LECTURE-2

Basics of Resurgence and Lefschetz thimbles in quantum mechanical path integrals.

The role of critical points at infinity, complex bion configurations, and hidden topological angle, Cheshire Cat resurgence.

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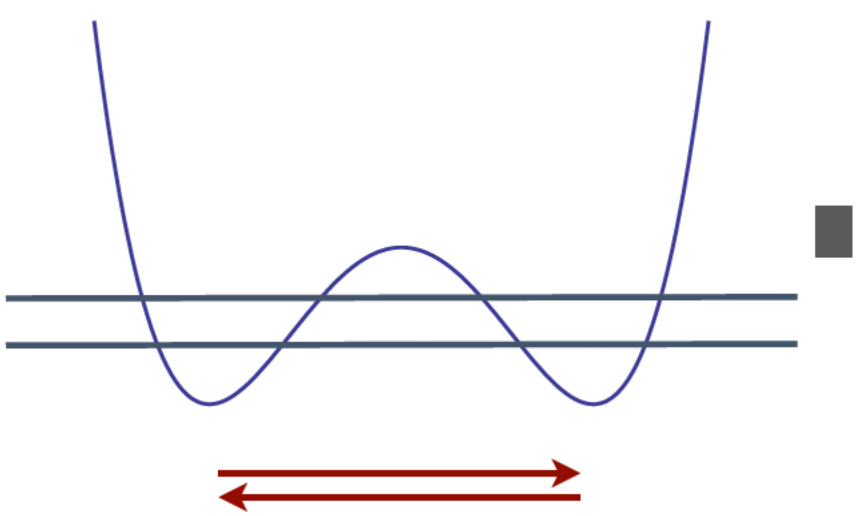
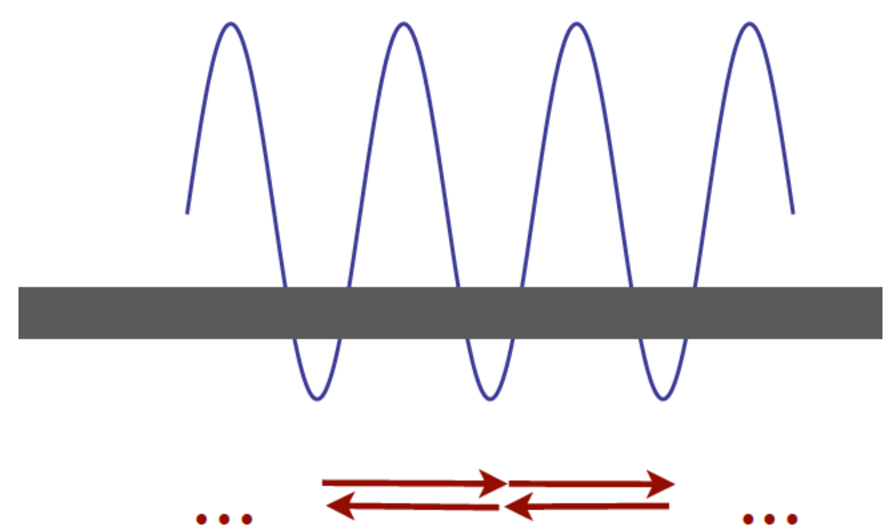




I will consider two classes of QM systems. Both classes are extremely interesting. (Many parallels with the saddles in semi-classical QFT.)

$$S = \frac{1}{g} \int dt \left(\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}(W')^2 \right)$$

$$S = \frac{1}{g} \int dt \left(\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}(W')^2 + \tfrac{1}{2}(\bar{\psi}_i \dot{\psi}_i - \dot{\bar{\psi}}_i \psi_i) + \tfrac{1}{2} W'' [\bar{\psi}_i, \psi_i] \right) , \qquad i = 1, \ldots, N_f \, .$$

$N_f = 1$ SUSY QM

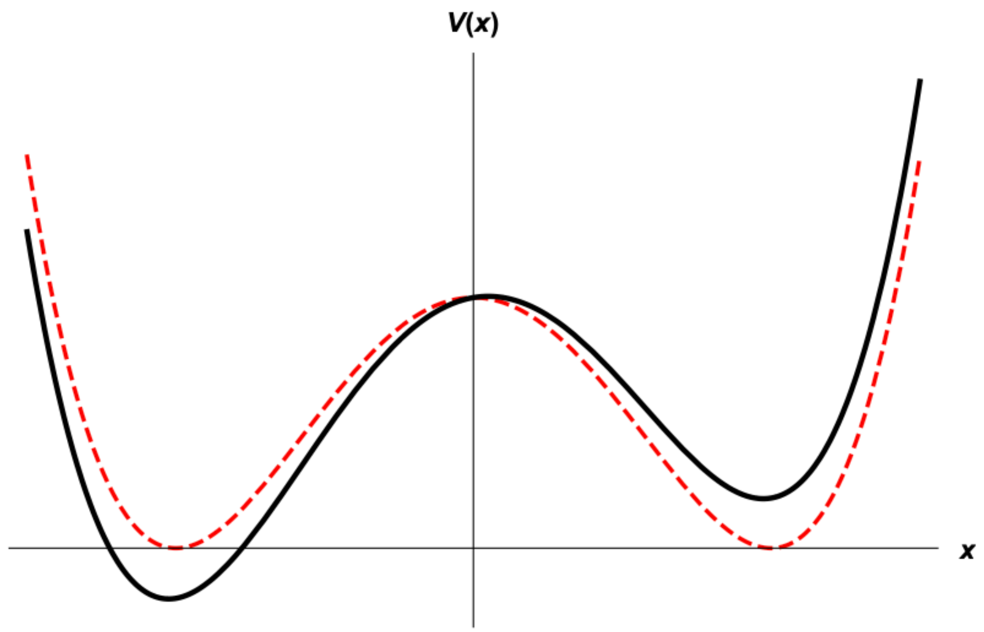
$N_f > 1$ related to QES systems. If exp[+W] or exp[-W] is normalizable, the lowest N_f states are exactly solvable. These systems are called Quasi-Exactly Solvable (QES), and to my mind, not less interesting than supersymmetric QM.

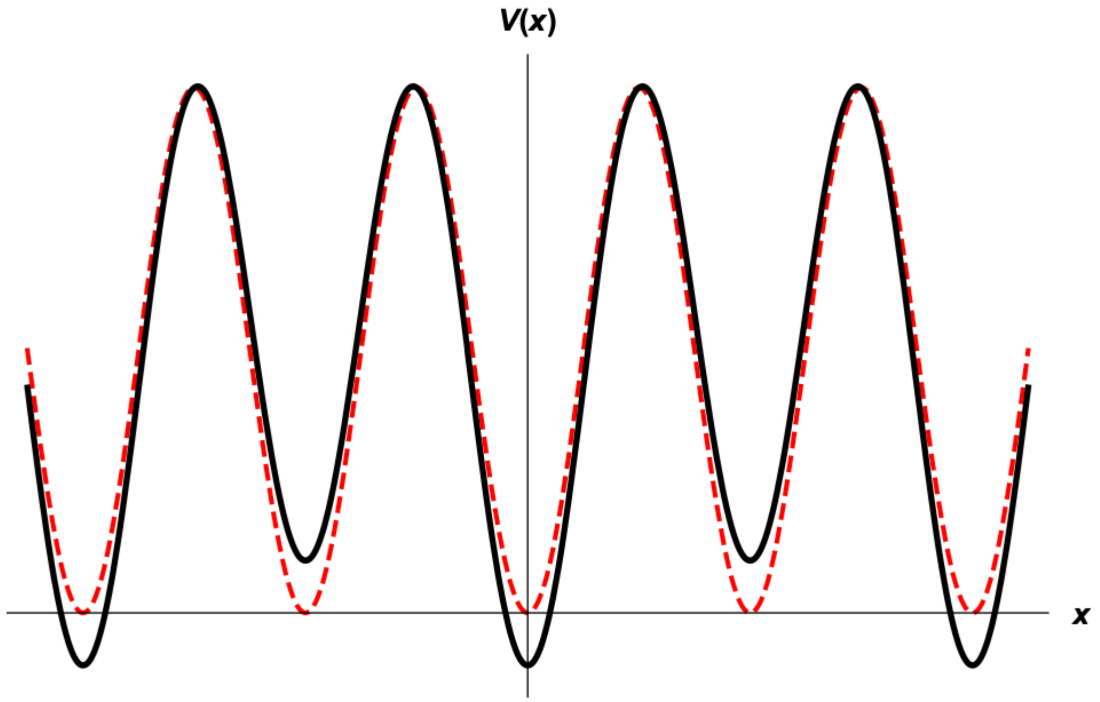
Quantizing the fermions, (or integrating them out), we end up with

$$\widehat{H} = \bigoplus_{k=0}^{N_f} \deg(\mathcal{H}_{(N_f,k)}) \widehat{H}_{(N_f,k)}, \qquad \deg(\mathcal{H}_{(N_f,k)}) = \binom{N_f}{k}$$

$$\hat{H}_{(N_f,k)} = \frac{g}{2}\,\hat{p}^2 + \frac{1}{2g}\Big((W')^2 + \zeta g W''\Big) \qquad \zeta = 2k - N_f, \qquad\qquad k = 0,\ldots,N_f\,.$$

$$\mathcal{L}_\zeta = \frac{1}{2g}\Big(\dot{x}^2 + (W')^2 \pm \zeta g W''\Big)$$

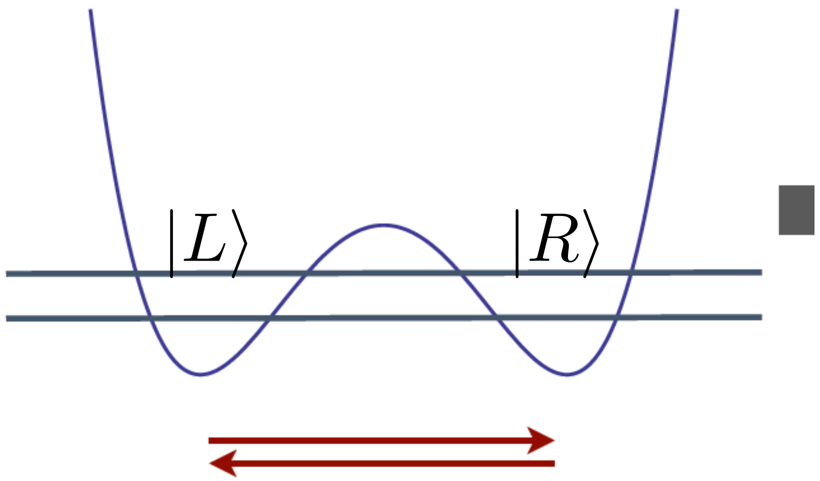




Note that the potential has a classical and quantum part. The tilting is a one-loop quantum effect, induced by integrating out fermions.

