Semi-classics, adiabatic continuity and resurgence in quantum theories

Mithat Ünsal
North Carolina State University

Some of the work presented here is done in collaboration with :
Yaffe, Shifman, Argyres, Poppitz, Schaefer, Dunne, Cherman, Sulejmanpasic, Tanizaki

Motivation: Can we make sense out of QFT? When is there a continuum definition of QFT?

Dyson(50s),
't Hooft (77),

Quoting from M. Douglas comments, in Foundations of QFT, talk at String-Math 2011

"A good deal of mathematical work starts with the Euclidean functional integral. There is no essential difficulty in rigorously defining a Gaussian functional integral, in setting up perturbation theory, and in developing the BRST and BV formulations (Costello).

A major difficulty, indeed many mathematicians would say the main reason that QFT is still "not rigorous," is that standard perturbation theory only provides an asymptotic (divergent) expansion. There is a good reason for this, namely exact QFT results are not (often) analytic in a finite neighborhood of zero coupling.

The situation is actually worse than described by Douglas.
In fact, this is only first and artificially isolated item in a longer list of problems.
For example,

Yang-Mills/QCD/SYM and standard/old problems

1) Perturbation theory is an asymptotic (divergent) expansion *even after regularization and renormalization*. Is there a meaning to perturbation theory?

2) Invalidity of the semi-classical dilute instanton gas approximation in asympotically free theories. Dilute instanton gas assumes inter-instanton separation is much larger than the instanton size, but the latter is a moduli, hence no meaning to the assumption.

3) ``Infrared embarrassment",e.g., large-instanton contribution to vacuum energy is IR-divergent, see Coleman's lectures.

4**)** Incompatibility of large-N results with instantons.(better be so!)

5**)** The renormalon ambiguity, ('t Hooft,79), deeper, to be explained.

You may be surprised to hear that all of the above may very well be interconnected according to the resurgence theory.

In order to say something new on an old problem, we must have new physical perspective and mathematical tools.
Few “recent” ideas from physics and mathematics:

- Resurgence theory and Trans-series
- Complex Morse Theory (or Picard-Lefschetz theory) and complexification of path integral
- Adiabatic Continuity (Avatar of large-N Volume independence)
- Reliable Semi-classics (calculability in gauge theories on R_3 x S_1)

LECTURE-1

Basics structure of perturbation theory
Resurgence
Lefschetz thimbles

in Exponential Integrals

The nature of perturbation theory

- Consider energy level in some generic problem, λ some small parameter:

$$E(\lambda) = E_0 + E_1\lambda + E_2\lambda^2 + E_3\lambda^3 + \cdots$$

- In almost all interesting cases, in QM and QFT, this sum starts to look better and better, but eventually it almost always diverges, $E_n \propto n!$
- Regardless of how small λ is, n! will always render the series divergent.
- Perturbation theory yields divergent asymptotic series.
- But it works!

Perturbation theory works

QED perturbation theory:

$$\frac{1}{2}(g-2) = \frac{1}{2}\left(\frac{\alpha}{\pi}\right) - (0.32848\ldots)\left(\frac{\alpha}{\pi}\right)^2 + (1.18124\ldots)\left(\frac{\alpha}{\pi}\right)^3 - (1.7283(35))\left(\frac{\alpha}{\pi}\right)^4 + \ldots$$

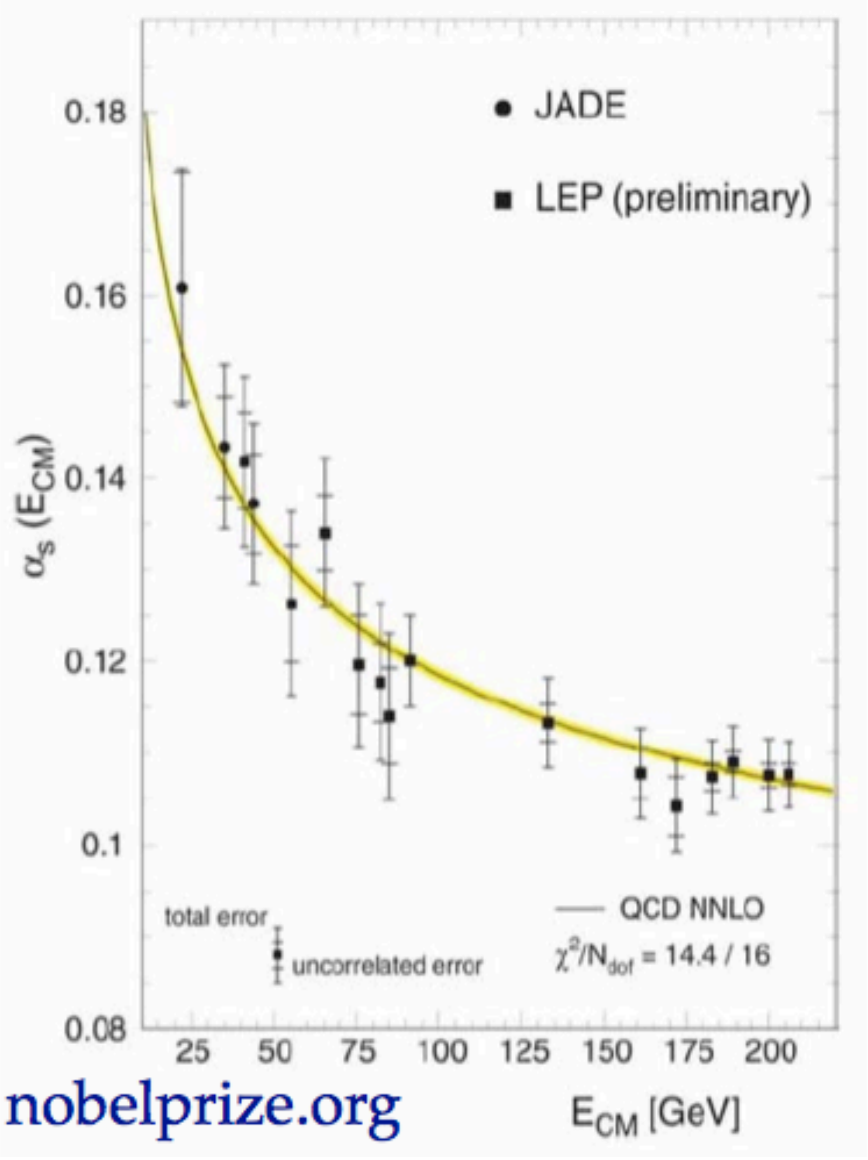
$$\left[\tfrac{1}{2}(g-2)\right]_{\text{exper}} = 0.001\,159\,652\,180\,73(28)$$

$$\left[\tfrac{1}{2}(g-2)\right]_{\text{theory}} = 0.001\,159\,652\,184\,42$$

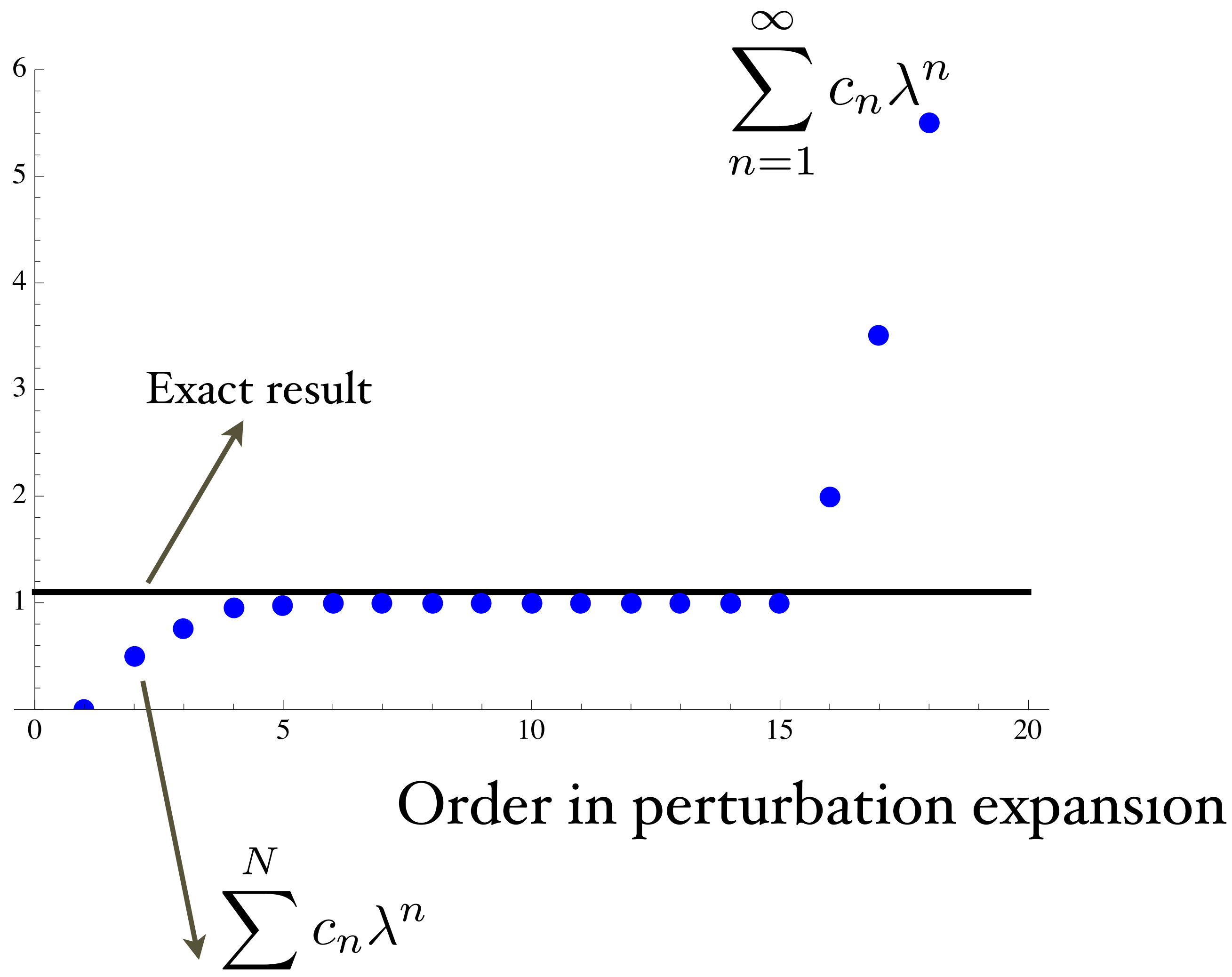
Magnetic moment of an electron, the best theory/experiment agreement in physics. Based on perturbation theory in QED.

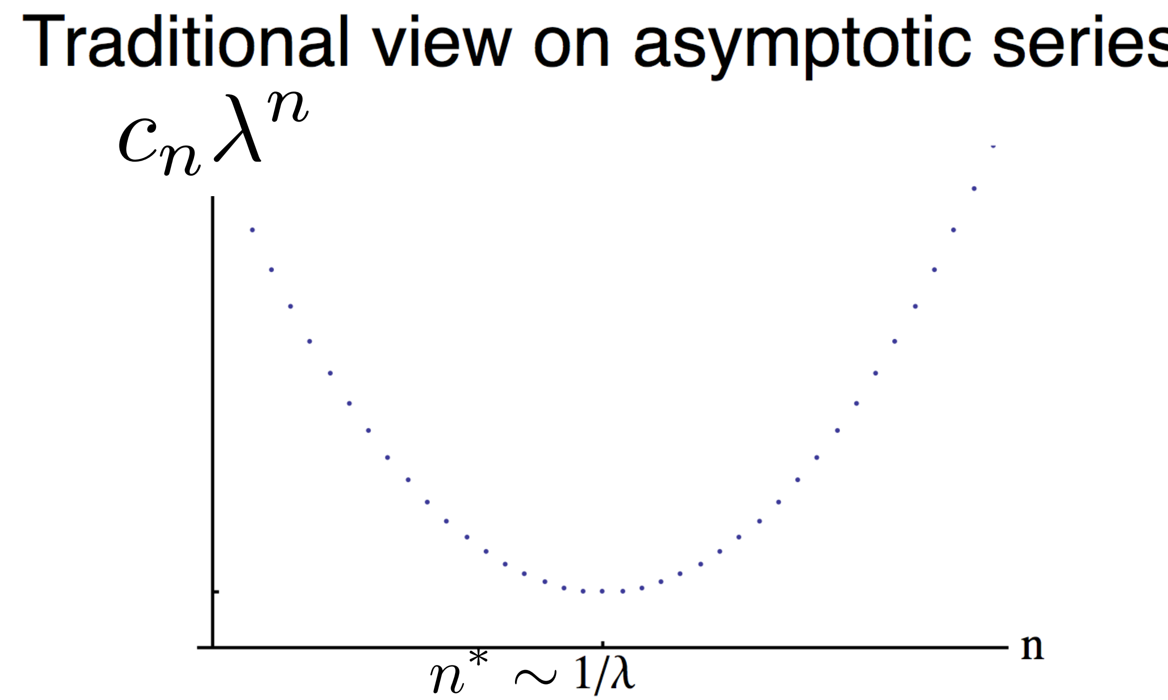
QCD: asymptotic freedom Remarkable agreement

$$\beta(g_s) = -\frac{g_s^3}{16\pi^2}\left(\frac{11}{3}N_C - \frac{4}{3}\frac{N_F}{2}\right)$$



