Bipartite Euler systems for certain Galois representations

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Notations

- $E/\mathbb{Q}$ be an elliptic curve, $\mathrm{Cond}(E) = N$
- $p \geq 5$ be an odd prime with good ordinary reduction
- T_p= Tate module
- $\overline{\rho}_E : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_p)$ be the residual representation
- K be an imaginary quadratic field (satisfying some hypothesis)
- K_{acyc} be the anticyclotomic $\mathbb{Z}_p$ extension
- $G_K = \mathrm{Gal}(\overline{K}/K), \quad \Gamma := \mathrm{Gal}(K_{acyc}/K) \cong \mathbb{Z}_p$
- $\Lambda := \mathbb{Z}_p[[\Gamma]]$ be the Iwasawa algebra of Γ
- $\mathrm{loc}_w : H^1(K, T_p) \longrightarrow H^1(K_w, T_p)$ by restriction

Anticyclotomic main conjecture

$S :=$ compactified Selmer group $\subset H^1(K_{acyc}, T_p)$,
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Conjecture (A)
S and X both have Λ-rank one, and $\exists \kappa_1 \in S$, such that

$$\mathrm{Char}_\Lambda(X_{tors}) = c.\,\mathrm{Char}_\Lambda(S/\Lambda\kappa_1)^2.$$

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Conjecture (B)

$$\mathrm{Char}_\Lambda(\mathbb{X}_{tors})\Lambda^{ur} = (L_p(E/K_{acyc}, s))$$

where $\mathbb{X}$ is dual of a modified Selmer group, $L_p(E/K_{acyc}, s)$ is a suitable p-adic L-function.

Motivation

Definite:= root number +1

Indefinite:= root number −1.

$\overline{\rho}_E$-surjective	Indefinite [H1,2004] (one divisibility)	Definite[BD,2005] (one divisibility)
$\overline{\rho}_E$-irreducible	Indefinite [BCC,2021]	Definite [BCC, 2021]
$\overline{\rho}_E$ – *reducible*	Indefinite [CGLS,2020] [CGS,2023]	Definite (?)

Notations

Goal: To extend the work on Bipartite Euler system, developed by (Mazur-Rubin) Howard for $\overline{\rho}_E$ *irreducible* to $\overline{\rho}_E$ *reducible.*

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- $(A, \mathfrak{m}) := (\mathbb{Z}/p^k\mathbb{Z}, p)$.
- T-free A-module of rank 2, with a continuous G_K-action.
- $\mathcal{L}$ be an infinite set of primes in K satisfying some conditions
- $\mathcal{N}$ be squarefree product of primes in $\mathcal{L}$
- $(T/\mathfrak{m}T)^{G_K} = 0$
- For a f.g A-mod M, $\mathrm{ind}(x, M) := \max\{j \leq \infty | x \in \mathfrak{m}^j M\}$
- for $c \in H^1(K, T)$, there exist infinitely many $\mathfrak{l} \in \mathcal{L}$ s.t. $\mathrm{loc}_{\mathfrak{l}}(c) \neq 0$.

Selmer modules

Definition
Given a Selmer structure $\mathcal{F}$ on T, define *Selmer module* $\mathrm{Sel}_{\mathcal{F}} = \mathrm{Sel}_{\mathcal{F}}(K, T)$ associated to $\mathcal{F}$ by

$$0 \longrightarrow \mathrm{Sel}_{\mathcal{F}} \longrightarrow H^1(K, T) \xrightarrow{\oplus_w \mathrm{loc}_w} \bigoplus H^1(K_w, T)/H^1_{\mathcal{F}}(K_w, T).$$

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Theorem (Howard)

$$\textit{For} \quad \mathfrak{n} \in \mathcal{N}: \quad \mathrm{Sel}_{\mathcal{F}(\mathfrak{n})} \cong A^{e(\mathfrak{n})} \oplus M_{\mathfrak{n}} \oplus M_{\mathfrak{n}} \quad \textit{with } e(\mathfrak{n}) \in \{0, 1\}.$$

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Definition (Stub modules)
Let $\mathcal{N}^{even} = \{\mathfrak{n} \in \mathcal{N} : e(\mathfrak{n}) = 0\}$, $\mathcal{N}^{odd} = \{\mathfrak{n} \in \mathcal{N} : e(\mathfrak{n}) = 1\}$.