If the tilting is rendered classical, the story changes quite a bit. (Needs another lecture of its own.) But such quantum induced potential appears naturally by integrating out fermions both in QM and QFT, it is worthwhile to discuss this system for its own right.

Perturbative vs. non-perturbative physics in QM: e.g. Double-well potential.



$$V(x) = \frac{x^2}{2}(1 - gx)^2$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|L\rangle + |R\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|L\rangle - |R\rangle)$$

If one forgets tunneling effects, two fold degeneracy of ground state to all orders in Rayleigh-Schrödinger perturbation theory. These are L and R states.

$$E_L = E_R = \frac{\hbar\omega}{2}\left(1 + a_1 g^2 + a_2 g^4 + \ldots + a_n g^{2n} + \ldots\right)$$

Taking into account tunneling events (= instantons, next pages)
Vacuum is unique Parity-even state $|+\rangle$

Gap? Energy needed to excite the system from the ground state to the first excited state.

$$\Delta E = E_- - E_+ \sim e^{-1/(6g^2)}$$

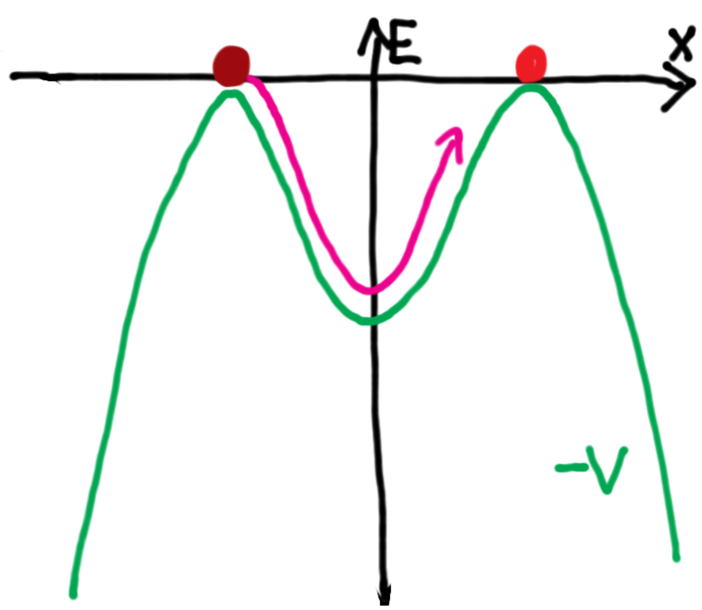
Tunnelling = Instantons

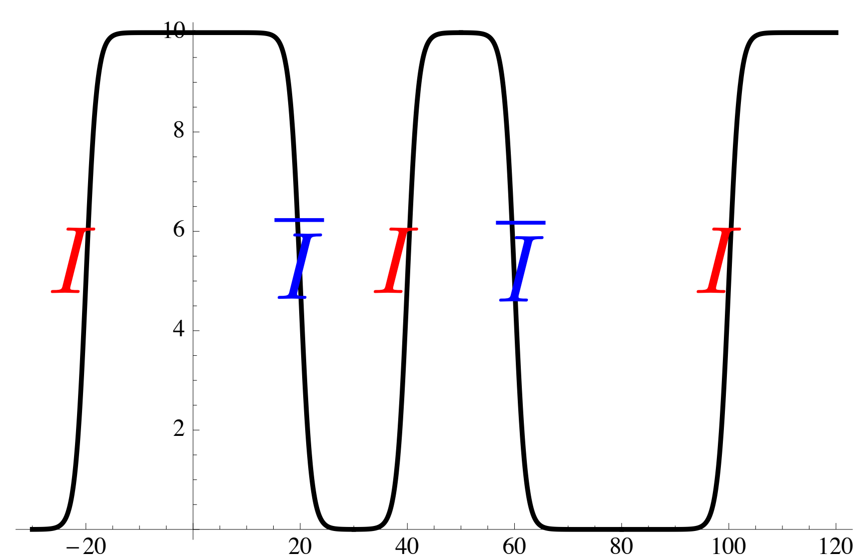
An equivalent representation of QM is Feynman path integrals. In the evaluation of path integrals, one usually works with (imaginary time) Euclidean space instead of real time Minkowski space. Same data can be extracted from both.

Saddle points of Euclidean path integrals = instantons = tunneling

Instanton amplitude= tunneling rate.

$$I \sim e^{-1/(6g^2)}$$

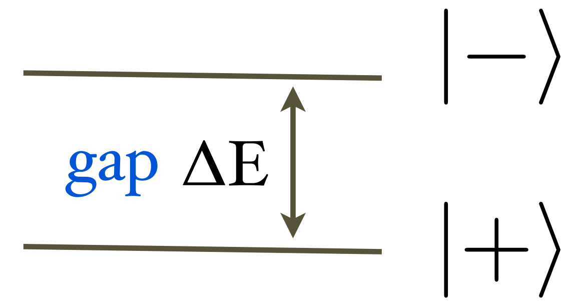




Euclidean Vacuum: Dilute gas of instantons: $I \; \bar{I} \; I \; \bar{I} \; I \; \bar{I} \; I \; \bar{I} \; I \; \bar{I}$

Same as vacuum being the superposition of L and R states.

Perturbative vs. non-perturbative physics in QM: e.g. Double-well potential.



If we (naively) Taylor expand ΔE in small-g^2, we obtain

$$e^{-1/(6g^2)} \to 0 + 0 + 0 + \ldots$$

Thus, tunneling=instanton effects cannot be captured in ordinary perturbation theory (at least in "text-book" sense).

However, as you anticipate by now, the reality is far more subtle and beautiful!

$$E_L = E_R = \frac{\hbar\omega}{2}\left(1 + a_1 g^2 + a_2 g^4 + \ldots + a_n g^{2n} + \ldots\right)$$

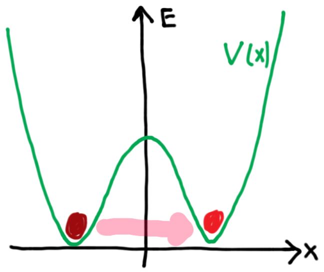
is actually a **divergent (asymptotic) series.** And it does not diverge in some arbitrary way. There is structure to it, and it can be decoded: In particular, the way it diverges knows the existence of instantons, and other saddles!

Quantum mechanics, perturbation theory and tunneling amplitudes

Consider perturbation theory in QM. If one applies Stokes' method to p.t., one obtains the "intrinsic vagueness" of the perturbation theory = |tunneling (instanton) amplitude|^2, for any sensible quantum mechanical systems with degenerate minima. In our example,

$\exp[-1/3g^2]$!

Recall: the gap ~ $\exp[-1/6g^2]$.

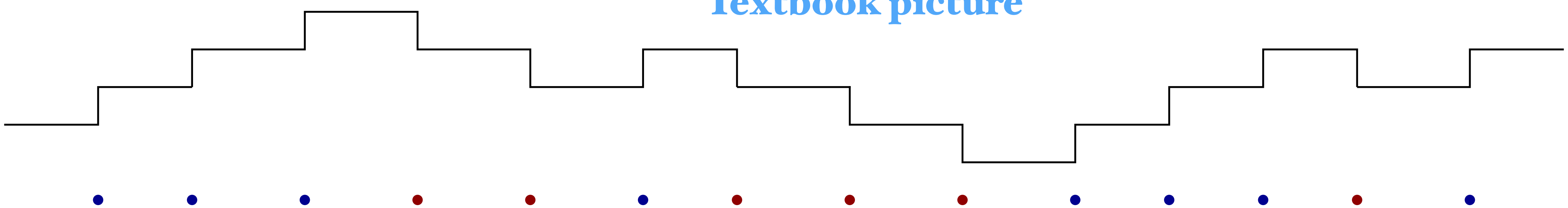


Large-orders in perturbation theory knows something about instanton pairs (Bogomolny-Zinn-Justin, 80s)!

The divergent perturbative expansion, in its late terms, has the knowledge of the instanton effects in a coded-form. This needs to be decoded.