Universal behavior of perturbation theory

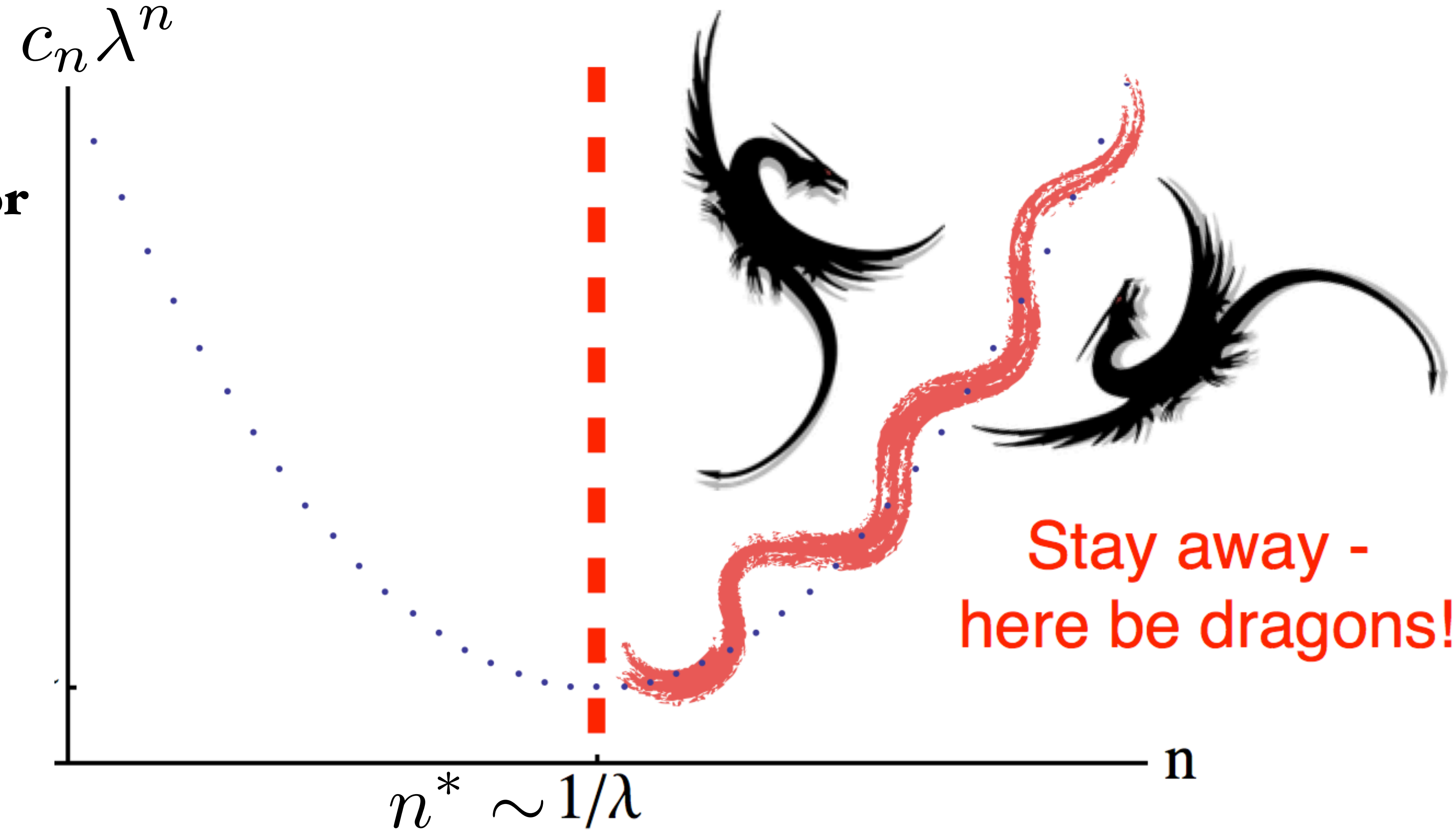




Approximate value at a given order in perturbative expansion

Stokes (~1850s) brilliant realization:
There is an optimal order at which the error is minimized!

Traditional view on asymptotic series



Two major points:

Stokes (1850) **truncates** the series when the error is minimal (least term or optimal truncation) and accepts that there is an **intrinsic vagueness.**

This vagueness turns out to be physically (extremely) interesting and deeper.

Watch carefully. This is important and easy.

Error: The deficit between exact result and the best perturbation theory can do.

$$\text{Error} \sim n^*!\lambda^{n^*} \qquad \text{use Stirling} - \text{approximation}$$

$$\sim \left(\frac{n^*}{e}\right)^{n^*} \lambda^{n^*} \qquad \text{use} \qquad n^* \sim 1/\lambda$$

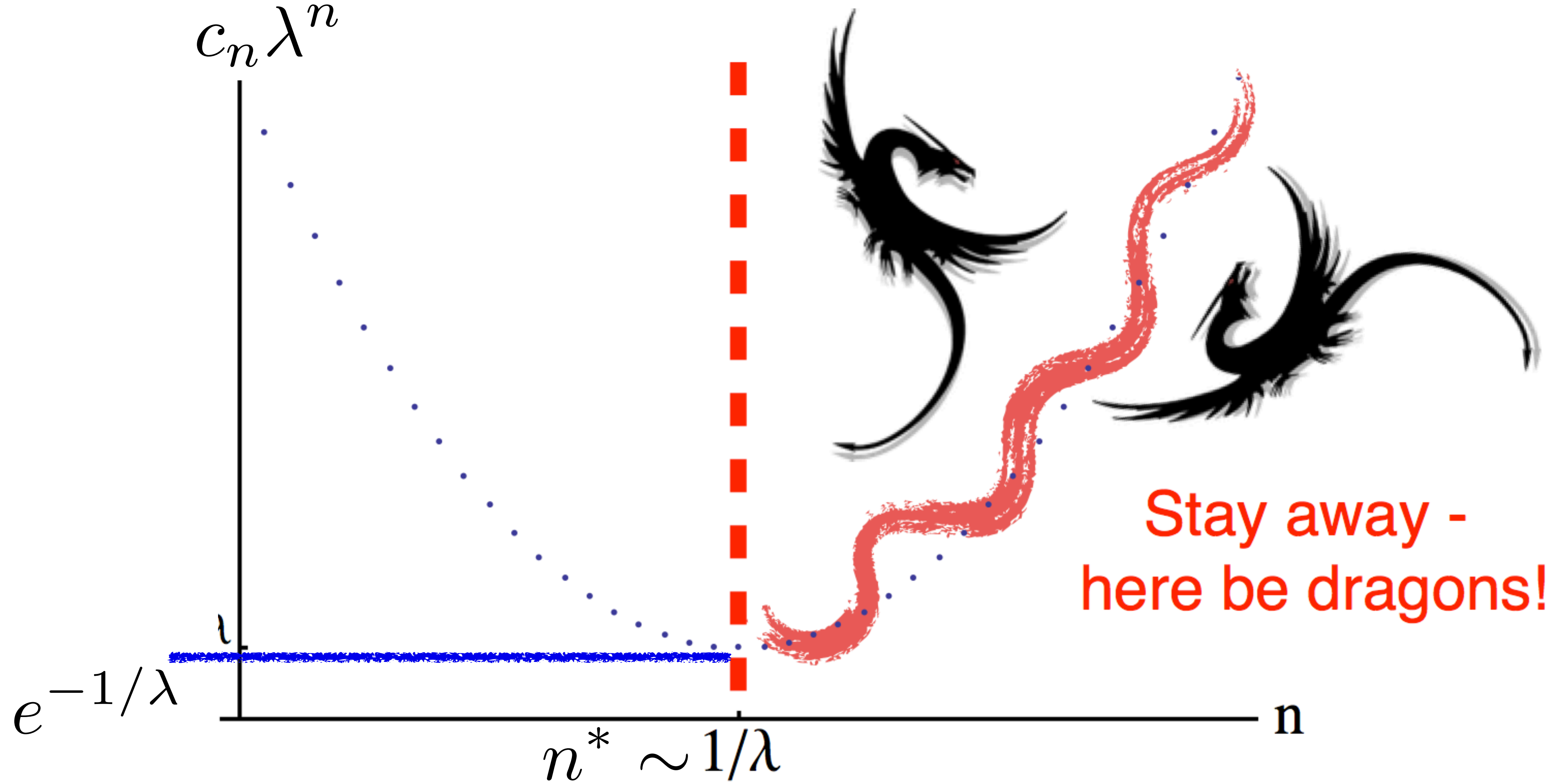
$$\sim e^{-1/\lambda}$$

Intrinsic (irremovable) error in perturbation theory is **non-perturbative!**

Exp[-1/λ] has essential singularity at zero, not describable in terms of pert. expansion. If you try to do Taylor expansion, you obtain **0+0+0+0 ……** ad infinitum.

This is one reason why perturbative vs. non-perturbative phenomena in books are in different sections and not so much in relation to each other.

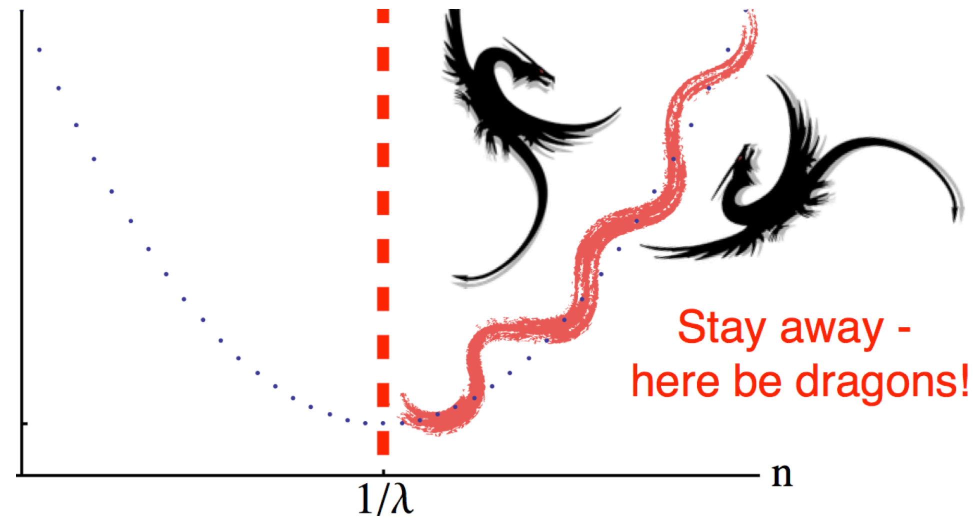
Traditional view on asymptotic series



~1850, Stokes observed something even deeper. There is another saddle in the problem which contributes exactly as $\exp[-1/\lambda]$!

This is actually interesting. The intrinsic vagueness of perturbation theory is related to the existence of another saddle in the problem and its non-perturbative contribution!

More than a century after Stokes:



People *started* to understand what Stokes did.
Genuine (sporadic) improvements of his ideas by mathematical physicists and mathematicians.

Robert Dingle: Universality of factorial divergence (50s-60s)
Jean Ecalle: (Resurgent) Algebraic structure in late terms [non-linear ODEs solutions](80s)
Michael Berry: Hyperasymptotic improvements (90s to today)
Chris Howls: Hyperasymptotics (90s to today)

Berry-Howls discovered, for simple ordinary integrals, something extremely remarkable (and something sufficiently explicit that physicist can appreciate.)

Not just a mathematical curiosity

Some of the most interesting phenomena in atomic and molecular physics, condensed matter physics, particle physics are non-perturbative $\mathrm{Exp}[-1/\lambda]$ effects.

Tunneling in quantum mechanics

Band-structures in solid state physics

Superconductivity

Your body mass and the mass of everything you see around you!

=proton and neutron mass, according to QCD

D-branes in string theory

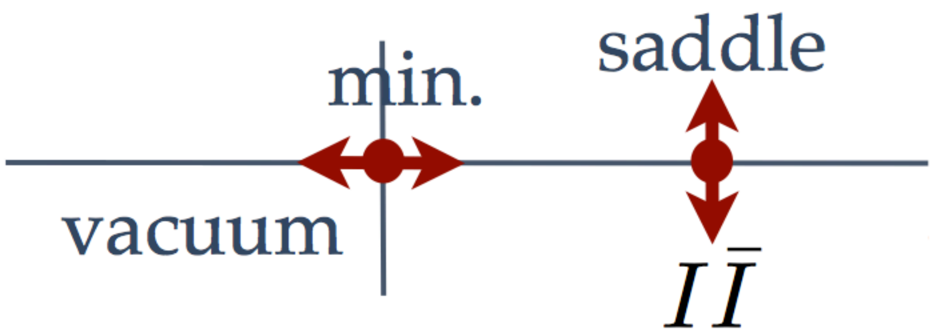
. . . .
So, **the "error" is important. (as discovered in many context, many times).**
A more systematic approach is called for.

Simple example: 2 saddles

$d = 0$ partition function for periodic potential $V(z) = \sin^2(z)$

$$Z(\lambda) = \frac{1}{\sqrt{\lambda}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, e^{-\frac{1}{2\lambda}\sin^2(x)}$$

two saddle points: $z_0 = 0$ and $z_1 = \frac{\pi}{2}$.



Perturbative expansion around a saddle

Perturbation theory around saddle-1:

$$\mathfrak{Z}_1(\lambda) = i e^{-\frac{A_1}{\lambda}} \Phi_1(\lambda) = i \sum_{n=0}^{\infty} a_n^{(1)} \lambda^n \equiv e^{-\frac{1}{2\lambda}} \Phi_1(\lambda)$$

$$= i e^{-\frac{1}{2\lambda}} \sqrt{2\pi} \left(1 - \frac{1}{2}\lambda + \frac{9}{8}\lambda^2 - \frac{75}{16}\lambda^3 + \frac{3675}{128}\lambda^4 - \frac{59535}{256}\lambda^5 + \dots \right)$$

Large-order of perturbation theory around saddle-0:

$$\mathfrak{Z}_0(\lambda) \equiv e^{-\frac{A_0}{\lambda}} \Phi_0(\lambda) = \sum_{n=0}^{\infty} a_n^{(0)} \lambda^n$$

$$a_n^{(0)} \sim \sqrt{2\pi} \frac{(n-1)!}{(A_{10})^n} \left(1 - \frac{\frac{1}{2} A_{10}}{(n-1)} + \frac{\frac{9}{8}(A_{10})^2}{(n-1)(n-2)} - \frac{\frac{75}{16}(A_{10})^3}{(n-1)(n-2)(n-3)} + \dots \right), \quad n \to \infty$$

Clearly, the divergence of perturbation theory is not a nuisance or something to be ignored.