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$$\mathrm{Stub}_{\mathfrak{n}} := \begin{cases} \mathfrak{m}^{length(M_{\mathfrak{n}})} A, & \text{if } \mathfrak{n} \in \mathcal{N}^{even} \\ \mathfrak{m}^{length(M_{\mathfrak{n}})} \mathrm{Sel}_{\mathcal{F}(\mathfrak{n})}, & \text{if } \mathfrak{n} \in \mathcal{N}^{odd} \end{cases}$$

Definition
A *bipartite Euler system of odd type* for $(T, \mathcal{F}, \mathcal{L})$ is a pair of families

$$\{\kappa_{\mathfrak{n}} \in \mathrm{Sel}_{\mathcal{F}(\mathfrak{n})}(K, T) \mid \mathfrak{n} \in \mathcal{N}^{odd}\} \text{ and } \{\lambda_{\mathfrak{n}} \in A \mid \mathfrak{n} \in \mathcal{N}^{even}\}$$

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related by the following reciprocity laws:

(i) for $\mathfrak{n}\mathfrak{l} \in \mathcal{N}^{odd}$,

$$A/(\lambda_{\mathfrak{n}}) \cong H^1_{ord}(K_{\mathfrak{l}}, T)/A.\,\mathrm{loc}_{\mathfrak{l}}(\kappa_{\mathfrak{n}\mathfrak{l}}),$$

(ii) for $\mathfrak{n}\mathfrak{l} \in \mathcal{N}^{even}$,

$$A/(\lambda_{\mathfrak{n}\mathfrak{l}}) \cong H^1_{unr}(K_{\mathfrak{l}}, T)/A.\,\mathrm{loc}_{\mathfrak{l}}(\kappa_{\mathfrak{n}}).$$

Definition (Free Euler systems)
An Euler system of *odd* type is *free* if for every $\mathfrak{n} \in \mathcal{N}^{odd}$, there is a *free rank one* A-module $\kappa_{\mathfrak{n}} \in C_{\mathfrak{n}} \subset \mathrm{Sel}_{\mathcal{F}(\mathfrak{n})}(K, T)$.

Bounding the length of a Selmer module

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Theorem (Aribam,K,2023)
Let $\mathrm{length}(A) > e$*,* $(T, \mathcal{F}, \mathcal{L})$*- free Euler system. Then*

$$\mathrm{length}(M_{\mathfrak{n}}) \leq \begin{cases} \mathrm{ind}(\lambda_{\mathfrak{n}}, A) + \varepsilon & \textit{if} \quad \mathfrak{n} \in \mathcal{N}^{even} \\ \mathrm{ind}(\kappa_{\mathfrak{n}}, \mathrm{Sel}_{\mathcal{F}(\mathfrak{n})}(K, T)) + \varepsilon & \textit{if} \quad \mathfrak{n} \in \mathcal{N}^{odd}. \end{cases}$$

where ε *is some constant depending on* ρ_E*.*

Euler system sheaf

- $\mathcal{X} := (\mathcal{V}, \mathcal{E})$ be a graph
- vertices $\mathcal{V} := \{v(\mathfrak{n}) \mid \mathfrak{n} \in \mathcal{N}\}$, edges $\mathcal{E} := \{e(\mathfrak{n}, \mathfrak{n}\mathfrak{l}) \mid \mathfrak{n}\mathfrak{l} \in \mathcal{N}\}$

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Definition (Euler system sheaf)
Attach to the graph an *Euler system sheaf* of A-modules, which is defined as:

$$\mathrm{ES}(v) = \begin{cases} \mathrm{Sel}_{\mathcal{F}(\mathfrak{n})} & \text{if } \mathfrak{n} \in \mathcal{N}^{odd} \\ A & \text{if } \mathfrak{n} \in \mathcal{N}^{even} \end{cases}; \quad \mathrm{ES}(e) = \begin{cases} H^1_{unr}(K_{\mathfrak{l}}, T) & \text{if } \mathfrak{n} \in \mathcal{N}^{odd} \\ H^1_{ord}(K_{\mathfrak{l}}, T) & \text{if } \mathfrak{n} \in \mathcal{N}^{even}. \end{cases}$$

with a vertex-to-edge map $ES(v) \longrightarrow ES(e)$.

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with a vertex-to-edge map $ES(v) \longrightarrow ES(e)$.

Define $u := \min\{\mathrm{length}(M_{\mathfrak{n}}) | \mathfrak{n} \in \mathcal{N}\}$.

Absolute core vertices

Definition

Let $\mathfrak{n} \in \mathcal{N}^{even}$, then we say $\mathfrak{n}$ is *universally trivial* if