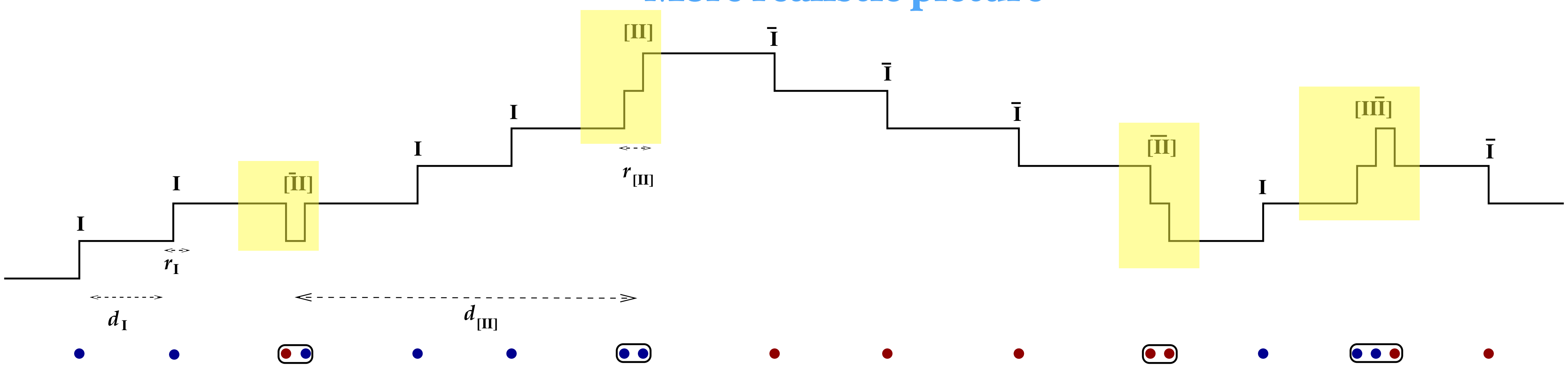
Instanton gas for periodic potential

Textbook picture

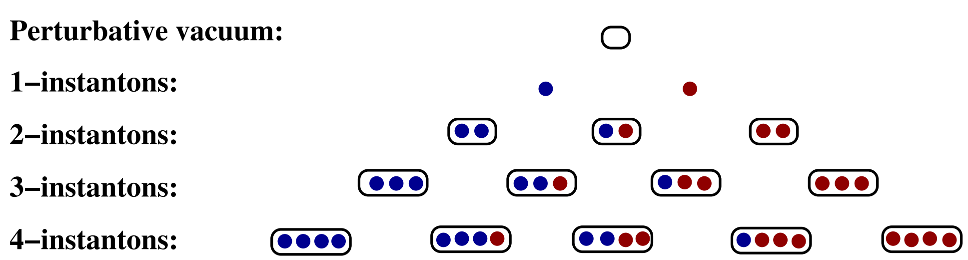


a) Dilute gas of 1−instantons

More realistic picture



b) Dilute gas of 1−instantons, 2−instantons, and other molecular−events



$$\begin{array}{ccccccc} r_{\rm I} & \ll & r_{[\rm II]} \sim \ell_{\rm qzm} & \ll & d_{\rm I} & \ll & d_{[\rm II]}, \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ L & \ll & L\,\log\left(\frac{1}{g^2}\right) & \ll & L\,e^{S_0} & \ll & L\,e^{2S_0}. \end{array}$$

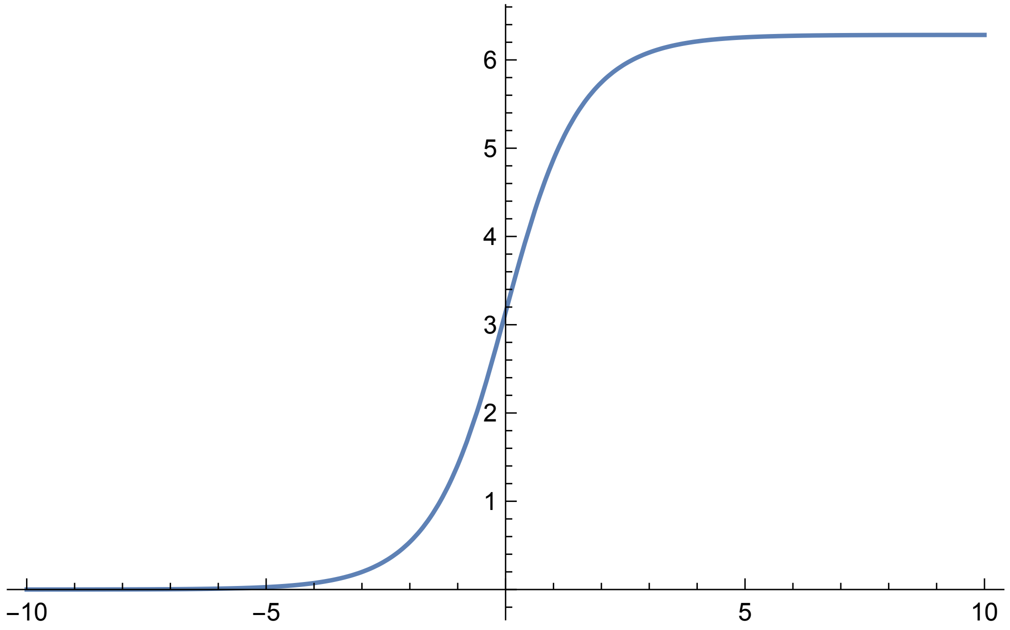
c) Representatives of n−instanton events, sketched according to the resurgence triangle.

Basics of instantons- 1

$$\int \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}(W'(x))^2 = \int \tfrac{1}{2}\underbrace{(\dot{x} \mp W'(x))^2}_{\geq 0} \pm \dot{x}W' \geq \left| \int dW \right| = |W(x_2) - W(x_1)|$$

$\dot{x} = \pm W'(x)\,.$ Instanton equation

$W(x) = 4\cos\left(\frac{x}{2}\right) \Longrightarrow x_I(\tau) = 4\arctan\left(\exp[\tau - \tau_c]\right)\ ,$ Instanton solution



Basics of instantons-2

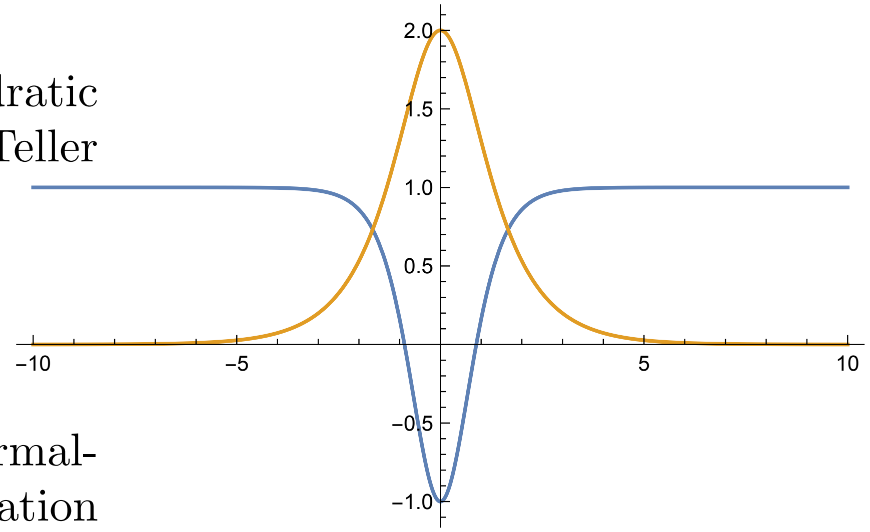
The instanton amplitude:

$$\mathcal{I} \equiv \xi = J_{\tau_c}\, e^{-S_I} \left[\frac{\widehat{\det}\,\mathcal{M}_I}{\det \mathcal{M}_0}\right]^{-\frac{1}{2}} P_I(g)\,,$$

- The overall amplitude: density of the instantons. Characteristic separation between instantons: $\sim e^{+S_I}$, dilute instanton gas.
- $J_{t_c} = \sqrt{S_I/(2\pi)}$: Jacobian associated with the bosonic zero mode.
- $\mathcal{M}_I = -\frac{d^2}{d\tau^2} + V''(x)|_{x=x_I(t)} = -\frac{d^2}{d\tau^2} + 1 - 2\,\mathrm{sech}^2(\tau - \tau_c)$,: quadratic fluctuation operator in the background of the instanton. (Pöschl-Teller form). Exact zero mode is given by

$$\Psi_0(\tau) = \dot{x}_I(\tau) = \frac{2}{\cosh(\tau - \tau_c)}$$

  The "hat": the zero mode has to be removed, and $\det \mathcal{M}_0$ is a normalization factor, which we take to be the corresponding free fluctuation operator.



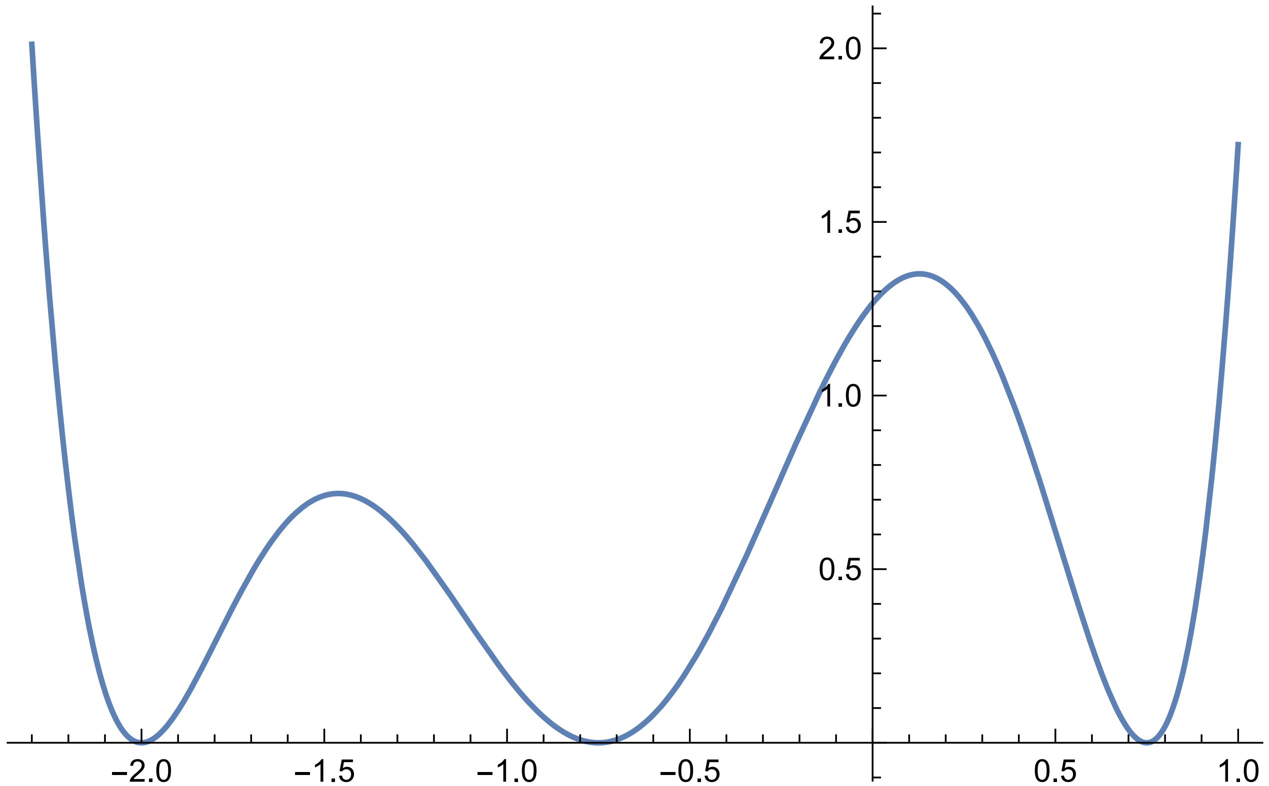
- Perturbative expansion around instanton:

$$P_I(g) = \sum_{n=0}^{\infty} b_{I,n} g^n,$$

  which is a formal asymptotic series, which is in general not Borel summable.
- The determinant of the instanton fluctuation operator can be computed using the Gel'fand-Yaglom (GY) method. (See Marino's book).