The divergent asymptotic part is coded information about the other saddle in the problem, at least for one dimensional integrals! (Berry-Howls 90s).

Michael Berry: ICTP, Trieste, 50th year celebration talk,2014

"Divergent series: From Thomas Bayes to resurgence via rainbow."
(You can see it on Youtube. Search: Michael Berry physics strongly recommended)

From Intro of his talk: "Understanding divergence has been a thread running through mathematics for several centuries. The subject has been repeatedly reborn, more deeply each time, and it is happening again now."

"A divergent series is not meaningless, or a nuisance, but an essential and informative coded representation of the function."

From the final part: "Now, and to my great surprise, there is another rebirth, appears in applications in QFT and string theory.

The difficulty, the technical difficulty, is immense. Because it is not just a question of double, quadrupole integrals, it is integrals in field theory with infinitely many variables, and it could well be that things are a bit different there."

In QM path integral: infinitely many coupled exponential integrals

Large-order of perturbation theory around perturbative vacuum for ground state N=0 in periodic potential:

$$a_n(N=0) \sim -\frac{1}{\pi}\frac{n!}{(2S_I)^{n+1}}\left(1 - \frac{5}{2}\cdot\frac{(2S_I)^1}{n} - \frac{13}{8}\cdot\frac{(2S_I)^2}{n(n-1)} + \ldots\right)$$

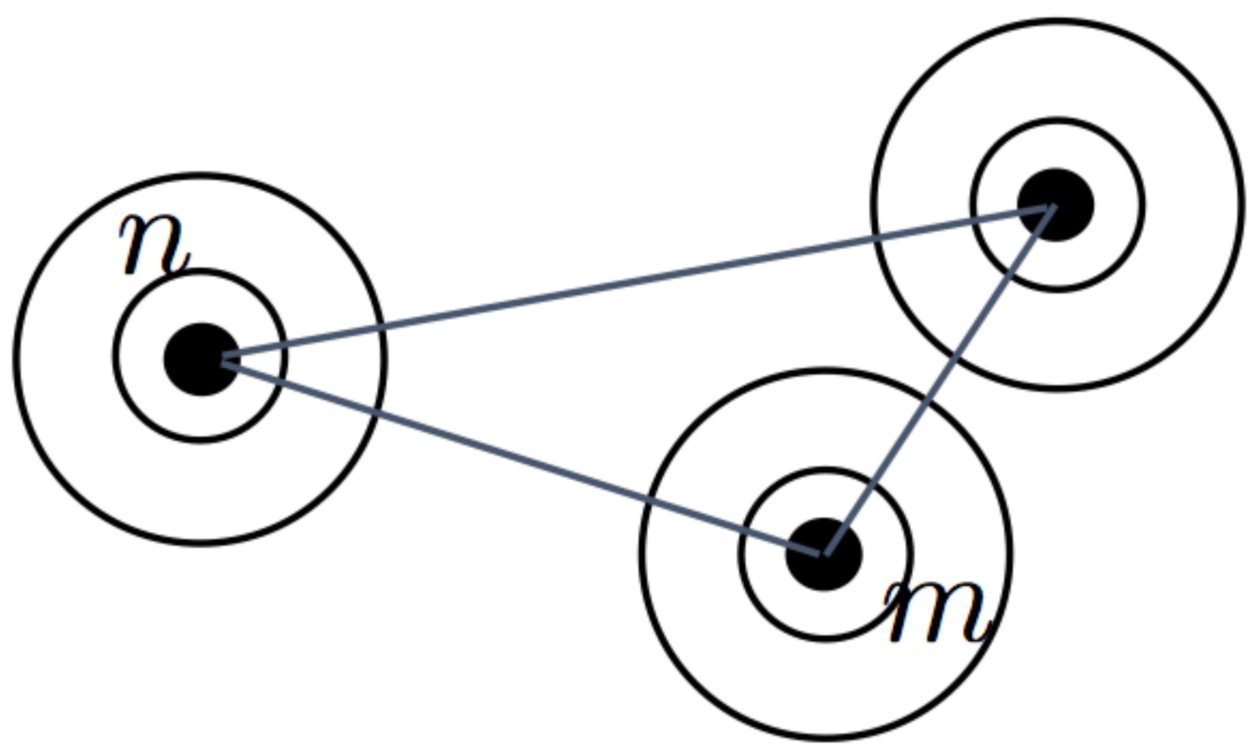
Contribution of instanton-antiinstanton critical point at infinity (a type of saddle that I will make precise) to ground state energy.

$$\mathrm{Im}[\mathcal{I}\bar{\mathcal{I}}]_{\pm} \sim \pm\pi e^{-(2S_I)/g}\left(1 - \frac{5}{2}\cdot g - \frac{13}{8}\cdot g^2 \ldots\right)$$

The leading terms obtained in Bogomolny and Zinn-Justin early 80s, but not sufficiently appreciated. The overall structure was obtained in 2014, in Gerald Dunne and MU. Why is this happening?

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle, 1980



Simpler question: Can we make sense of the semi-classical expansion of QFT?

Argyres, MÜ, Dunne, MÜ, 2012

$$f(\lambda\hbar) \sim \sum_{k=0}^{\infty} c_{(0,k)} \, (\lambda\hbar)^k + \sum_{n=1}^{\infty} (\lambda\hbar)^{-\beta_n} \, e^{-n\, A/(\lambda\hbar)} \sum_{k=0}^{\infty} c_{(n,k)} \, (\lambda\hbar)^k$$

pert. th. — n-instanton factor — pert. th. around n-instanton

All series appearing above are asymptotic, i.e., divergent as $c_{(0,k)} \sim k!$.

The combined object is called trans-series following resurgence terminology.

Trans-series well-defined under analytic continuation.

All trans-series coefficients are correlated in a precise sense.

Borel transform and resummation

Let $P(g^2)$ denote a perturbative asymptotic series that satisfy

$$P(g^2) = \sum_{q=0}^{\infty} a_q g^{2q}, \qquad \text{Gevrey} - 1: \quad |a_q| \leq C R^q q!$$

for some positive constants C and R, i.e., it diverges factorially.

Borel transform of $P(g^2)$ by $BP(t)$:

$$BP(t) := \sum_{q=0}^{\infty} \frac{a_q}{q!} t^q .$$

A finite radius of convergence.

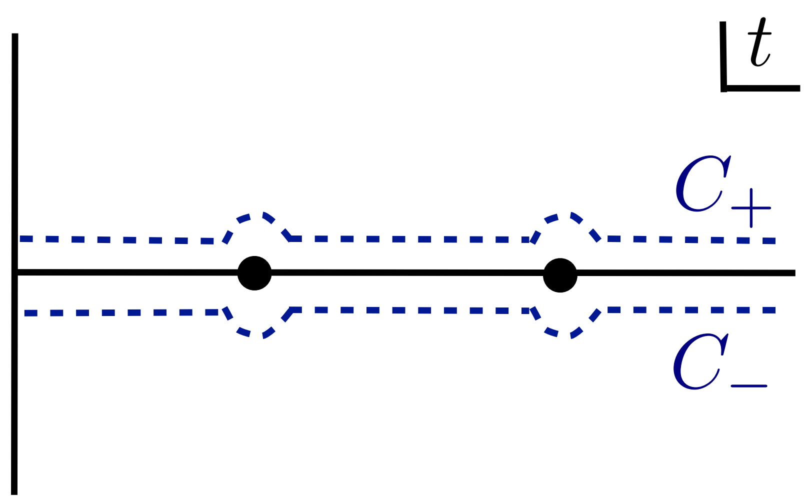
Borel resummation: The Borel resummation of $P(g^2)$, when it exists

$$\mathbb{B}(g^2) = \frac{1}{g^2} \int_0^{\infty} BP(t) e^{-t/g^2} dt \, .$$

If $BP(t)$ has no singularities on $\mathbb{R}^+$, then, we say, $\mathbb{B}(g^2)$ is the (unique) Borel resummation of $P(g^2)$.

Lateral Borel sums and ambiguity

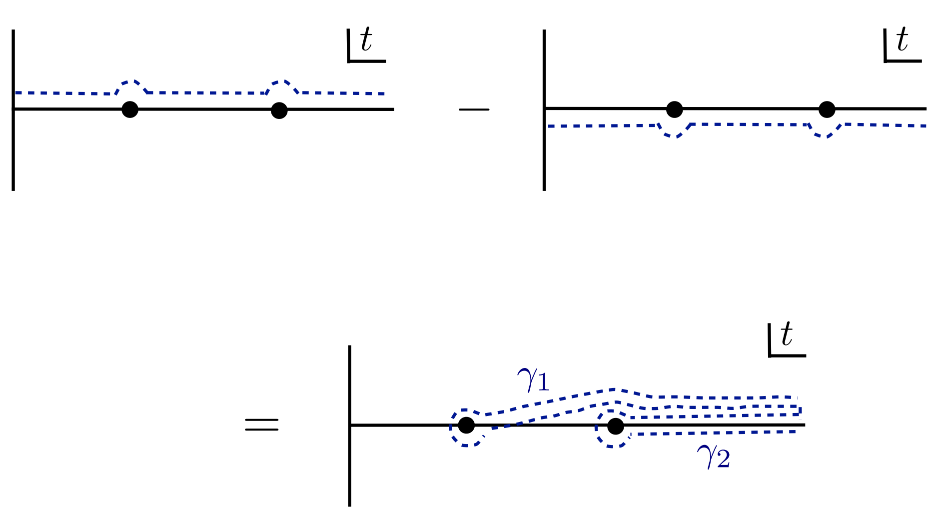
Directional (sectorial) Borel sum. $\mathcal{S}_\theta P(g^2) \equiv \mathbb{B}_\theta(g^2) = \frac{1}{g^2}\int_0^{\infty\cdot e^{i\theta}} BP(t)\, e^{-t/g^2}\, dt$



left and right Borel sums

$$\mathbb{B}_{0\pm}(|g^2|) = \mathrm{Re}\,\mathbb{B}_0(|g^2|) \pm i\,\mathrm{Im}\,\mathbb{B}_0(|g^2|), \qquad \mathrm{Im}\,\mathbb{B}_0(|g^2|) \sim e^{-2S_I} \sim e^{-2A/g^2}$$

The *non-equality* of the left and right Borel sum means the series is *non-Borel summable or ambiguous*. The ambiguity has the same form of a 2-instanton factor (not 1) in QM. The measure of ambiguity (Stokes automorphism/jump in g-space interpretation):



$$\mathcal{S}_{\theta+} = \mathcal{S}_{\theta-} \circ \underline{\mathfrak{S}}_\theta \equiv \mathcal{S}_{\theta-} \circ (\mathbf{1} - \mathrm{Disc}_{\theta-})\,,$$

$$\mathrm{Disc}_{\theta-}\,\mathbb{B} \sim e^{-t_1/g^2} + e^{-t_2/g^2} + \ldots \qquad t_i \in e^{i\theta}\mathbb{R}^+$$

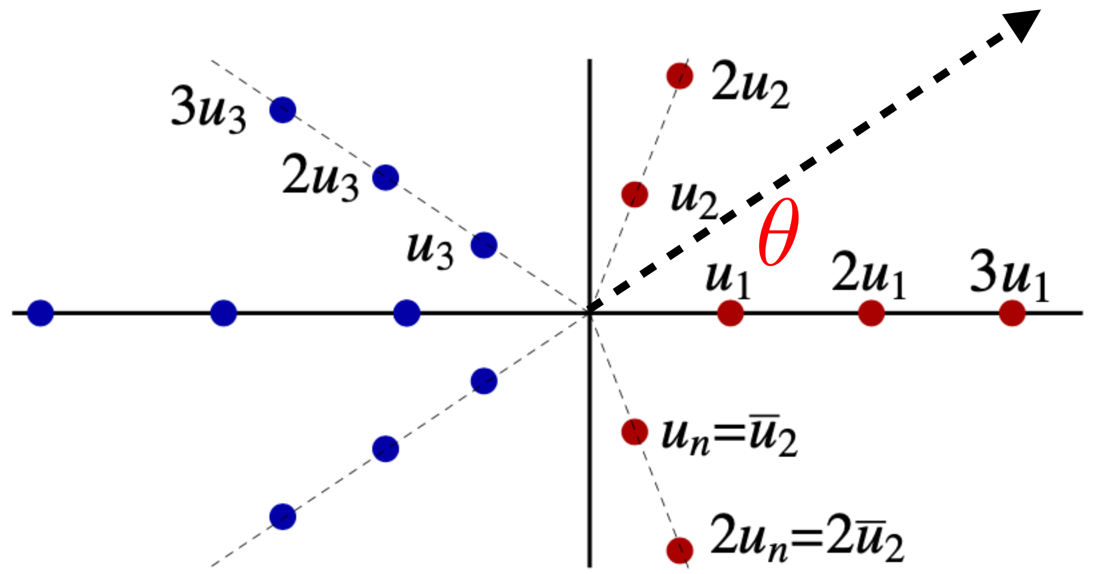
Jean Ecalle, 80s

Borel triangle

$$P(g^2) \xrightarrow{\text{Borel-trans.}} BP(t)$$

Borel-sum: $P(g^2) \searrow \mathcal{S}_\theta P(g^2)$

Directional Laplace trans.: $BP(t) \swarrow \mathcal{S}_\theta P(g^2)$



If theta is a non-singular direction, all is good.