$$\begin{aligned} & \mathrm{loc}_{\mathfrak{l}}(\mathrm{Sel}_{\mathcal{F}(\mathfrak{n})}) = 0 \,\forall\, \mathfrak{l} \nmid \mathfrak{n}, \\ \text{whenever } & \mathrm{loc}_{\mathfrak{l}_1}(\mathrm{Sel}_{\mathcal{F}(\mathfrak{n}\mathfrak{l})}) \cong A, \mathrm{loc}_{\mathfrak{l}_2}(\mathrm{Sel}_{\mathcal{F}(\mathfrak{n}\mathfrak{l}\mathfrak{l}_1)}) = 0 \,\forall\, \mathfrak{l}_2 \nmid \mathfrak{n}, \\ & \cdots \quad \cdots \quad \cdots \quad , \\ \text{whenever } & \mathrm{loc}_{\mathfrak{l}_{2k-1}}(\mathrm{Sel}_{\mathcal{F}(\mathfrak{n}\mathfrak{l}\mathfrak{l}_1\ldots\mathfrak{l}_{2k-2})}) \cong A, \mathrm{loc}_{\mathfrak{l}_{2k}}(\mathrm{Sel}_{\mathcal{F}(\mathfrak{n}\mathfrak{l}\mathfrak{l}_1\ldots\mathfrak{l}_{2k-1})}) = 0 \,\forall \mathfrak{l}_{2k} \nmid \mathfrak{n}, \\ & \cdots \quad \cdots \quad \cdots \end{aligned}$$

Similarly we define universally trivial odd vertices.

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Similarly we define universally trivial odd vertices.

Theorem
Let $\mathfrak{n}$ and $\mathfrak{a}$ be universally trivial. Then $\mathrm{length}(M_{\mathfrak{n}}) = \mathrm{length}(M_{\mathfrak{a}})$.
Conversely, if $\mathrm{length}(M_{\mathfrak{n}}) = u$, *then $\mathfrak{n}$ is universally trivial.*

Rigidity Theorem

Definition
A vertex $v = v(\mathfrak{n})$ is called *absolute core* if $\mathrm{length}(M_{\mathfrak{n}}) = u$.

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Theorem (The Rigidity theorem,A-K,2023)
$(T, \mathcal{F}, \mathcal{L})$-free Euler system. There is a unique integer δ, independent of $\mathfrak{n} \in \mathcal{N}$, such that

$$< \pi^{\varepsilon} \lambda_{\mathfrak{n}} >= \mathfrak{m}^{\delta}\, \mathrm{Stub}_{\mathfrak{n}}, \textit{ for } \mathfrak{n} \in \mathcal{N}^{even},$$
$$< \pi^{\varepsilon} \kappa_{\mathfrak{n}} >= \mathfrak{m}^{\delta}\, \mathrm{Stub}_{\mathfrak{n}}, \textit{ for } \mathfrak{n} \in \mathcal{N}^{odd}.$$

Euler systems over Λ

- ϵ-quadratic character attached to K
- $N = N^+N^-$
- for $\mathfrak{n}$ there exists $n \in \mathbb{Z}$ s.t $n\mathcal{O}_K = \mathfrak{n}$ (Heegner Hypothesis)
- $\mathcal{N}^{definite} := \{\mathfrak{n} \in \mathcal{N} | \epsilon(nN^-) = -1\}$, $\quad \mathcal{N}^{indefinite} := \{\mathfrak{n} \in \mathcal{N} | \epsilon(nN^-) = 1\}$

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Definition
A bipartite Euler system over Λ is a pair:

$$\{\kappa_{\mathfrak{n}} \in \mathcal{S}_{\mathfrak{n}}(K^{ac}, E[p^k]) | \mathfrak{n} \in \mathcal{N}_k^{indefinite}\} \quad \text{and} \quad \{\lambda_{\mathfrak{n}} \in \Lambda/p^k\Lambda | \mathfrak{n} \in \mathcal{N}_k^{definite}\}$$

satisfying the reciprocity laws.

Theorem (Aribam,K,2023)
Assume that $\lambda^{\infty} \neq 0$ *and* $\kappa^{\infty} \neq 0$ *. Then*

(i)

$$\mathrm{rank}_{\Lambda} S = \mathrm{rank}_{\Lambda} X = \begin{cases} 0 & \textit{if} \quad \epsilon(N^{-}) = -1 \\ 1 & \textit{if} \quad \epsilon(N^{-}) = 1. \end{cases}$$

(ii) *For any height one prime* $\mathfrak{P}$ *of* Λ *one has*

$$\mathrm{ord}_{\mathfrak{P}}(\mathrm{Char}(X_{\Lambda_{tor}})) \leq 2. \begin{cases} \mathrm{ord}_{\mathfrak{P}}(\lambda^{\infty}) & \textit{if} \quad \epsilon(N^{-}) = -1 \\ \mathrm{ord}_{\mathfrak{P}}(\mathrm{Char}(S/\Lambda\kappa^{\infty})) & \textit{if} \quad \epsilon(N^{-}) = 1. \end{cases}$$

(iii) $s \in \mathbb{N}$, *if for all* $t \geq s$

$$\{\lambda_{\mathfrak{n}} \in \Lambda/p^t\Lambda | \mathfrak{n} \in \mathcal{N}_t^{definite}\}$$

contains an element nonzero in $\Lambda/(\mathfrak{P}, p^s)$, *then* $'' = ''$ *holds in* (ii).

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Thank you.