Side remarks: Do instantons always contribute to physical observables?

In almost all books and texts, you will see the discussion of double-well or periodic potential, but not a more generic potential with harmonic degenerate minima as shown in figure. Why not?



Despite the fact that there are exact instanton solutions, for generic potential of this type, they typically do not contribute to the spectrum at the exp[-S] order, rather, the first NP contribution appears at order exp[-2S], related to the concept of critical point at infinity (which I will explain).

The reason instantons do not contribute at leading order is that the determinant of fluctuation operator is infinite unless the frequency in two consecutive well are the same.

Therefore, in QM, instanton contributing to spectrum is exception instead of being a rule.

Perturbation theory by Bender-Wu method

Bender-Wu Mathematica package written by Tin Sulejmanpasic:
https://library.wolfram.com/infocenter/MathSource/9479/.

Description

The BenderWu package allows for analytic computation of the perturbative series in 1D quantum mechanics around a harmonic minimum of the potential. The code is based on the method pioneered by Bender and Wu.

$$
\begin{aligned}
E^{\text{pert}}(N,g) &\sim \sum_{n=0}^{\infty} \hbar^n a_n(N) \\
&\sim \left[N+\frac{1}{2}\right] - \frac{g}{16}\left[\left(N+\frac{1}{2}\right)^2+\frac{1}{4}\right] - \frac{g^2}{16^2}\left[\left(N+\frac{1}{2}\right)^3+\frac{3}{4}\left(N+\frac{1}{2}\right)\right] \\
&- \frac{g^3}{16^3}\left[\frac{5}{2}\left(N+\frac{1}{2}\right)^4+\frac{17}{4}\left(N+\frac{1}{2}\right)^2+\frac{9}{32}\right] \\
&- \frac{g^4}{16^4}\left[\frac{33}{4}\left(N+\frac{1}{2}\right)^5+\frac{205}{8}\left(N+\frac{1}{2}\right)^3+\frac{405}{64}\left(N+\frac{1}{2}\right)\right] - \dots
\end{aligned}
$$

$$a_n(N) \sim -\frac{2^{2N}}{\pi\,(N!)^2}\frac{\Gamma(n+2N+1)}{(2S_I)^{n+2N+1}}$$

Large-order factorial growth for harmonic level N

$$a_n(N=0) \sim -\frac{1}{\pi}\frac{n!}{(2S_I)^{n+1}}\left(1-\frac{5}{2}\cdot\frac{(2S_I)^1}{n}-\frac{13}{8}\cdot\frac{(2S_I)^2}{n(n-1)}+\dots\right)$$

Large-order factorial growth for ground state.

Instanton interactions

Since instanton equations and Euclidean eq of motion are non-linear, two instanton configurations is not a solution at finite separation.

$$x_{\mathcal{I}\mathcal{I}}(\tau) = x_{\mathcal{I}}(\tau - \tau_1) + x_{\mathcal{I}}(\tau - \tau_2),$$
$$x_{\mathcal{I}\bar{\mathcal{I}}}(\tau) = x_{\mathcal{I}}(\tau - \tau_1) - x_{\mathcal{I}}(\tau - \tau_2),$$

$$S_{\mathcal{I}\mathcal{I}}(\tau_{12}) = 2S_I + \frac{A}{g}e^{-\tau_{12}}, \qquad \text{repulsive},$$
$$S_{\mathcal{I}\bar{\mathcal{I}}}(\tau_{12}) = 2S_I - \frac{A}{g}e^{-\tau_{12}}, \qquad \text{attractive}$$

Attractive/repulsive are just words, inheritance from old literature. Caused too much confusion in past. This formula just means that these combos are not exact solution for finite separation. That is all. Tau direction is called quasi-moduli space.

Cluster expansion

In the $\beta \to \infty$ limit, we can write Z as

$$Z = e^{-\beta E_0 P_0(g)} \left(1 + \frac{\xi}{1!} \int d\tau_1 + \frac{\xi^2}{2!} \int d\tau_1 d\tau_2 \; e^{-V_{12}} + \frac{\xi^3}{3!} \int d\tau_1 d\tau_2 d\tau_3 \; e^{-V_{123}} + \ldots \right) .$$

whee $\xi \sim e^{-S_I}$ is the instanton amplitude.

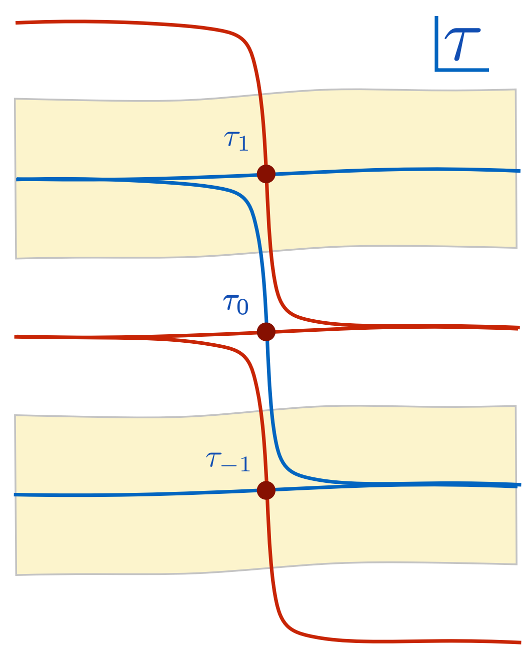
$$Z_{\text{dilute}} = e^{-\beta \left(E_0 P_0(g) - [\mathcal{I}] - [\bar{\mathcal{I}}] - [\mathcal{I}^2] - [\bar{\mathcal{I}}^2] - [\mathcal{I}\bar{\mathcal{I}}]_\pm - [\bar{\mathcal{I}}\mathcal{I}]_\pm - [\mathcal{I}^3] - [\mathcal{I}^2\bar{\mathcal{I}}] \ldots \right)}$$

$$= e^{-\beta E_0 P_0(g)} \left(\sum_{n_{\mathcal{I}}=0}^{\infty} \frac{\beta^{n_{\mathcal{I}}} [\mathcal{I}]^{n_{\mathcal{I}}}}{n_{\mathcal{I}}!} \right) \left(\sum_{n_{\bar{\mathcal{I}}}=0}^{\infty} \frac{\beta^{n_{\bar{\mathcal{I}}}} [\mathcal{I}]^{n_{\bar{\mathcal{I}}}}}{n_{\bar{\mathcal{I}}}!} \right) \left(\sum_{n_{\mathcal{I}\bar{\mathcal{I}}}=0}^{\infty} \frac{\beta^{n_{\mathcal{I}\bar{\mathcal{I}}}} [\mathcal{I}\bar{\mathcal{I}}]_\pm^{n_{\mathcal{I}\bar{\mathcal{I}}}}}{n_{\mathcal{I}\bar{\mathcal{I}}}!} \right) \ldots$$

Compactify $\mathbb{R} \to S^1_\beta$ in order to study $Z(\beta) = \text{Tr}\left[e^{-\beta H}\right]$.
The interaction between two events is modified in a fairly obvious way into:

$$S(\tau) = \pm \frac{A}{g}\left(e^{-\tau} + e^{-(\beta-\tau)}\right)$$

$$[\mathcal{I}\bar{\mathcal{I}}] = \left(\frac{1}{2}\int_0^\beta d\tau\; e^{\frac{A}{g}\left(e^{-\tau}+e^{-(\beta-\tau)}\right)} - \beta/2\right)[\mathcal{I}][\bar{\mathcal{I}}]$$



The Lefschetz thimbles for the $\mathcal{I}\bar{\mathcal{I}}$ saddle, showing the downward flows (blue curves) connecting τ_0 to $\tau_{\pm 1}$ when $g \to g\, e^{i\theta}$ with $\theta \to 0^+$. The directions are flipped about the imaginary axis for $\theta \to 0^-$.

Borel-Ecalle summability in bosonic theory

$$[\mathcal{I}\bar{\mathcal{I}}]_{\pm} = \left(\mp i\pi - \gamma - \log\left(\frac{A}{g}\right) + \ldots \right) [\mathcal{I}][\bar{\mathcal{I}}]$$

$$[\mathcal{I}\bar{\mathcal{I}}]_{\pm} \sim \left(\mp i\pi - \gamma - \log\left(\frac{A}{g}\right) + \ldots \right) e^{-(2S_I)/g} \left(1 - \frac{5}{2}\cdot g - \frac{13}{8}\cdot g^2 \ldots \right)$$

$$a_n(N=0) \sim -\frac{1}{\pi}\frac{n!}{(2S_I)^{n+1}} \left(1 - \frac{5}{2}\cdot\frac{(2S_I)^1}{n} - \frac{13}{8}\cdot\frac{(2S_I)^2}{n(n-1)} + \ldots \right)$$

$$\mathrm{Im}\,\mathbb{B}_{0,\theta=0^{\pm}} + \mathrm{Im}\,[\mathcal{I}\overline{\mathcal{I}}]_{\theta=0^{\pm}} = 0\ , \qquad \text{up to } O(e^{-4S_I})$$