If theta is a singular direction, at this stage, it naively looks like we traded one pathology (divergence) with another (complex imaginary ambiguity).

It looks like we did not gain much, except that, we realize the ambiguity is related to another saddle in the problem.

Saddle points and Lefschetz thimbles-I

Next, I will describe a geometric perspective on Borel resummation.
First, we need to discuss saddle point method properly.

$$\int_\Sigma dx_1 \ldots dx_N \, e^{-S}$$

- The critical points (saddles) $\rho_\sigma, \qquad \sigma = 1, \ldots, N_{\text{saddle}}$ found by $\frac{\partial S}{\partial z_i} = 0$
- The critical point cycles (Lefschetz thimbles) $\mathcal{J}_\sigma$ associated with them.

Thimbles can be thought as forming a complete basis over a vector space,
any integration can be expressed as a linear combinations of them.
(This is called homology cycle decomposition.)

$$\Sigma = \sum_\sigma n_\sigma \mathcal{J}_\sigma, \qquad \dim_{\mathbb{R}}(\mathcal{J}_\sigma) = N.$$

Complex gradient flow equations

$$\frac{\partial z_i}{\partial \tau} = \frac{\partial \bar{S}}{\partial \bar{z}_i}, \qquad \frac{\partial \bar{z}_i}{\partial \tau} = \frac{\partial S}{\partial z_i}, \qquad (i = 1, \ldots, N),$$

Saddle point method and Lefschetz thimbles-I

Using complex gradient flow equation, prove that the imaginary part of the action is *invariant* under the flow.

$$\frac{\partial \mathrm{Im}[S]}{\partial \tau} = \frac{1}{2i}\left(\frac{\partial S}{\partial z_i}\frac{\partial z_i}{\partial \tau} - \frac{\partial \bar{S}}{\partial \bar{z}_i}\frac{\partial \bar{z}_i}{\partial \tau}\right) = 0$$

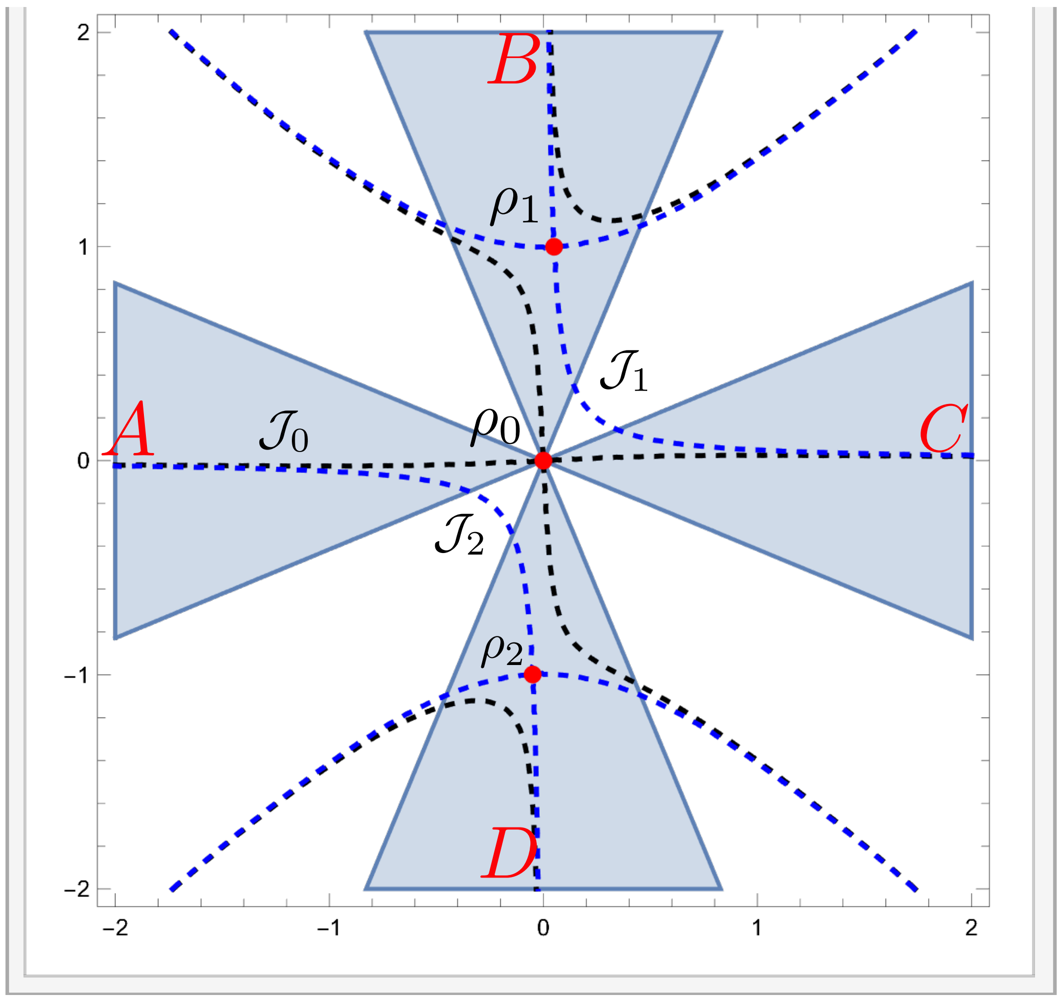
$$\mathrm{Im}[S(z_1, \ldots, z_n)] = \mathrm{Im}[S_\sigma]$$

This is the reason the 1d version of this story is sometimes called *stationary phase* method. The real part obeys

$$\frac{\partial \mathrm{Re}[S]}{\partial \tau} = \frac{1}{2}\frac{\partial}{\partial \tau}(S + \bar{S}) = \left|\frac{\partial S}{\partial z_i}\right|^2 > 0$$

and exponent e^{-S} is ever decreasing. And this is the reason that it is also called *steepest descent* method. Guarantees the convergence of integration over the cycle $\mathcal{J}_\sigma$.

Two simple examples-I: Polynomial potential



$$I(w) = \int_{-\infty}^{\infty} dz\, e^{-(wz^2 + \frac{1}{2}z^4)},$$

$$\theta = \arg(w)$$

$\theta = 0^-$ in the plot.

$$I_\sigma(\lambda) = \underbrace{\int_{\mathcal{J}_\sigma(\theta)} dz_1 \dots dz_n\, e^{-\frac{1}{\lambda}S(z_i)}}_{\text{integration over thimble}} = \underbrace{e^{-\frac{1}{\lambda}S_\sigma} \mathcal{S}_\theta \Phi_\sigma(\lambda)}_{\text{Borel resummation}}$$

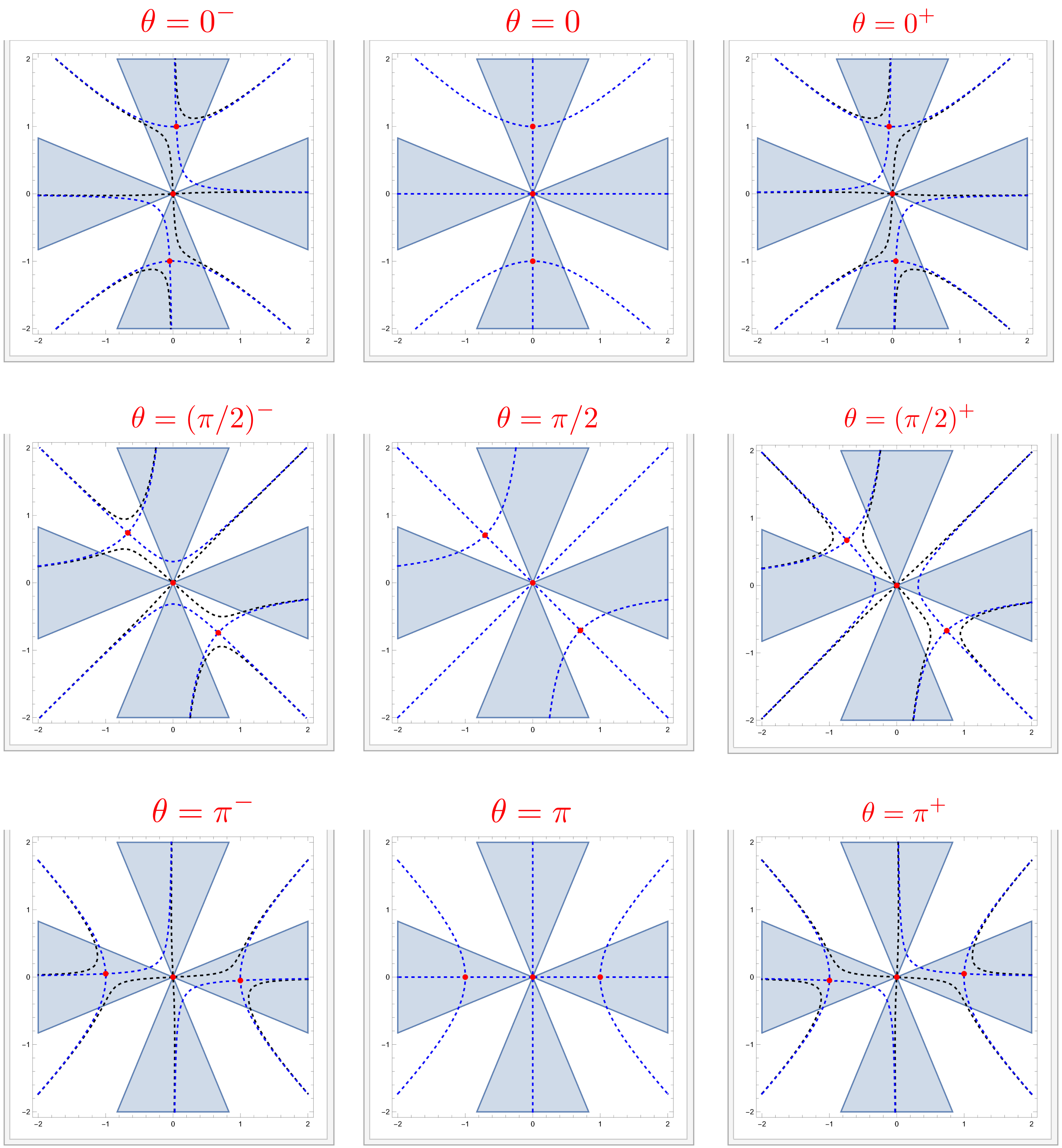
$$I(\lambda) = \sum_\sigma n_\sigma(\theta) \mathrm{Int}[\mathcal{J}_\sigma(\theta)] = \sum_\sigma n_\sigma(\theta) e^{-\frac{S_\sigma}{\lambda}} \mathcal{S}_\theta \Phi_\sigma(\lambda)$$

Thimble decomposition   Transseries expansion

- $\mathcal{J}_a(\theta)$ is piece-wise continuous. It is discontinuous at Stokes line.
- $n_a(\theta)$ is piece-wise constant. It is discontinuous at Stokes line. Number of active saddles changes crossing the Stokes line.
- The Stokes line associated with ρ_0 saddle is at $\arg(w) = \theta = \pi/2, 3\pi/2$. The Stokes line associated with $\rho_{1,2}$ saddles is at $\arg(w) = \theta = 0, \pi$.
- The two discontinuities are present to make the function $I(w)$ well-defined through the integral $\int_{\Gamma_{AC}}$ continuous across Stokes line.

$$\Gamma_{AC} = \begin{cases} \mathcal{J}_0(\theta) & \theta \in (0, \frac{\pi}{2}) \\ \mathcal{J}_1(\theta) + \mathcal{J}_0(\theta) + \mathcal{J}_2(\theta) & \theta \in (\frac{\pi}{2}, \pi) \end{cases}$$

Stokes lines and phenomena



$$\Gamma_{AC} = \begin{cases} \mathcal{J}_0(\theta) & \theta \in (0, \frac{\pi}{2}) \\ \mathcal{J}_1(\theta) + \mathcal{J}_0(\theta) + \mathcal{J}_2(\theta) & \theta \in (\frac{\pi}{2}, \pi) \end{cases}$$

Passing remark: Gradient flow vs. instantons

You may also realize that the complex gradient flow equation is actually instanton equation in extended supersymmetric (N=2) quantum mechanics with superpotential W(z)= S(z).

$$\frac{\partial z}{\partial \tau} = \frac{\partial \overline{W}}{\partial \bar{z}}$$

This is not an accident. Instanton solutions in 1D QM are related to Lefschetz thimbles in the ordinary integrals. But I will not describe this in detail.

Similarly, the real gradient flow equation is instanton equation in minimal supersymmetric (N=1) quantum mechanics with superpotential W(x)= S(x).

This also has a bearing in higher dimension. For example, the real gradient flow equation in 3d where action is Chern-Simons functional is the instanton equation in 4d.

But these relations will not be discussed here.

