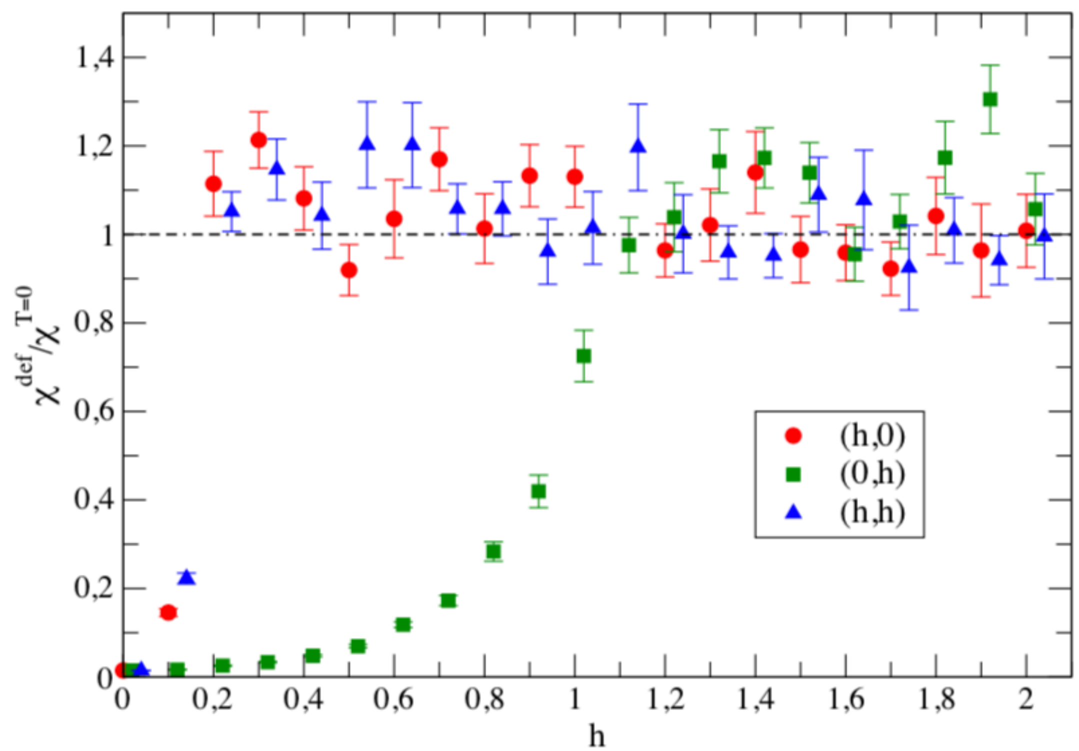


FIG. 11: Ratio between the topological susceptibility χ computed in the deformed theory and the one at $T = 0$ computed in Ref. [16] for different values of the deformation parameters h_1 and h_2. Results are obtained on the 6×32^3 lattice at bare coupling $\beta = 11.15$.

Bonati, Cardinali, D'Elia, Mazziotti, 2019

The simulation results strongly suggest us that we should carefully think about deformed YM. Clearly, it knows something deep about YM on R4!

IR-Renormalon problem in Yang-Mills theory *'t Hooft(79)*

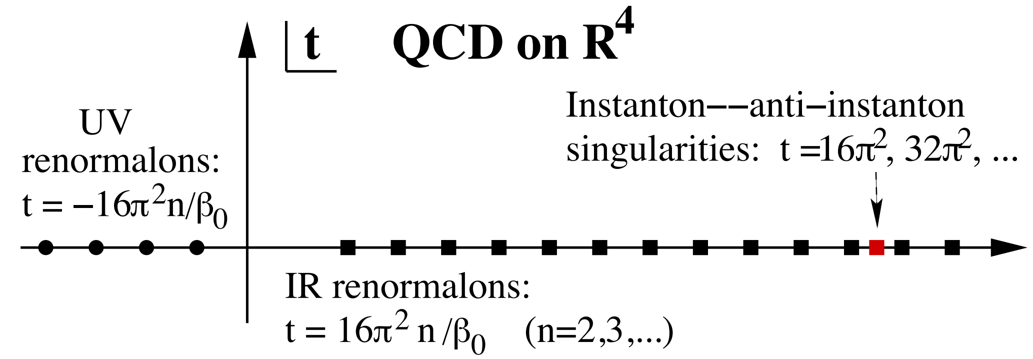
There is a very famous and important problem in Yang-Mills theory, attributed to 't Hooft, which is described in a famous set of lectures "*Can we make sense out of QCD?* "

$[\mathcal{I}\bar{\mathcal{I}}]$ contribution, calculated in some way, gives an $\pm i \exp[-2S_I]$.
Lipatov(77): Borel-transform BP(t) has singularities at $t_n = 2n\, g^2 S_I$.