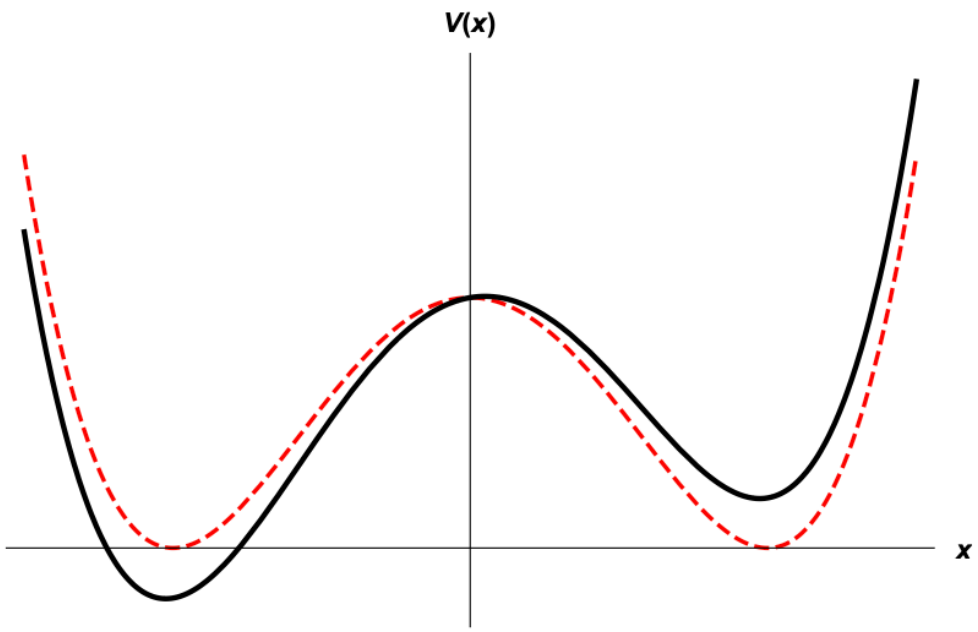
The leading terms (structures) obtained in Bogomolny and Zinn-Justin early 80s, but not sufficiently appreciated. The interesting thing is, B-ZJ was not an unknown work. The problem was that their methods in the derivation did not convince people. I was personally fascinated by what they did, and was convinced that their main claim was correct.

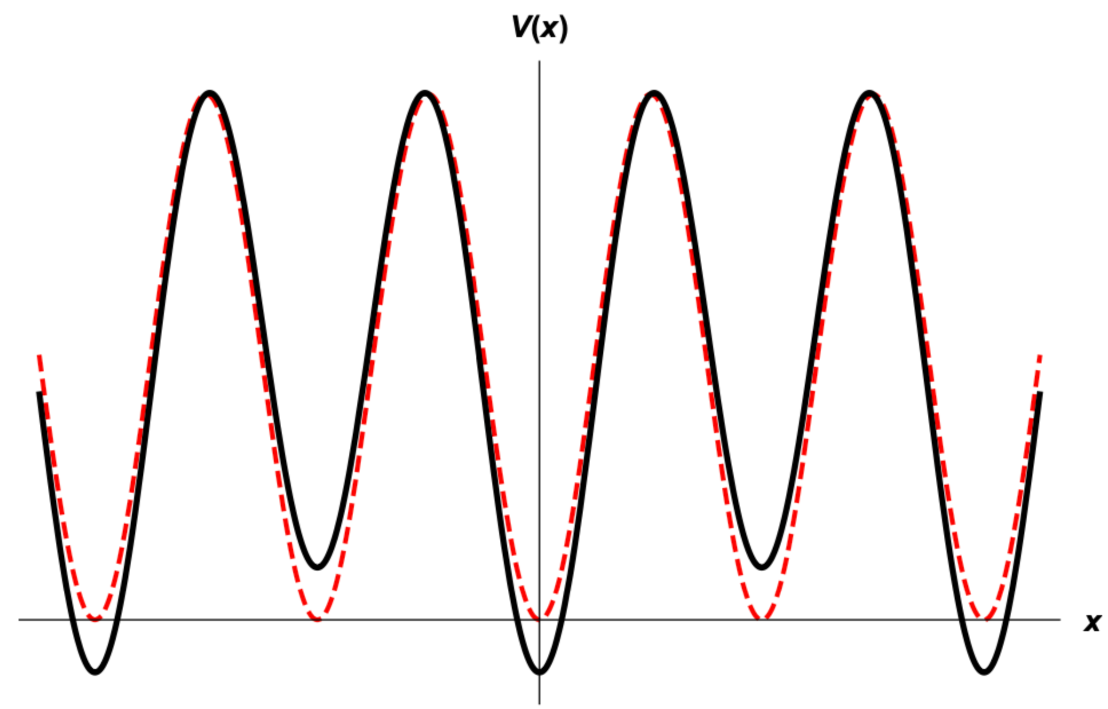
The overall structure was obtained in 2014, in Gerald Dunne and MU.

SUSY, QES and in between

$$S = \frac{1}{g}\int dt\left(\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}(W')^2 + \tfrac{1}{2}(\bar{\psi}_i\dot{\psi}_i - \dot{\bar{\psi}}_i\psi_i) + \tfrac{1}{2}W''[\bar{\psi}_i,\psi_i]\right)\,, \qquad i = 1,\ldots,N_f\,.$$

$$\mathcal{L}_\zeta = \frac{1}{2g}\left(\dot{x}^2 + (W')^2 \pm \zeta g W''\right)$$





Instanton interactions in the presence of fermions or quantum tilting

$$S_{\mathcal{I}\mathcal{I}}(\tau) = \overbrace{+\frac{A}{g}\left(e^{-\tau} + e^{-(\beta-\tau)}\right)} + \overbrace{\zeta\,\tau}\,.$$

$$S_{\mathcal{I}\bar{\mathcal{I}}}(\tau) = \underbrace{-\frac{A}{g}\left(e^{-\tau} + e^{-(\beta-\tau)}\right)}_{\text{classical}} + \underbrace{\zeta\,\tau}_{\text{quantum}}$$

Concept of critical point at infinity and non-Gaussian critical points

$$I_+(\zeta,g) \equiv \int_{\Gamma^{\theta=0\pm}_{\rm QZM}} d\tau \, e^{-\frac{A}{g}\left(e^{-\tau}+e^{-(\beta-\tau)}\right)} e^{-\zeta\tau}$$

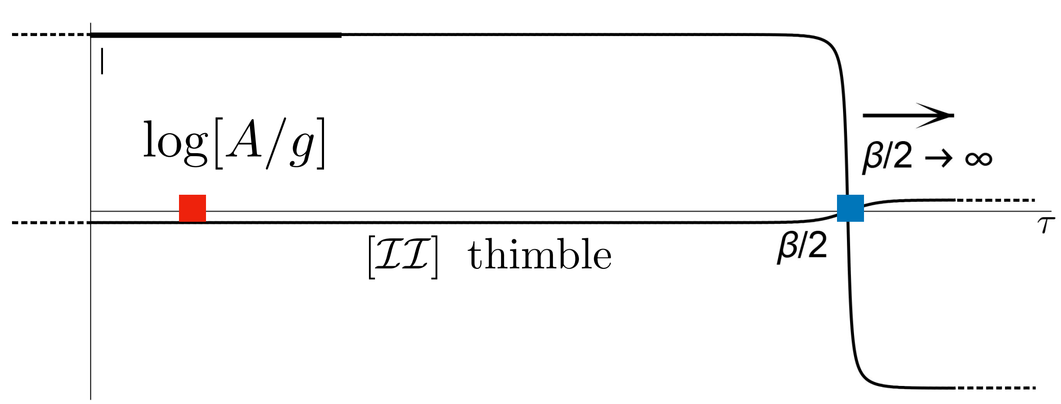
$$I_-(\zeta,g) \equiv \int_{\Gamma^{\theta=0\pm}_{\rm QZM}} d\tau \, e^{\frac{A}{g}\left(e^{-\tau}+e^{-(\beta-\tau)}\right)} e^{-\zeta\tau}$$

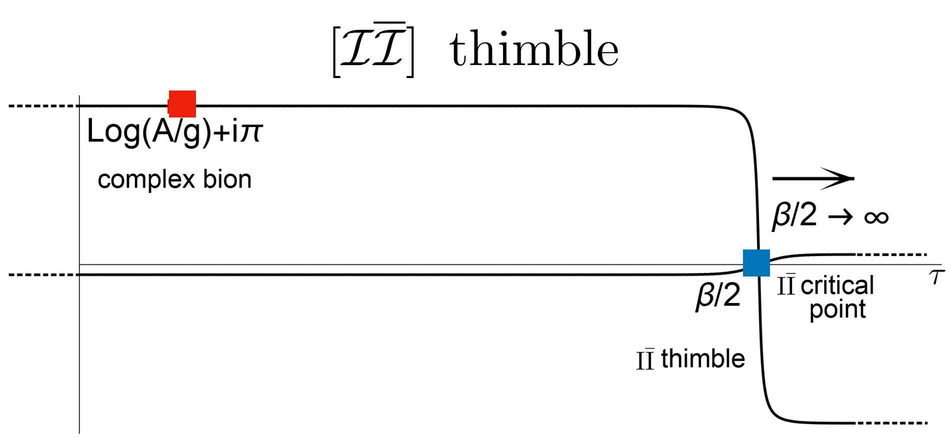
$$\lim_{\beta\to\infty} e^{-\frac{2A}{g}\left(e^{-\beta/2}\right)} e^{-\zeta\beta/2} = 0\,.$$

$$\lim_{\beta\to\infty} e^{\frac{2A}{g}\left(e^{-\beta/2}\right)} e^{-\zeta\beta/2} = 0$$

$$\begin{aligned}[\mathcal{I}\mathcal{I}] &= I_+(\zeta,g) \times [\mathcal{I}]^2 \\ &= \left(\frac{g}{A}\right)^{\zeta} \Gamma(\zeta) \times \frac{S_I}{2\pi}\left[\frac{\widehat{\det}\,\mathcal{M}_I}{\det \mathcal{M}_0}\right]^{-1} e^{-2S_I} \\ &= \frac{1}{2\pi}\left(\frac{g}{32}\right)^{\zeta-1} \Gamma(\zeta) e^{-2S_I}\,.\end{aligned}$$

$$\begin{aligned}[\mathcal{I}\bar{\mathcal{I}}]_\pm &= I_-(\zeta,g) \times [\mathcal{I}]^2 \\ &= e^{\pm i\pi\zeta}\left(\frac{g}{A}\right)^{\zeta} \Gamma(\zeta) \times \frac{S_I}{2\pi}\left[\frac{\widehat{\det}\,\mathcal{M}_I}{\det \mathcal{M}_0}\right]^{-1} e^{-2S_I} \\ &= \frac{1}{2\pi}\left(\frac{g}{32}\right)^{\zeta-1} \Gamma(\zeta) e^{-2S_I} e^{\pm i\pi\zeta}\,.\end{aligned}$$