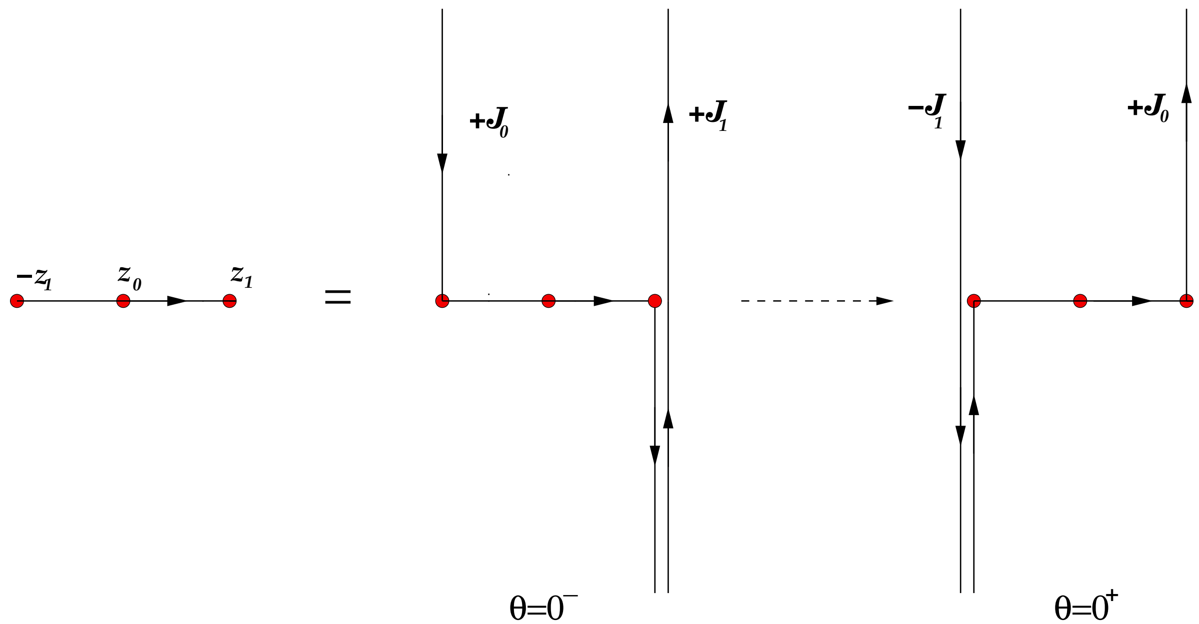
Stokes phenomena, ambiguities and their cancellations.

$$Z^{0\mathrm{d}}(\lambda) = \frac{1}{\sqrt{\lambda}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \, e^{-\frac{1}{2\lambda}\sin^2(x)}$$

$$\frac{dS}{dz} = 0 \Longrightarrow \text{critical points: } \{z_0, z_1\} = \left\{0, \frac{\pi}{2}\right\} .$$

To each saddle, there is a unique steepest descent path (Lefschetz thimble). Thimbles form natural basis for integration and analytic continuation. (P-saddle and NP-saddle.)

Original cycle = linear combination of these thimbles.
But on the Stokes line, thimble decomposition is multi-fold ambiguous!



$$I = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow \Sigma = \begin{cases} \mathcal{J}_0(0^-) + \mathcal{J}_1(0^-) \\ \mathcal{J}_0(0^+) - \mathcal{J}_1(0^+) \end{cases}$$

Geometrization of the ambiguity:
The direction of the tail of J_0 flips.

$$\mathrm{Im} S(z)|_{\mathcal{J}_i} = \mathrm{Im} S(z_i),$$

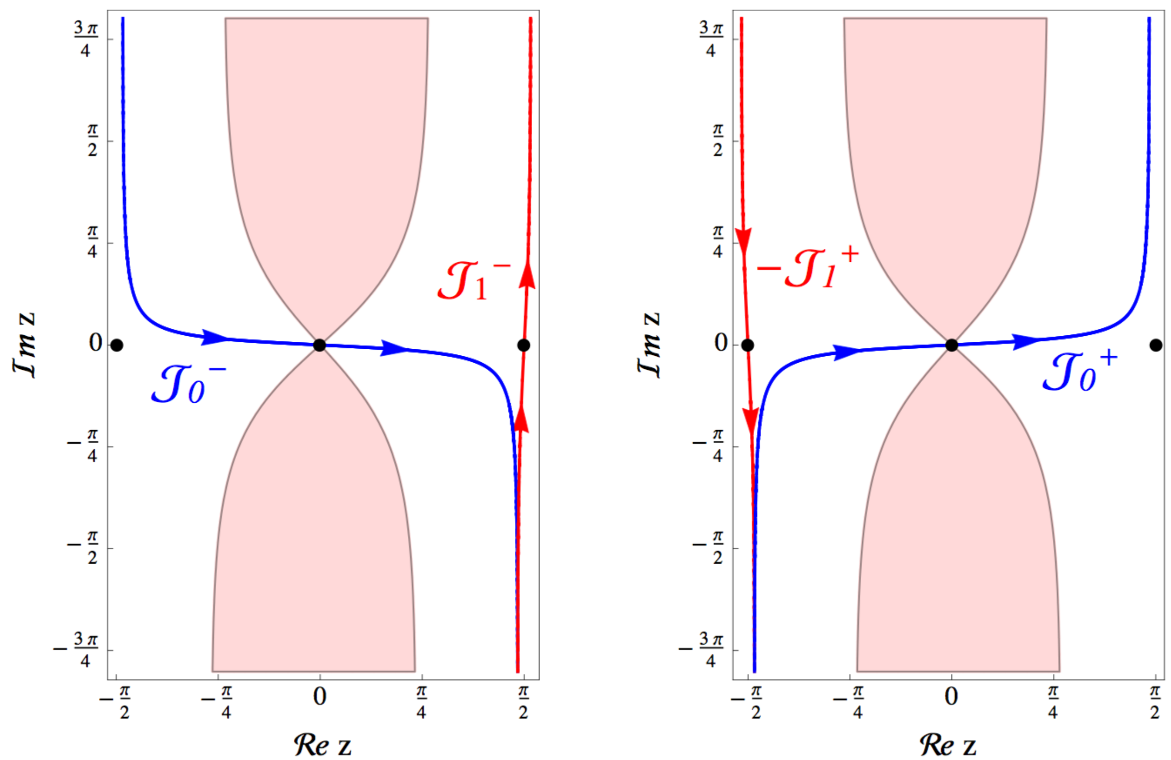
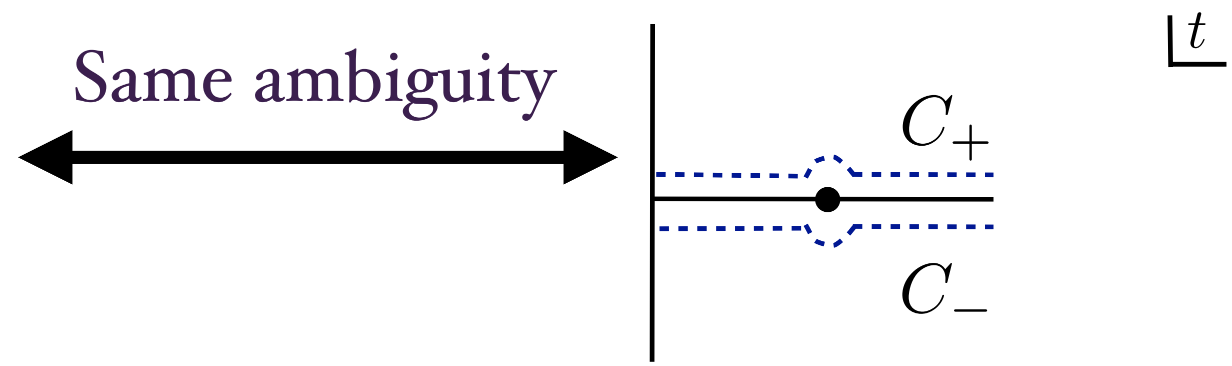


Figure 1. Left: Lefschetz thimbles at $\lambda = e^{i\theta}$ with $\theta = 0^-$: $\mathcal{J}_0 + \mathcal{J}_1$. **Right:** At $\theta = 0^+$. $\mathcal{J}_0 - \mathcal{J}_1$. We take $\theta = \mp 0.1$ to ease visualization.



$$\mathbb{B}_{0^\pm}(\lambda) = \int_{\mathcal{J}_0^\pm} e^{-\frac{1}{\lambda} S}$$

Giving an elegant geometric meaning to Borel analysis:

Left/right Borel sum = Integration over Lefschetz thimble!

Borel ambiguity=Ambiguity in the choice of the cycle on a Stokes line

$$I(\lambda) = \sum_\sigma n_\sigma(\theta) \mathrm{Int}[\mathcal{J}_\sigma(\theta)] = \sum_\sigma n_\sigma(\theta) e^{-\frac{S_\sigma}{\lambda}} \mathcal{S}_\theta \Phi_\sigma(\lambda)$$

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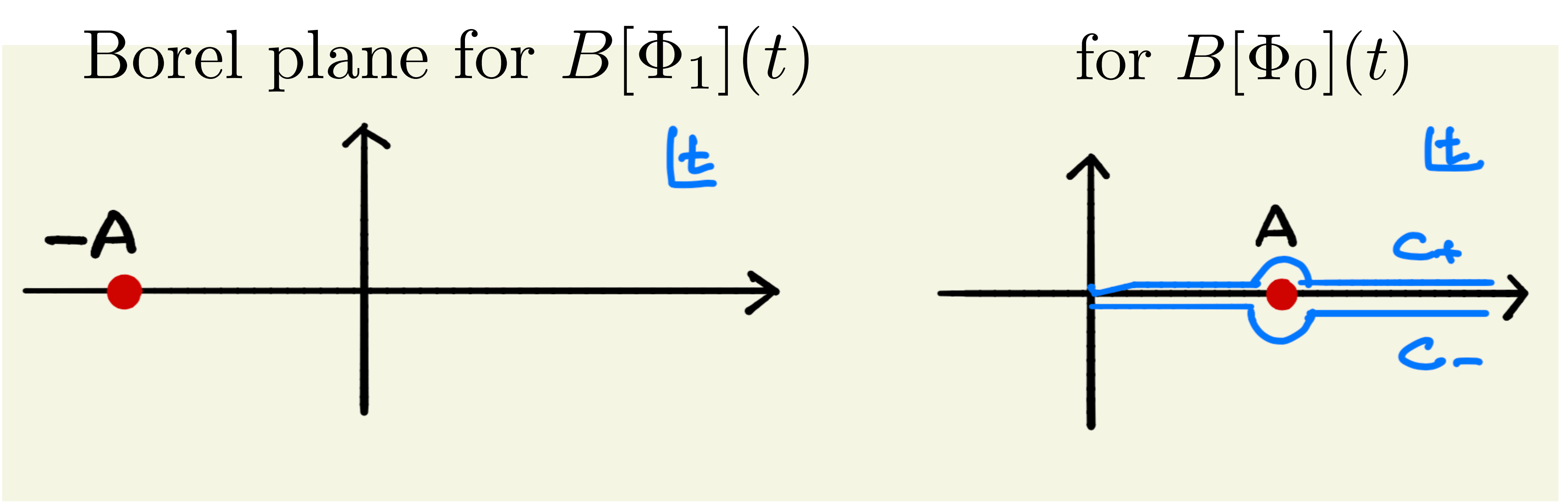
- The Stokes line associated with ρ_0 saddle is at $\arg(\lambda) = 0$. The Stokes line associated with ρ_1 saddle is at $\arg(\lambda) = \pi$.
- The two discontinuities are present to make the integral continuous crossing Stokes line.

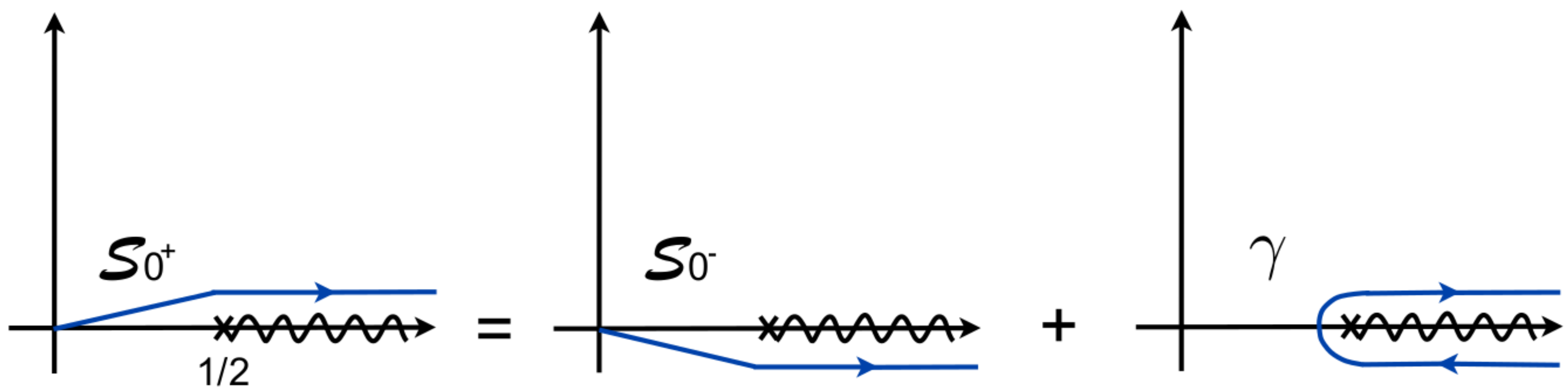
Borel analysis and Stokes Phenomena: very explicitly.

$$\mathfrak{Z}_0(\lambda) \equiv e^{-\frac{A_0}{\lambda}}\,\Phi_0(\lambda) = \sum_{n=0}^{\infty} a_n^{(0)}\lambda^n \equiv \sqrt{2\pi}\sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)^2 2^n}{n!\,\Gamma\left(\frac{1}{2}\right)^2}\lambda^n$$

$$\hat{\Phi}_0(t) \equiv B[\Phi_0](t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)^2}\frac{(2t)^n}{n!} = {}_2F_1\left(\frac{1}{2},\frac{1}{2},1;2t\right)$$

$$\mathcal{S}_\theta\Phi_i(\lambda) = \frac{1}{\lambda}\int_0^{e^{i\theta}\infty} dt\, e^{-t/\lambda}\, B[\Phi_i](t)$$





$$
\begin{aligned}
(\mathcal{S}_{0^+} - \mathcal{S}_{0^-})\Phi_0(\lambda) &= \frac{1}{\lambda}\int_\gamma dt\, e^{-t/\lambda}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2},1;2t\right) \\
&= \frac{1}{\lambda}\int_{1/2}^{\infty} dt\, e^{-t/\lambda}\left[\ {}_2F_1\left(\frac{1}{2},\frac{1}{2},1,2t+i\varepsilon\right) -\ {}_2F_1\left(\frac{1}{2},\frac{1}{2},1,2t-i\varepsilon\right)\right] \\
&= \frac{1}{\lambda}\int_{1/2}^{\infty} dt\, e^{-t/\lambda}\, 2i\ {}_2F_1\left(\frac{1}{2},\frac{1}{2},1,1-2t\right) \\
&= 2ie^{-1/(2\lambda)}\frac{1}{\lambda}\int_{0}^{\infty} dt\, e^{-t/\lambda}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2},1,-2t\right) \\
&= 2ie^{-1/(2\lambda)}\mathcal{S}_0\Phi_1(\lambda)\,.
\end{aligned}
$$

$$\mathrm{Disc}_0\Phi_0(\lambda) = -2ie^{-1/(2\lambda)}\Phi_1(\lambda)$$ **Remarkable relation**

Punchline: The discontinuity of Borel resummation of a series Φ_m associated by saddle ρ_m is determined by **only** the other series Φ_n associated by other saddles ρ_n in the problem and **nothing else**!!!