BUT, BP(t) has other (more important) singularities closer to the origin of the Borel-plane. (not due to factorial growth of number of diagrams, but due to phase space integration.)

't Hooft called these IR-renormalon singularities with the hope that they would be associated with a saddle point like instantons. **No such configuration is known!**

A real problem in QFT, means pert. theory, as is, ill-defined. How to cure starting from microscopic dynamics?



Leading IR singularity $\frac{4S_I}{\beta_0} = \frac{12 S_I}{11N}$

Standard view emanating from late 70s

e.g. : from Parisi(78)

If the theory is renormalizable, the Borel transform has new singularities which cannot be controlled by using semi-classical methods [5-8].

5 G. 't Hooft, Lectures given at Erice (1977)

6 B. Lautrup, Phys. Lett. 69B (1977) 109

7 G. Parisi, Lectures given at the 1977 Cargèse Summer School

8 P. Olesen, Nordita preprint NBI HE 77.48 (1977)

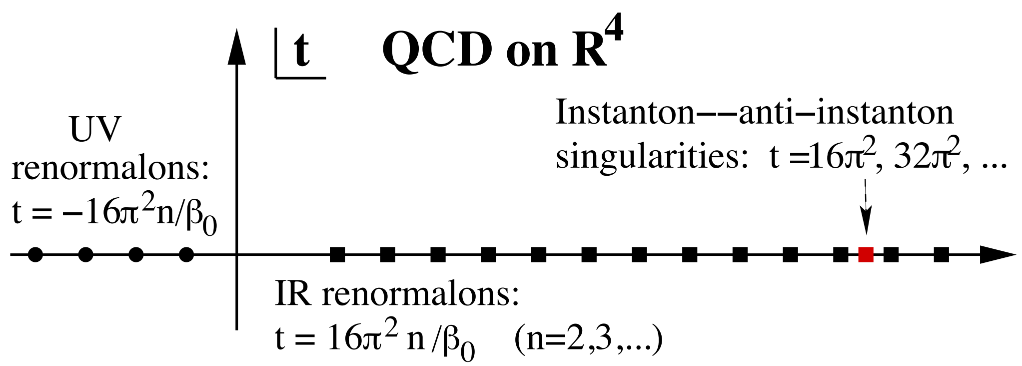
Change the Question: What happens if we can make in deformed Yang-Mills theory in the semi-classically calculable regime?

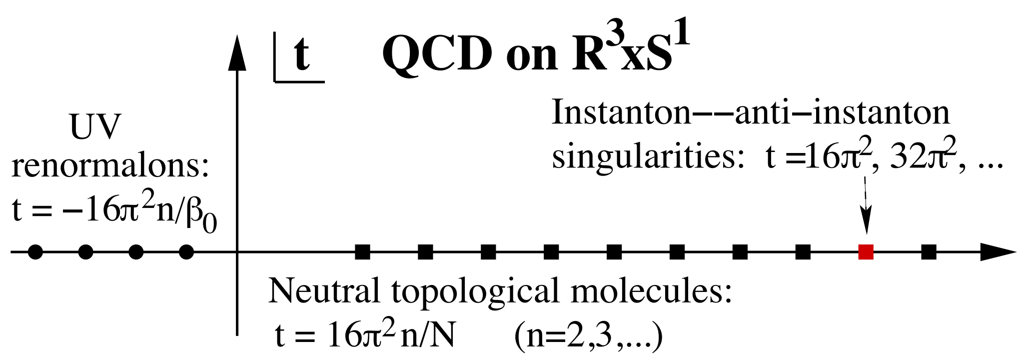
Calculating complex (neutral) bion amplitude similar to QM example:

$$\mathcal{B}_{ii,\theta=0^\pm} = [\mathcal{M}_i \overline{\mathcal{M}}_i]_{\theta=0^\pm}$$

$$\sim e^{-\frac{16\pi^2}{g^2 N}} \pm i e^{-\frac{16\pi^2}{g^2 N}}$$

This corresponds to an IR singularity in the Borel plane at $\frac{2S_I}{N}$





Important thing is 1/N parts match. Perhaps, as one moves from weak coupling to strong coupling, 2(S/N) flows to (12/11)(S/N). No one knows......