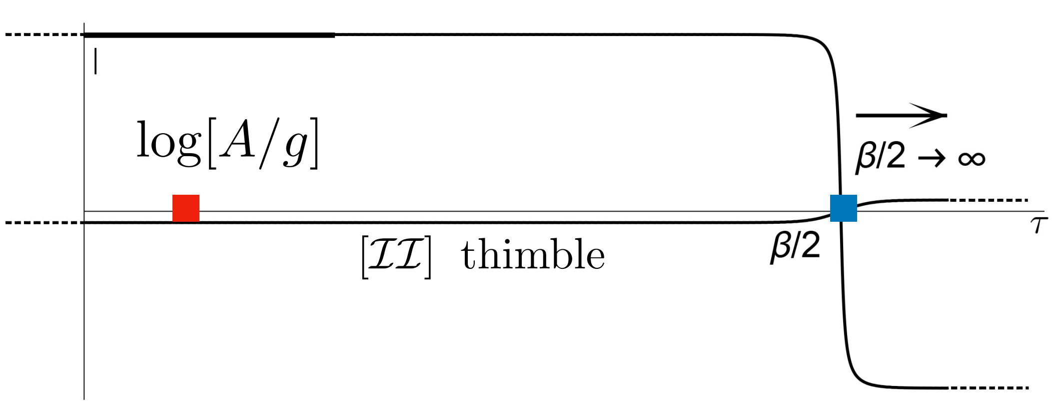
Concept of critical point at infinity and non-Gaussian critical points

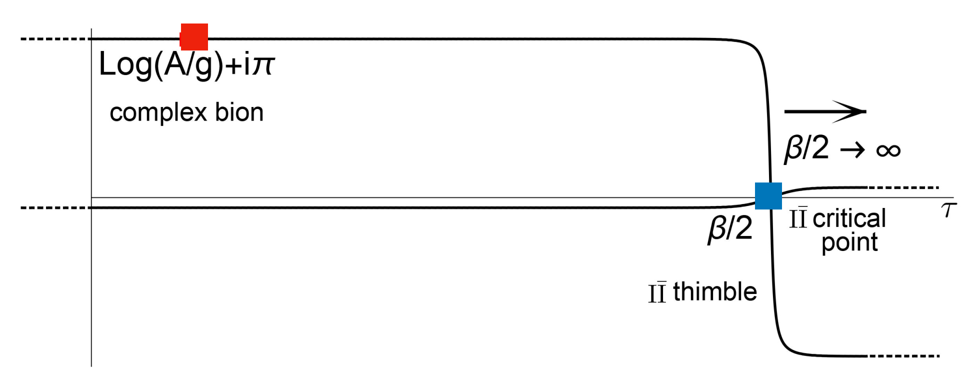
Unlike Gaussian critical point, the critical point at infinity itself does not contribute. However, its thimble gives major contribution.

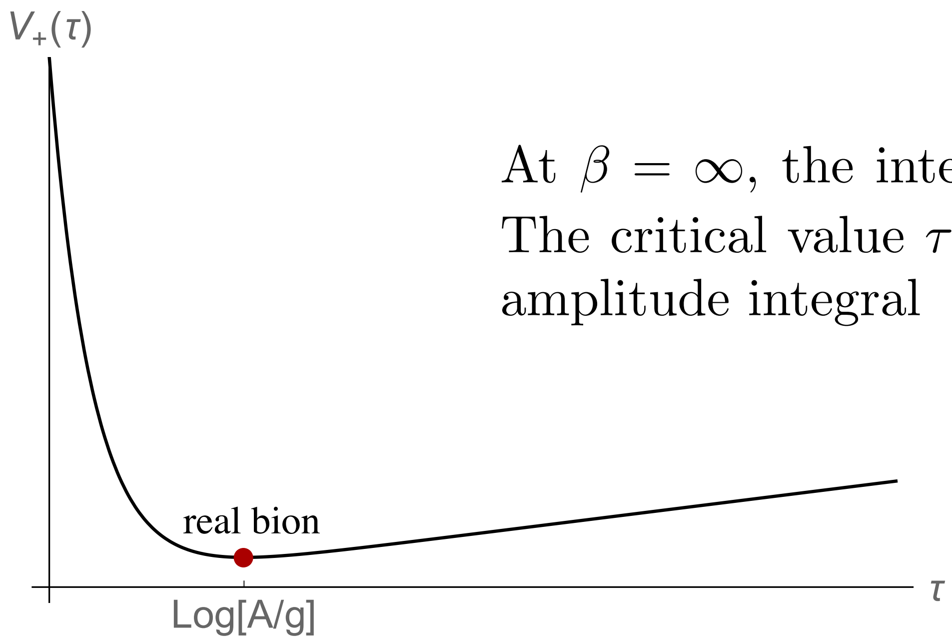
The major contribution on the thimble comes about from configurations (bions) which are exact solutions to quantum modified holomorphic equations of motions. The equations are for a holomorphic classical mechanical systems, and holomorphic version of Newton's equations. These are called real and complex bions and I will show you their plots.

$$\frac{d^2 z}{dt^2} = \frac{\partial V}{\partial z} \qquad \text{or equivalently} \qquad \frac{d^2 x}{dt^2} = +\frac{\partial V_{\mathrm{r}}}{\partial x}\,,$$

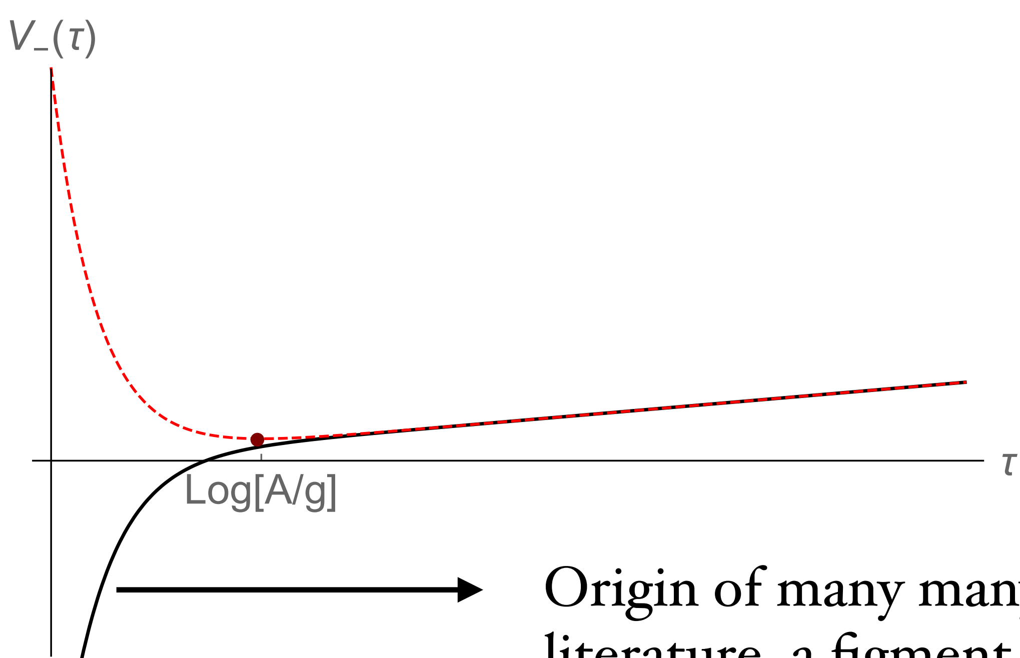
$$\frac{d^2 y}{dt^2} = -\frac{\partial V_{\mathrm{r}}}{\partial y}\,,$$







At $\beta = \infty$, the interaction potential between $\mathcal{I}$ and $\mathcal{I}$ is $V(\tau) = \frac{A}{g} e^{-\tau} + \zeta\, \tau$. The critical value $\tau^* = \ln{(A/g\, \zeta)}$ gives the dominant contribution to the $[\mathcal{I}\mathcal{I}]$ amplitude integral



Origin of many many confusions in literature, a figment of imagination.

(black-solid curve): For real values of the separation $\tau \in \mathbb{R}^+$, which is the naive (or customary) integration cycle, the interactions are completely attractive, and configuration is viewed as unstable. (red-dashed curve): the effective potential on the thimble. The value $\tau^* = \ln{(A/g\, \zeta)} + i\pi$ gives the dominant contribution to the $[\mathcal{I}\bar{\mathcal{I}}]$ amplitude integral.

Working of resurgence at arbitrary ζ

$$
\begin{aligned}
E^{\text{pert}}(N,g;\zeta) &\sim \sum_{n=0}^{\infty} a_n(N;\zeta) g^n \\
&\sim \left(N + \frac{1}{2} - \frac{\zeta}{2}\right) + \frac{1}{8}\Big(- \left[2N^2 + 2N + 1\right] + \left[2N + 1\right]\zeta\Big) g \\
&+ \frac{1}{64}\Big(- \left[4N^3 + 6N^2 + 6N + 2\right] + \left[6N^2 + 6N + 3\right]\zeta - \left[2N + 1\right]\zeta^2\Big) g^2 \\
&+ \frac{1}{256}\Big(- \left[10N^4 + 20N^3 + 32N^2 + 22N + 6\right] + \left[20N^3 + 30N^2 + 32N + 11\right]\zeta \\
&- \left[12N^2 + 12N + 6\right]\zeta^2 + \left[2N + 1\right]\zeta^3\Big) g^3 + \dots
\end{aligned}
$$

Thanks to Tin Sulejmanpasic for his BenderWu Mathematica package, this is possible as a symbolic calculation.