This is the essence of resurgence. The set of all series around all saddles are closed under Stokes jumps. This can be encoded into a type of *singularity derivative* called the *alien derivative* operation.

Borel-Ecalle summability and exact result from semi-classics

Two-term trans-series

$$Z(\lambda) = \begin{cases} \mathfrak{Z}_0(\lambda) + i\mathfrak{Z}_1(\lambda) = \Phi_0(\lambda) + ie^{-\frac{1}{2\lambda}}\Phi_1(\lambda) & -\pi < \theta < 0\,, \\ \mathfrak{Z}_0(\lambda) - i\mathfrak{Z}_1(\lambda) = \Phi_0(\lambda) - ie^{-\frac{1}{2\lambda}}\Phi_1(\lambda) & 0 < \theta < \pi\,, \end{cases}$$

Lefschetz thimble decomposition

$$Z = \frac{1}{\sqrt{\lambda}} \int_{\mathcal{J}_0(0^-)+\mathcal{J}_1(0^-)} e^{-S}$$

$$Z = \frac{1}{\sqrt{\lambda}} \int_{\mathcal{J}_0(0^+)-\mathcal{J}_1(0^+)} e^{-S}$$

Reality of the BE resummation for real coupling (approaching real line from below)

$$\begin{aligned}\Phi_0(\lambda) + ie^{-\frac{1}{2\lambda}}\Phi_1(\lambda) \xrightarrow{\text{BE}-\text{summation }\mathcal{S}_{0^-}} & \mathcal{S}_{0^-}\Phi_0 + ie^{-\frac{1}{2\lambda}}\mathcal{S}_{0^-}\Phi_1 \\ &= (\text{Re}\mathcal{S}_0\Phi_0 + i\,\text{Im}\mathcal{S}_{0^-}\Phi_0) + ie^{-\frac{1}{2\lambda}}\mathcal{S}_0\Phi_1 \\ &= \text{Re}\mathcal{S}_0\Phi_0 + i\left(\text{Im}\mathcal{S}_{0^-}\Phi_0 + e^{-\frac{1}{2\lambda}}\mathcal{S}_0\Phi_1\right) \\ &= \text{Re}\mathcal{S}_0\Phi_0\end{aligned}$$

This is the exact, real, unambiguous result,
and simple realization of Borel-Ecalle summability.

The reason for the late term/early term relation (and name resurgence)

Perturbative expansion parameters for saddle z_0 are: $a_n^{(0)} = \sqrt{2\pi}\frac{\Gamma\left(n+\frac{1}{2}\right)^2 2^n}{n!\,\Gamma\left(\frac{1}{2}\right)^2}$. In the $n \to \infty$ limit, we could (by brute force) see that:

$$a_n^{(0)} \sim \sqrt{2\pi}\frac{(n-1)!}{(A_{10})^n}\left(1 - \frac{\frac{1}{2}A_{10}}{(n-1)} + \frac{\frac{9}{8}(A_{10})^2}{(n-1)(n-2)} - \frac{\frac{75}{16}(A_{10})^3}{(n-1)(n-2)(n-3)} + \dots\right), \quad n \to \infty$$

Now, we understand why it had to be so. By Cauchy's thm. And using:

$$\mathrm{Disc}_0\Phi_0(\lambda) = -2ie^{-1/(2\lambda)}\Phi_1(\lambda)$$

$$F(z) = \frac{1}{2\pi i}\sum_a \int_0^{e^{i\theta_a}\infty} d\omega \frac{\mathrm{Disc}_{\theta_a}F(\omega)}{\omega - z}$$

$$a_n^{(0)} \sim \frac{\mathrm{s}}{2\pi i}\frac{\Gamma(n)}{(S_{10})^n}\left[a_0^{(1)} + a_1^{(1)}\frac{S_{10}}{(n-1)} + a_2^{(1)}\frac{(S_{10})^2}{(n-1)(n-2)} + \dots\right].$$

The information in the series expansion around the NP-saddle surges up, in a disguised form, in the expansion around the P-saddle and vice versa (Ecalle 80s)

Universality

1) This story with simple exponential integrals is very nice, but one may think that it is an ordinary integral that we can do exactly after all.....

2) But even in QM, we have infinitely many coupled exponential integrals, and in general infinitely many saddles.

3) QFT is even more involved. But.....

There is a genuinely universal behavior in the story I am telling you. It does not quite matter if we are dealing with exponential integral, path integral in QM or path integral in QFT.

The thing that changes is number of saddles. It may become infinite. But for closely knitted saddles that talk with each other, the ordinary exponential integral provides a remarkably useful prototype.

An example which captures some essence of more general cases.

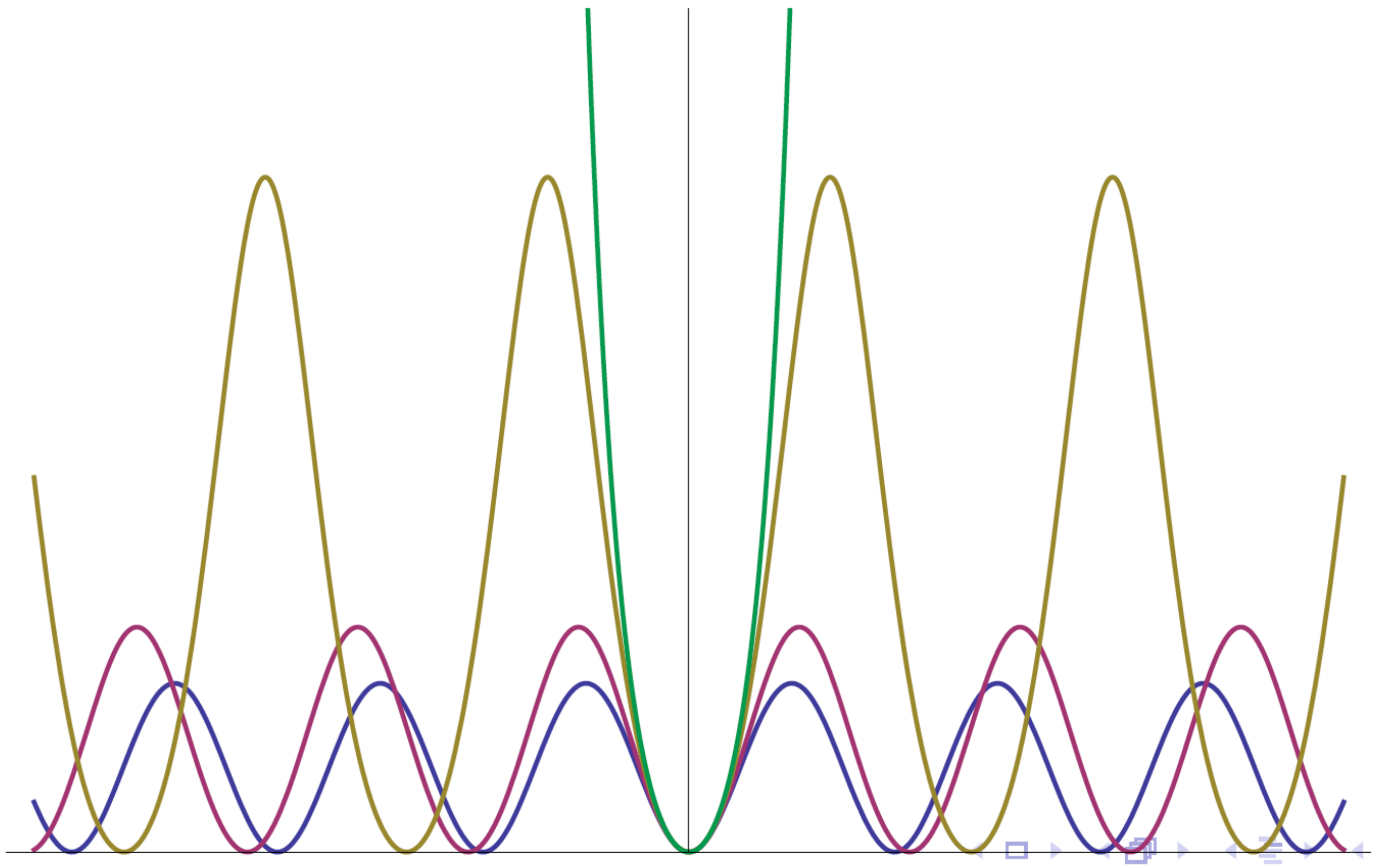
I borrowed the next 8 pages from a lecture of my collaborator Gerald Dunne.

Path integrals with complex saddles: "ghost instantons"

- elliptic potential: (Basar, GD, Ünsal, arXiv:1308.1108)

$$V(z|m) = \mathrm{sd}^2(x|m)$$

interpolates between Sine-Gordon ($m = 0$ and Sinh-Gordon ($m = 1$)



Path integrals with complex saddles: zero dim. prototype

$$V(z|m) = \frac{1}{g^2}\,\mathrm{sd}^2(g\,z|m)$$

- duality property:

$$V(z|m)|_{g^2} = V(z|1-m)|_{-g^2}$$

- perturbative series $\sum_n a_n(m) g^{2n}$ satisfies duality:

$$a_n(m) = (-1)^n a_n(1-m)$$

d=0 partition function:

$$\mathcal{Z}(g^2|m) = \frac{1}{g\sqrt{\pi}} \int_{-\mathbb{K}}^{\mathbb{K}} dz\, e^{-\frac{1}{g^2}\,\mathrm{sd}^2(z|m)}$$

Path integrals with complex saddles: zero dim. prototype

$$\mathcal{Z}(g^2|0)\Big|_{\text{pert}} = 1 + \frac{g^2}{4} + \frac{9g^4}{32} + \frac{75g^6}{128} + \frac{3675g^8}{2048} + \frac{59535g^{10}}{8192}$$

$$\mathcal{Z}(g^2|1)\Big|_{\text{pert}} = 1 - \frac{g^2}{4} + \frac{9g^4}{32} - \frac{75g^6}{128} + \frac{3675g^8}{2048} - \frac{59535g^{10}}{8192}$$

$$\mathcal{Z}\left(g^2\middle|\frac{1}{4}\right)\Big|_{\text{pert}} = 1 + \frac{g^2}{8} + \frac{9g^4}{64} + \frac{105g^6}{512} + \frac{1995g^8}{4096} + \frac{48195g^1}{32768}$$

$$\mathcal{Z}\left(g^2\middle|\frac{3}{4}\right)\Big|_{\text{pert}} = 1 - \frac{g^2}{8} + \frac{9g^4}{64} - \frac{105g^6}{512} + \frac{1995g^8}{4096} - \frac{48195g^1}{32768}$$

$$\mathcal{Z}\left(g^2\middle|\frac{1}{2}\right)\Big|_{\text{pert}} = 1 + 0g^2 + \frac{3g^4}{32} + 0g^6 + \frac{315g^8}{2048} + 0g^{10} + \dots$$

- duality relation: $\mathcal{Z}(g^2|m) = \mathcal{Z}(-g^2|1-m)$

  non-alternating for $m < \frac{1}{2}$ alternating for $m > \frac{1}{2}$

puzzles: Borel summable? "instantons" ?

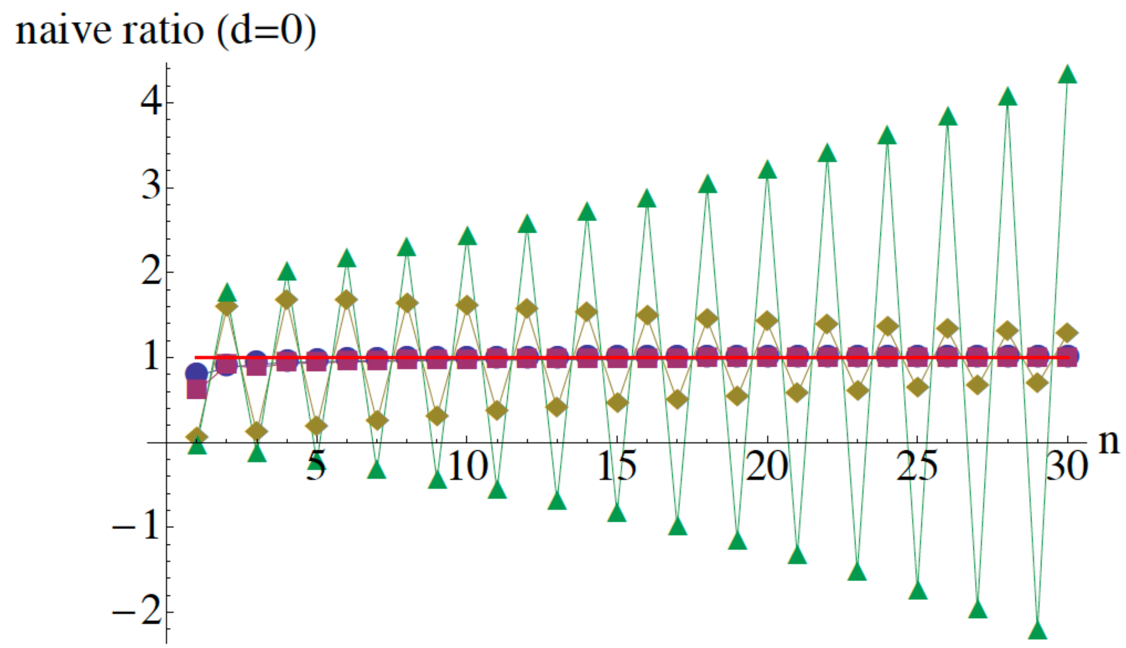
Path integrals with complex saddles: zero dim. prototype

$$\mathcal{Z}(g^2|m) = \frac{2}{g\sqrt{\pi}} \int_0^{\mathbb{K}} dz\, e^{-\frac{1}{g^2}\,\mathrm{sd}^2(z|m)}$$

- large-order behavior about 0 from saddle point $B = \mathbb{K}$:

$$S_B = \frac{1}{1-m} \qquad \Rightarrow \qquad a_n \sim \frac{(n-1)!}{\pi S_B^{n+1/2}}$$

- compare with actual series:

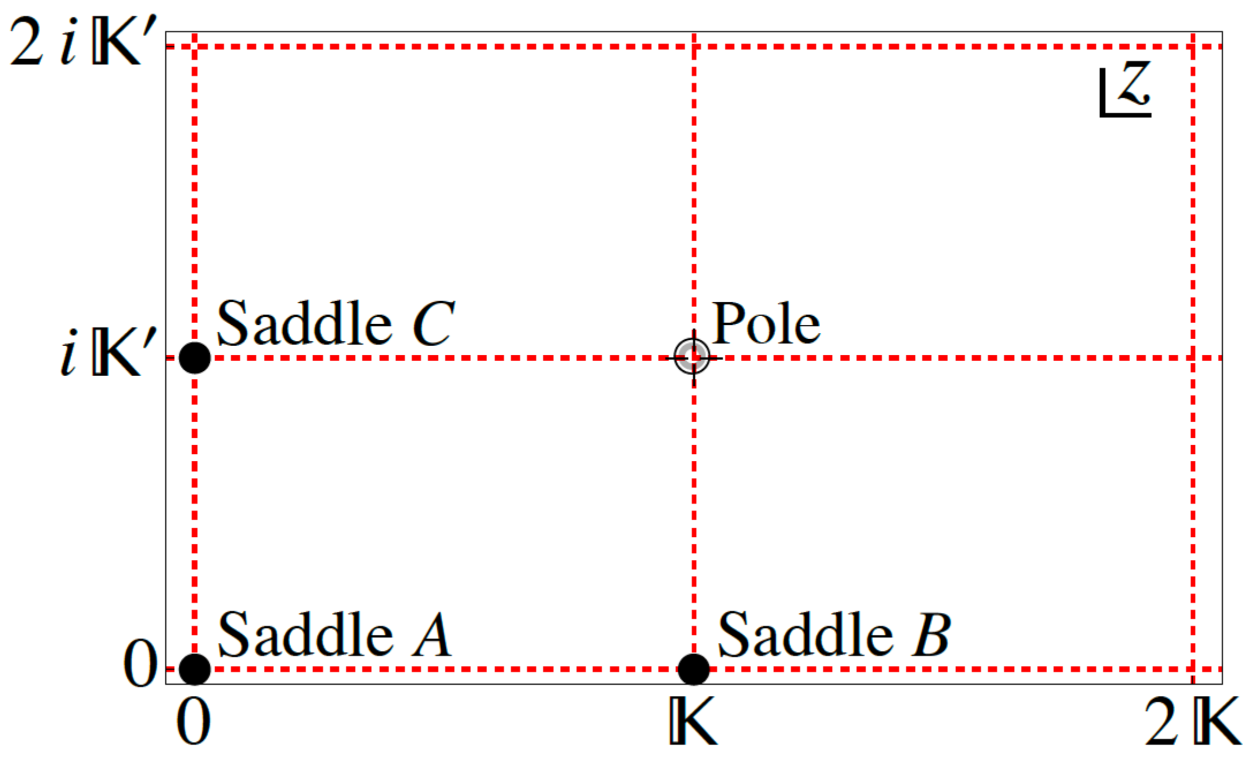


disaster !

Different curves refer to different values of the elliptic parameter m: m = 0 (blue circles), m =1/4 (red squares), m = 0.49 (gold diamonds), and m = 0.51 (green triangles). As m approaches 1/2 from below the agreement breaks down rapidly, showing that the contribution of the saddle B by itself is not sufficient to capture the large order growth.

Path integrals with complex saddles: zero dim. prototype

resolution: another saddle off the integration path!

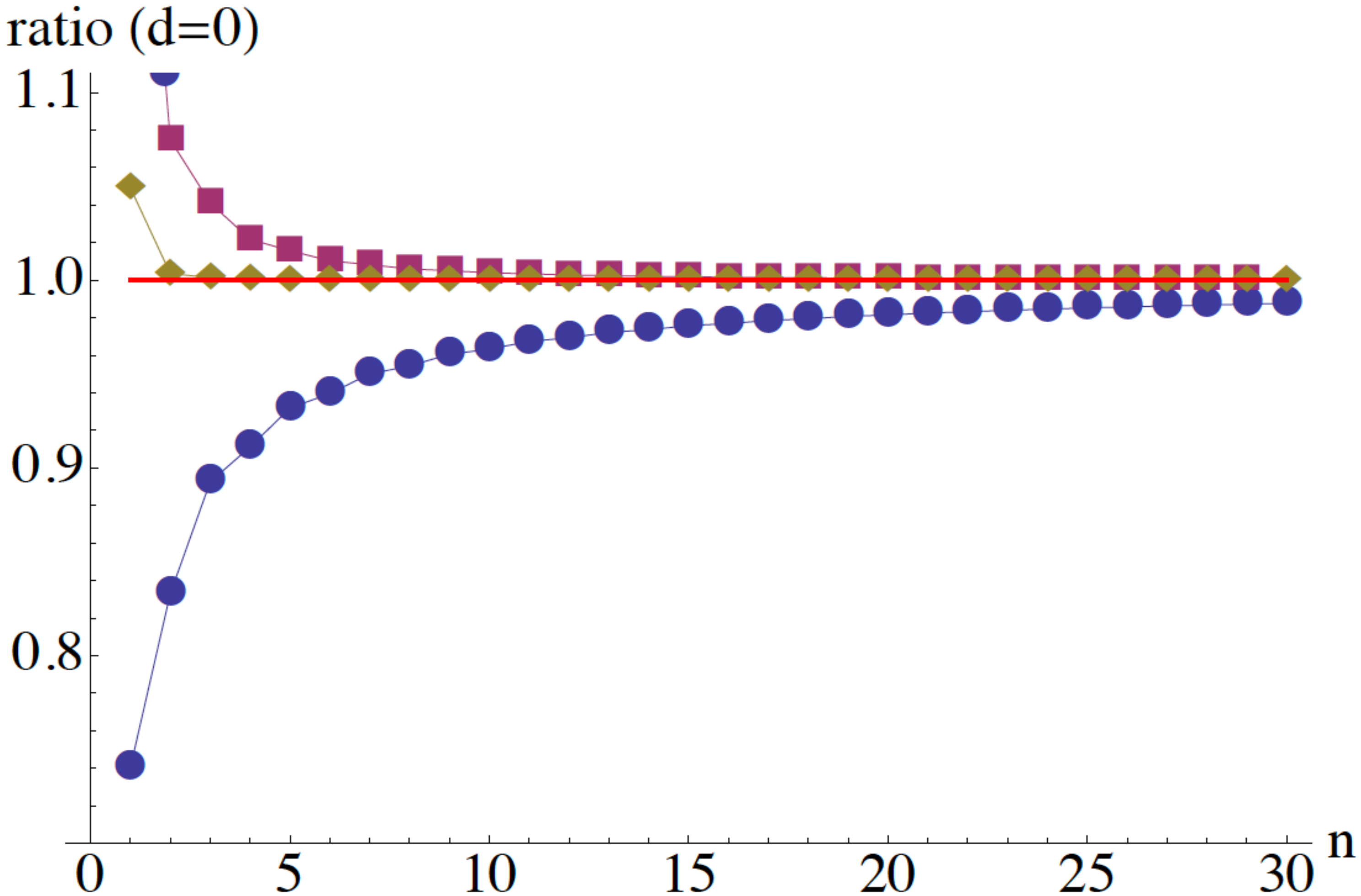


$$S_C = -1/m \qquad \Rightarrow \qquad a_n \sim \frac{(n-1)!}{\pi}\left(S_B^{n+1/2} + (-1)^n |S_C|^{n+1/2}\right)$$

Path integrals with complex saddles: zero dim. prototype

$$a_n \sim \frac{(n-1)!}{\pi}\left(S_B^{n+1/2} + (-1)^n |S_C|^{n+1/2}\right)$$

$\Rightarrow$ improved asymptotics:



conclusion: perturbation series feels *all* saddles, both real and complex

Path integrals with complex saddles: zero dim. prototype

the bigger picture:

- associated with each critical point z_i, there is a unique integration cycle $\mathcal{J}_i$, called a *Lefschetz thimble*, along which the phase remains stationary

- around each saddle there is a contribution of the form:

$$\mathcal{I}^{(k)}(\xi|m) = \frac{1}{\sqrt{\pi}}\sqrt{\xi}\int_{\mathcal{J}_k} dz\, e^{-\xi\,\mathrm{sd}^2(z|m)}$$

- expansions around different saddles are connected via

exact resurgence relation:

$$\mathcal{I}^{(A)}\left(\frac{1}{g^2}|m\right) = \frac{2}{2\pi i}\sum_{k\in\{B,C\}}\int_0^\infty \frac{dv}{v}\frac{1}{1-g^2\,v}\mathcal{I}^{(k)}(v|m)$$

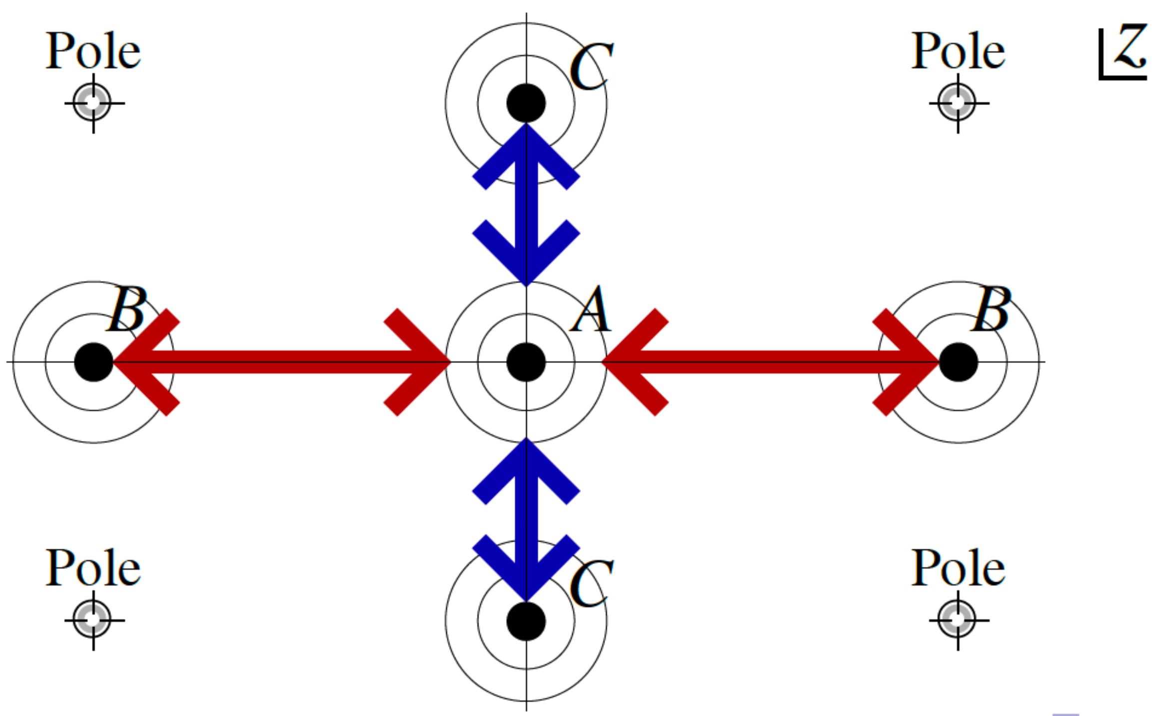
Path integrals with complex saddles: zero dim. prototype