Large-order behavior can be extracted: (Kozcaz, Sulejmanpasic, Tanizaki, MU, 2016)

$$
\begin{aligned}
a_n(N=0;\zeta) &\sim -\frac{1}{\pi}\frac{1}{(8)^{\zeta-1}}\frac{1}{\Gamma(1-\zeta)}\frac{(n-\zeta)!}{(S_b)^{n-\zeta+1}} \\
&\times \left(b_0(\zeta) + \frac{(S_b)\, b_1(\zeta)}{n-\zeta} + \frac{(S_b)^2\, b_2(\zeta)}{(n-\zeta)(n-\zeta-1)} + \dots \right)
\end{aligned}
$$

Working of resurgence at arbitrary ζ

Where b's are non-trivial polynomials of zeta.

$$b_0(\zeta) = 1$$
$$b_1(\zeta) = \frac{1}{8}\left(-5 + 5\zeta - \zeta^2\right)$$
$$b_2(\zeta) = \frac{1}{128}\Big(- 13 + 2\zeta + 15\zeta^2 - 8\zeta^3 + \zeta^4\Big),$$
$$b_3(\zeta) = \frac{1}{3072}\Big(- \zeta^6 + 9\zeta^5 - 10\zeta^4 - 51\zeta^3 - 10\zeta^2 + 381\zeta - 357\Big)$$

And using the NP contributions to the energy:

$$E_{\pm}^{\text{n.p.}}(N=0, g; \zeta) \sim -(2[\mathcal{RB}] + 2[\mathcal{CB}]_{\pm})$$

$$\sim \frac{1}{\pi}\left(\frac{g}{8}\right)^{\zeta-1}\Gamma(\zeta)(-1 - e^{\pm i\pi\zeta})e^{-S_b/g}\underbrace{\left(b_0(\zeta) + b_1(\zeta)g + b_2(\zeta)g^2 + b_3(\zeta)g^3 + \dots\right)}_{\mathcal{P}_{\text{fluc}}(N=0,g;\zeta)}$$

$$\text{Im}\left[\mathcal{S}_{\pm}E^{\text{pert.}}(N=0, g, \zeta) + [\mathcal{CB}]_{\pm}(N=0, g, \zeta)\right] = 0.$$

Quite remarkable, traditional form of resurgence

My emphasis in this lecture was traditional resurgence, an late term/early term relation.

There is actually a new and constructive version which applies to the systems we studied. I cannot make it justice here, but main point is, if you know 10 orders of perturbative expansion around perturbative saddle, you can derive 9 orders around instantons and bions!

$$[\mathcal{RB}] = (\dots)e^{-S_b/g}\mathcal{P}_{\text{fluc}}(\nu, g, \zeta)$$
$$[\mathcal{CB}]_\pm = (\dots)e^{-S_b/g}e^{\pm i\zeta\pi}\mathcal{P}_{\text{fluc}}(\nu, g, \zeta) \ .$$

where $\mathcal{P}_{\text{fluc}}(\nu, g, \zeta)$ is the perturbative expansion around the complex bion;

$$\mathcal{P}_{\text{fluc}}(\nu, g, \zeta) = \sum_{i=0}^{\infty} b_i(\nu, \zeta)\, g^i \ ,$$

The formal power-expansion of the energy in coupling g of the energy level ν is given by

$$E^{\text{pert.}}(\nu, g, \zeta) = a_0(\nu, \zeta) + a_1(\nu, \zeta)g + a_2(\nu, \zeta)g^2 + \cdots$$

We have the following remarkable relation:

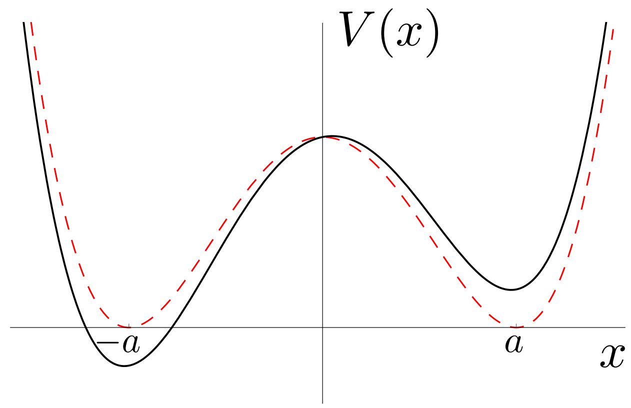
$$\mathcal{P}_{\text{fluc}}(\nu, g, \zeta) = \frac{\partial E^{\text{pert.}}}{\partial \nu} \exp\left[S_b \int_0^g \frac{dg}{g^2}\left(\frac{\partial E^{\text{pert.}}}{\partial \nu} - a_0'(\nu, \zeta) - a_1'(\nu, \zeta)g\right)\right].$$

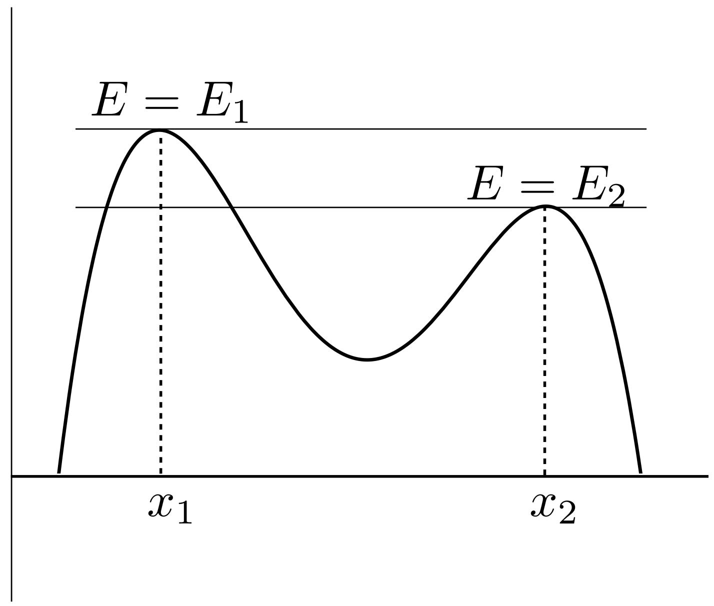
where the prime indicates differentiation with respect to ν.

Complex saddles and hidden topological angles

Supersymmetric QM and complex bions-I

Take Double-well susy QM. This system breaks susy spontaneously. (Witten, 81)
Quantize fermions and reduce the system to Bose-Fermi pair of Hamiltonians with tilted potential.





$$V_{\pm} = \frac{1}{2}(z^2 - 1)^2 \pm gz$$

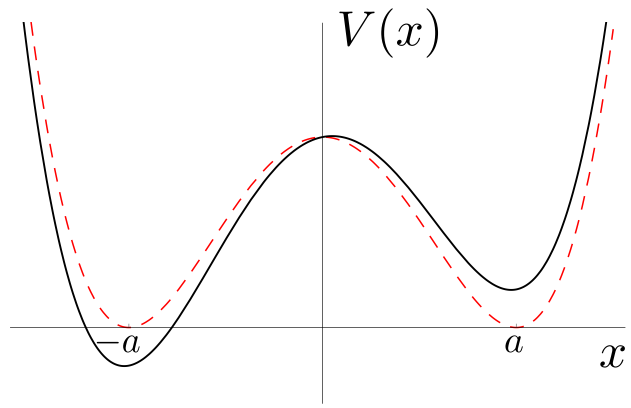
Ground state energy is zero to all orders in P.T. But is known to be lifted non-perturbatively. What causes it?

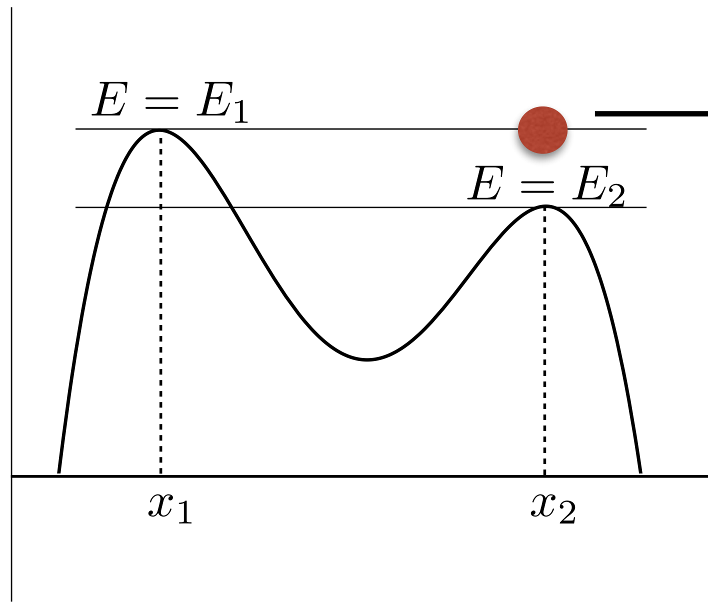
In the inverted potential, there is an obvious real bounce solution, but this is not related to ground state properties.

At level E1, the classical particle will fly of to infinity, infinite action, irrelevant. So, what causes the non-zero ground state energy in bosonized description?

Supersymmetric QM and necessity of complex bions!

Take Double-well susy QM. This system breaks susy spontaneously. (Witten, 81)
Quantize fermions and reduce the system to Bose-Fermi pair of Hamiltonians with tilted potential.



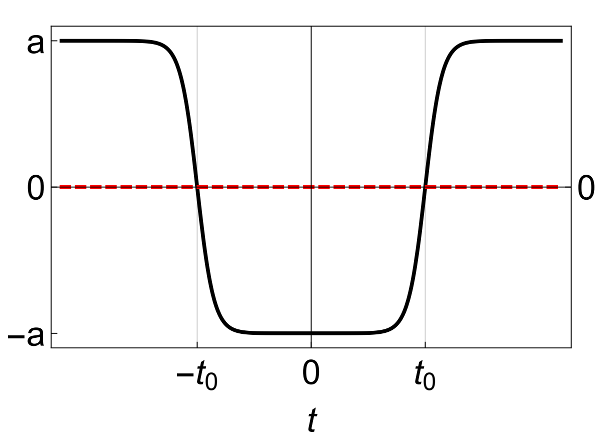


Complex conjugate turning points

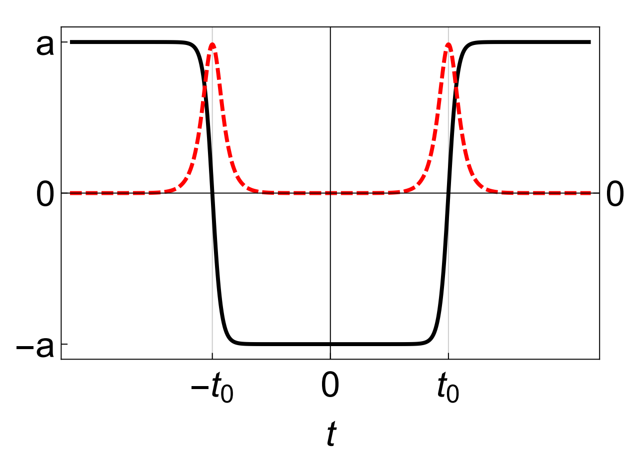
$$V_{\pm} = \frac{1}{2}(z^2 - 1)^2 \pm g z$$

If complex bion is not included, we would conclude Susy is unbroken. Contradiction!