- most general expansion is a three-term trans-series

$$\mathcal{Z}_{\mathfrak{C}}(g^2|m) \equiv \sigma_A \Phi_A(g^2) + \sigma_B e^{-S_B/g^2} \Phi_B(g^2) + \sigma_C e^{-S_C/g^2} \Phi_C(g^2)$$

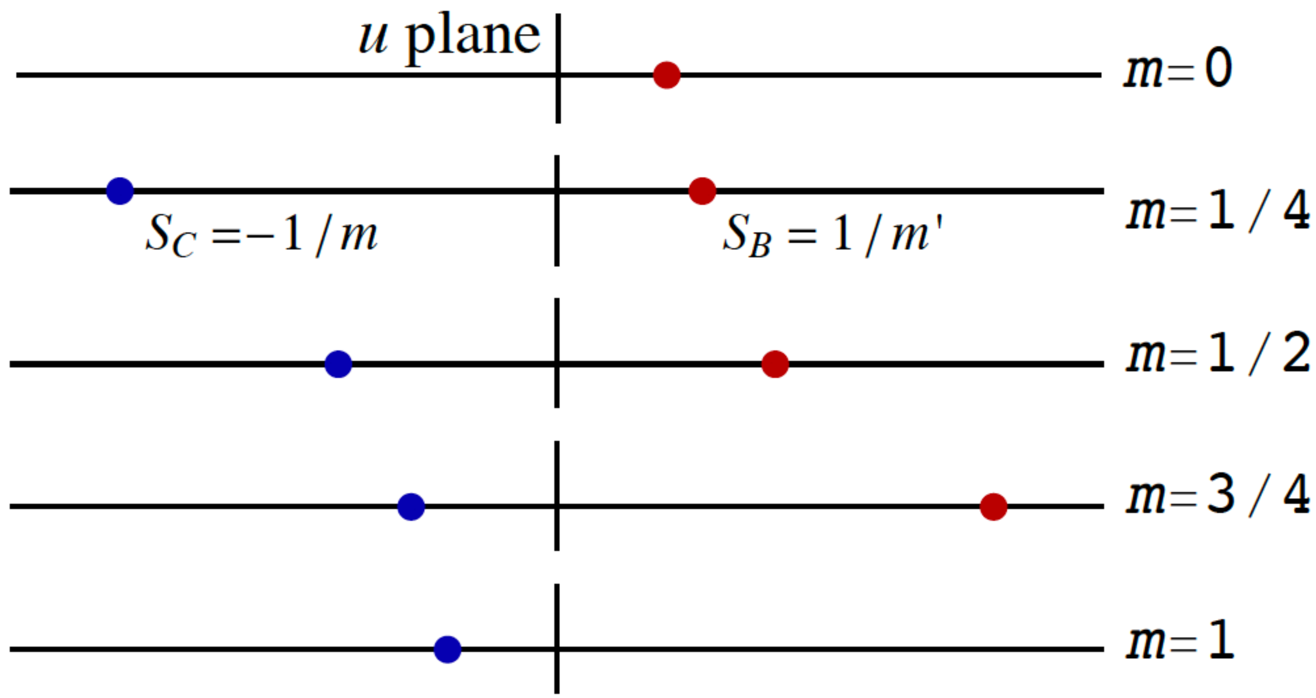
- coefficients of perturbative expansions are connected

$$a_n^{(A)}(m) = \sum_{j=0} \frac{(n-j-1)!}{\pi} \left(\frac{a_j^{(B)}(m)}{S_B^{n-j}} + \frac{a_j^{(C)}(m)}{S_C^{n-j}} \right)$$



Path integrals with complex saddles: zero dim. prototype

view from the Borel plane:



- 'distance' in Borel plane, $\Delta S = S_i - S_j$ ("relative action") controls divergence of perturbation series Φ_j

- $m > 1/2$: closest singularity on $\mathbb{R}^-$ $\Leftrightarrow$ alternating series Φ_A

- mimics structure of both UV and IR renormalons

Lefschetz thimbles and Stokes phenomena at theta=0

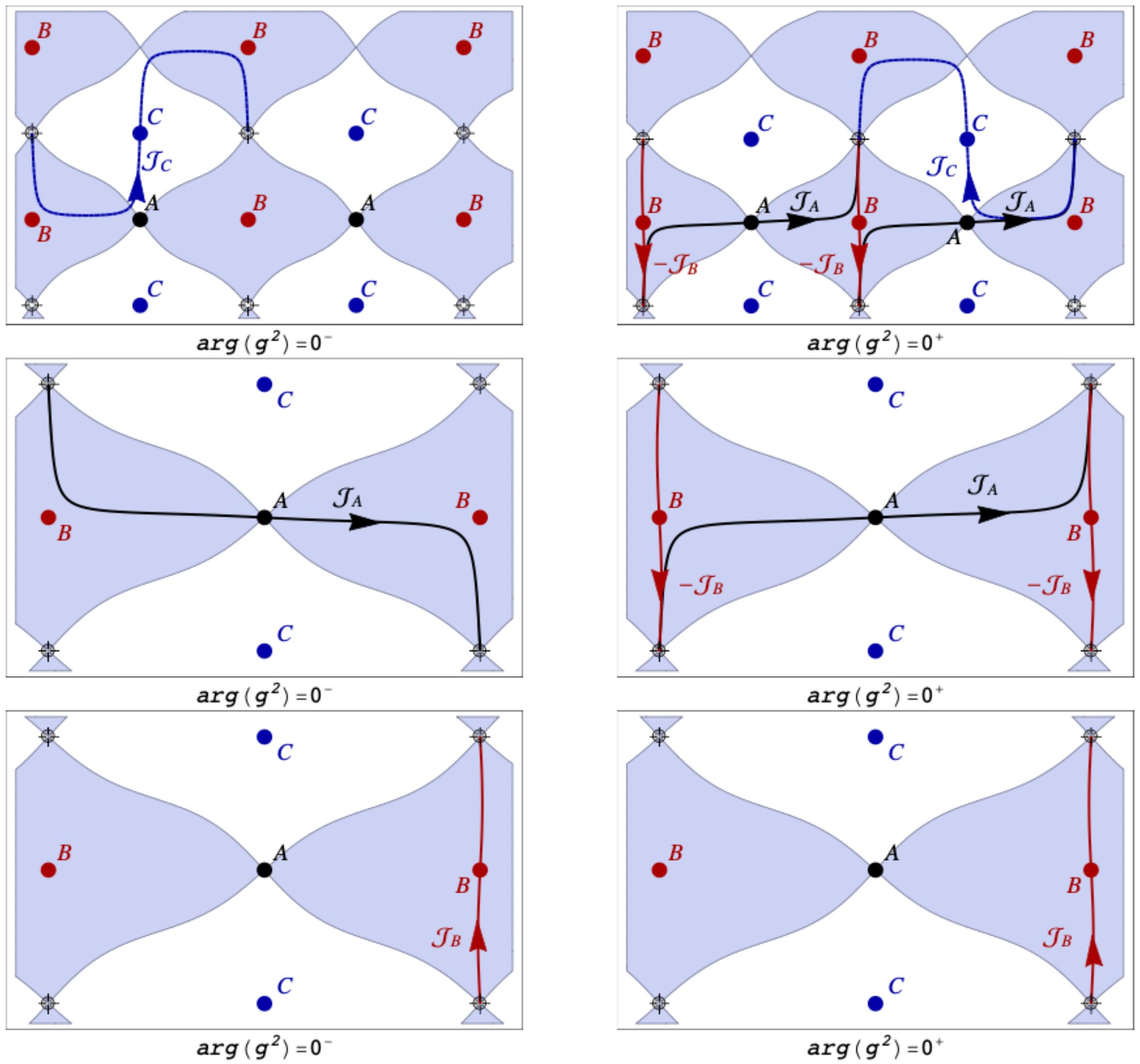


Figure 3. Lefschetz thimbles and Stokes phenomenon at $\theta = 0$. Order of dominance is $C > A > B$. Hence, there is no room for a downward gradient flow of B, hence B has no Stokes jump. A has a Stokes jump which gives birth to B. C has a a Stokes jump which gives birth to both A and B. See text.

Stokes phenomena at $\theta = 0$ ray: Now, the monodromy of the cycles crossing the Stokes ray $\theta = 0$ are:

$$\begin{aligned} \mathcal{J}_C &\longrightarrow \mathcal{J}_C + 2\mathcal{J}_A - 2\mathcal{J}_B \\ \mathcal{J}_A &\longrightarrow \mathcal{J}_A - 2\mathcal{J}_B \\ \mathcal{J}_B &\longrightarrow \mathcal{J}_B \end{aligned} \qquad \text{or} \quad \mathcal{J}_i \to U_{ij}(\theta = 0) J_j \qquad \text{with} \quad U_{\circlearrowleft}(0) = \begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

Lefschetz thimbles and Stokes phenomena at theta=pi

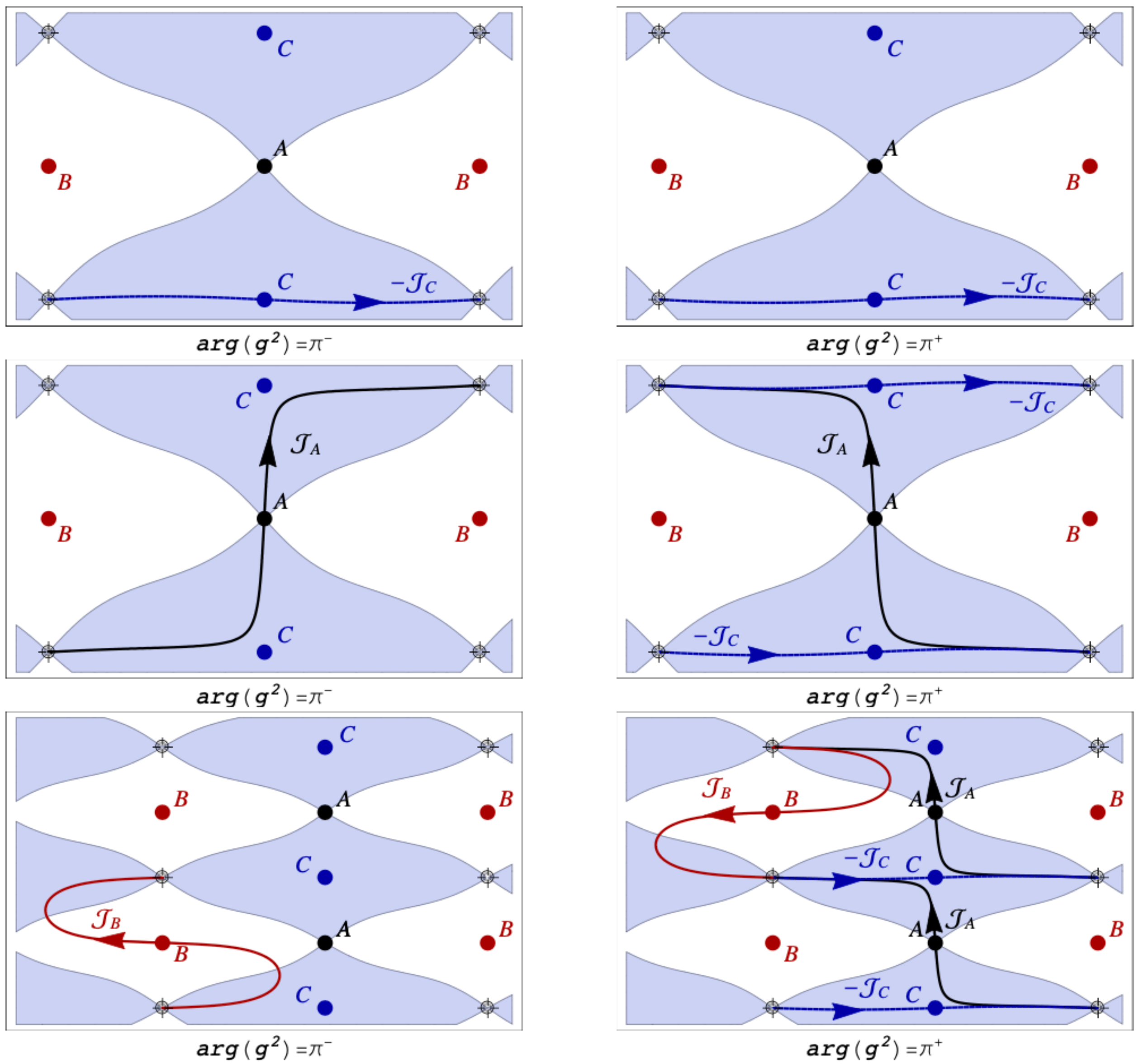


Figure 4. Lefschetz thimbles and Stokes phenomenon at $\theta = \pi$. Order of (maximal) dominance is $B > A > C$. Hence, C has no Stokes jump. A has a Stokes jump which gives birth to C. B has a a Stokes jump which gives birth to both A and B. See text.

Stokes phenomena at $\theta = \pi$ ray: The monodromy of the cycles crossing the Stokes ray $\theta = 0$ are:

$$\begin{array}{l} \mathcal{J}_C \longrightarrow \mathcal{J}_C \\ \mathcal{J}_A \longrightarrow \mathcal{J}_A - 2\mathcal{J}_C \\ \mathcal{J}_B \longrightarrow \mathcal{J}_B - 2\mathcal{J}_C + 2\mathcal{J}_A \end{array} \qquad \text{or} \;\; \mathcal{J}_i \to U_{ij}(\pi)\mathcal{J}_j \qquad \text{with} \;\; U_{\circlearrowleft}(\pi) = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 2 & 1 \end{pmatrix}$$