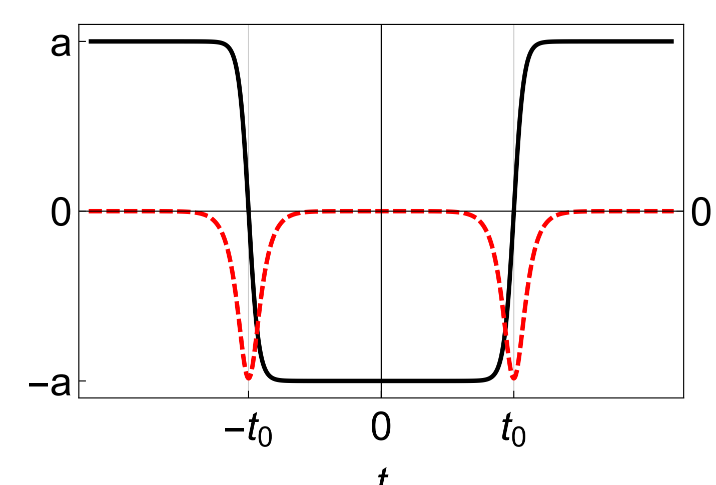
Exact bounce



Exact complex bion-1

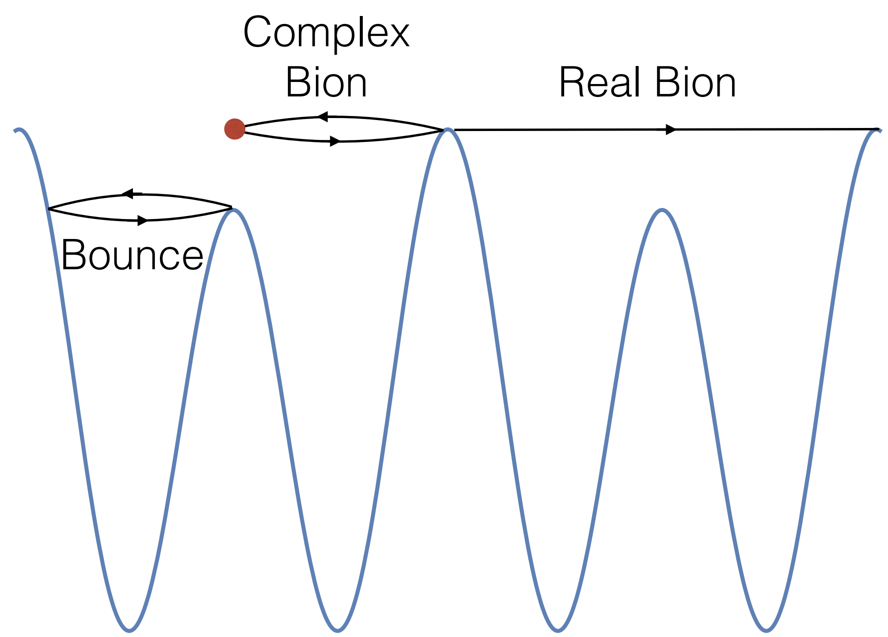


Exact complex bion-2



Periodic potential, real and complex bions

This system has Witten index zero but susy is known to be unbroken. Two ground states, Bose-Fermi paired.

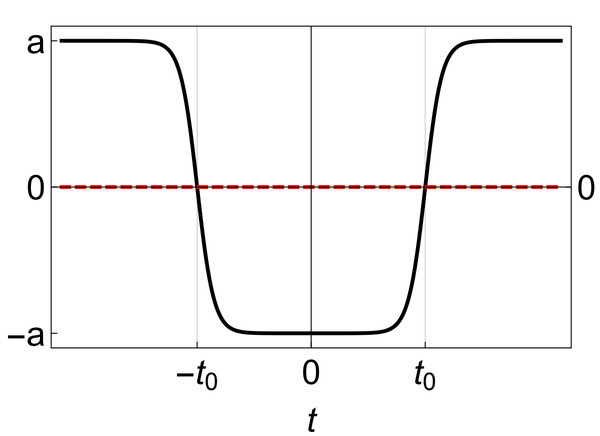


Inverted potential

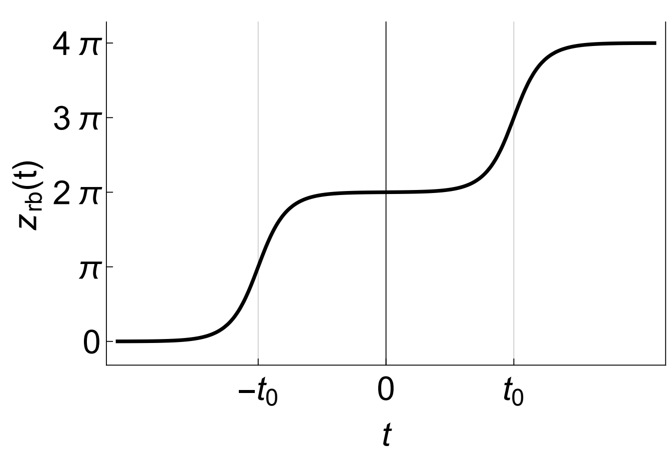
⇒If complex bion is not included, real bion renders ground state energy negative.
In violation of Susy algebra. (a would-be genuine disaster)!

⇒Complex bion is strictly necessary. But it is not only multi-valued, but also singular. Yet, its action is finite.
Imaginary part of action $i\pi$. This is the hidden topological angle (HTA) (Behtash et.al.2015)
This is the sense in which we have to go through a change of perspective in path integrals! These are legit configurations contributing to path integral.

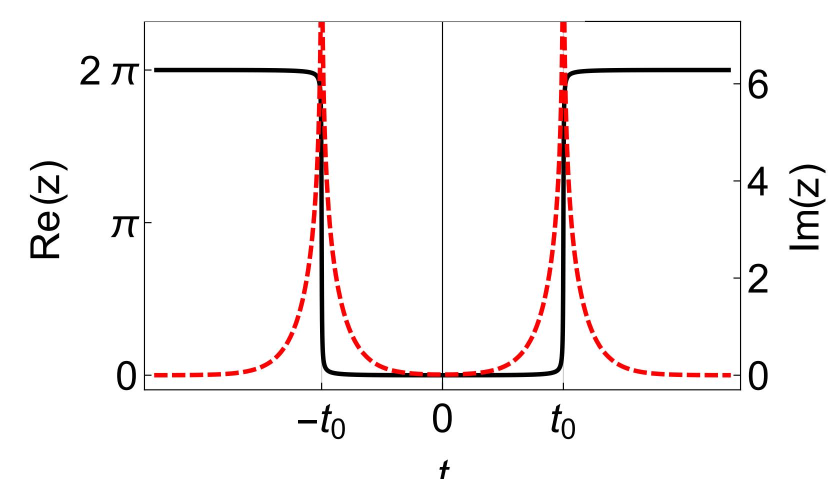
Exact bounce



Real bion



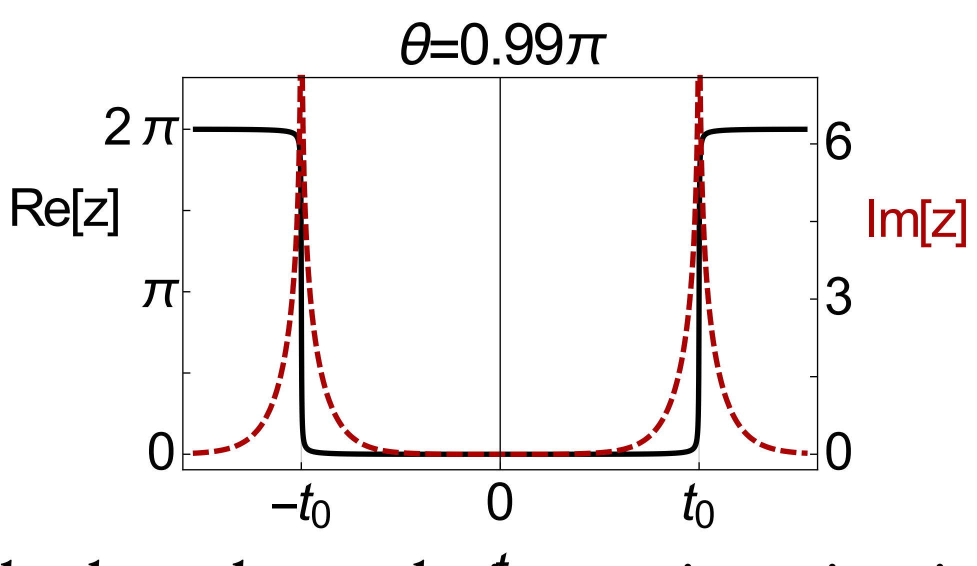
Exact complex bion



More on complex bion solution

The action is finite, and its real part is same as smooth real bion action.
Despite singular behavior of solution!

$$S_{\rm cb} \simeq \left(\frac{16}{g} + p \ln \frac{32}{pg} + \ldots \right) \pm i\, p\, \pi$$



Singularity smoothed out by analytic continuation in θ.
The solution is multi-valued, singular, complex.

If you look any standard textbook or discussion, you will see that these are all big “sins”.

From current point of view, this is the natural realization of semi-classics.

SUSY-QM vacuum: Dilute bion gas

Ground state: Dilute gas of complex and real bions

$$E_{gs} \sim -e^{-S_{\rm cb}} - e^{-S_{\rm rb}} = -e^{\pm i\pi} e^{-2S_{\rm rb}} - e^{-2S_{\rm rb}} = 0$$

Supersymmetry consistent with semi-classics thanks to multivalued complex saddle.