Measurement-Induced Phase Transition in a Quantum Ising System

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ICTS-QT, 30 Jan 2025

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Outline

- History of ``MIPT''
- Description of Measurement Protocol for our scheme
- Derivation of recursion relation for survival probability
- Results for survival probability and the transition
- Scaling with size of the system
- Relation between MIPT and Quantum Phase Transition

History

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1. arXiv:2501.00547 [pdf, other] cond-mat.stat-mech

Measurement-Induced Phase Transition in State Estimation of Chaotic Systems and the Directed Polymer

Authors: Federico Gerbino, Guido Giachetti, Pierre Le Doussal, Andrea De Luca

Abstract: We introduce a solvable model of a measurement-induced phase transition (MIPT) in a deterministic but chaotic dynamical system with a positive Lyapunov exponent. In this setup, an observer only has a probabilistic description of the system but mitigates chaos-induced uncertainty through repeated measurements. Using a minimal representation via a branching tree, we map this problem to the directed... ▽ More

Submitted 31 December, 2024; **originally announced** January 2025.

Comments: 14 pages, 5 figures

2. arXiv:2412.11097 [pdf, other] quant-ph cond-mat.dis-nn cond-mat.stat-mech

Topology and Spectrum in Measurement-Induced Phase Transitions

Authors: Hisanori Oshima, Ken Mochizuki, Ryusuke Hamazaki, Yohei Fuji

Abstract: Competition among repetitive measurements of noncommuting observables and unitary dynamics can give rise to a rich variety of entanglement phases. We here characterize topological phases in monitored quantum systems by their spectrum and many-body topological invariants. We analyze (1+1)-dimensional monitored circuits for Majorana fermions, which have topological and trivial area-law entangled pha... ▽ More

Submitted 23 December, 2024; **v1** submitted 15 December, 2024; **originally announced** December 2024.

Comments: 7 + 9 pages, 3 + 4 figures, v3: corrected the reference list and figures

Report number: RIKEN-iTHEMS-Report-24

3. arXiv:2412.06440 [pdf, other] cond-mat.stat-mech quant-ph

106. arXiv:0809.1150 [pdf, ps, other] physics.atom-ph doi 10.1103/PhysRevA.78.062509

Lifetime of the A(v'=0) state and Franck-Condon factor of the A-X(0-0) transition of CaF measured by the saturation of laser-induced fluorescence

Authors: T. E. Wall, J. F. Kanem, J. J. Hudson, B. E. Sauer, D. Cho, M. G. Boshier, E. A. Hinds, M. R. Tarbutt

Abstract: We describe a method for determining the radiative decay properties of a molecule by studying the saturation of laser-induced fluorescence a the associated power broadening of spectral lines. The fluorescence saturates because the molecules decay to states that are not resonant with the lase The amplitudes and widths of two hyperfine components of a spectral line are measured over a range of… ▽ More

Submitted 22 December, 2008; **v1** submitted 6 September, 2008; **originally announced** September 2008.

Comments: 10 pages, 6 figures. Minor revisions following referee suggestions

Journal ref: Phys. Rev. A 78, 062509 (2008)

107. arXiv:physics/0612154 [pdf, ps, other] physics.atom-ph doi 10.1103/PhysRevLett.98.033001

Storage-ring measurement of the hyperfine induced 47Ti18+(2s 2p 3P0 -> 2s2 1S0) transition rate

Authors: S. Schippers, E. W. Schmidt, D. Bernhardt, D. Yu, A. Mueller, M. Lestinsky, D. A. Orlov, M. Grieser, R. Repnow, A. Wolf

Abstract: The hyperfine induced 2s 2p 3P0 > 2s2 1S0 transition rate AHFI in berylliumlike 47Ti18+ was measured. Resonant electron-ion recombination i heavy-ion storage ring was employed to monitor the time dependent population of the 3P0 state. The experimental value AHFI=0.56(3)/s is almost 60% l than theoretically predicted.

Submitted 15 December, 2006; **originally announced** December 2006.

Comments: 4 pages. 3 figures, 1 table, accepted for publication in Physical Review Letters

Journal ref: Physical Review Letters 98 (2007) 033001

108. arXiv:cond-mat/0203521 [pdf, ps, other] cond-mat doi 10.1103/PhysRevLett.89.018301

Measurement induced quantum-classical transition

Authors: D. Mozyrsky, I. Martin

Abstract: A model of an electrical point contact coupled to a mechanical system (oscillator) is studied to simulate the dephasing effect of measurement o quantum system. The problem is solved at zero temperature under conditions of strong non-equilibrium in the measurement apparatus. For linear coup between the oscillator and tunneling electrons, it is found that the oscillator dynamics becomes dam… ▽ More

Submitted 1 April, 2002; **v1** submitted 25 March, 2002; **originally announced** March 2002.

Comments: in RevTex, 1 figure, corrected notation

Journal ref: Phys. Rev. Lett. 89, 018301 (2002)

98. arXiv:1606.05721 [pdf, other] quant-ph doi 10.1103/PhysRevLett.117.190503

Measurement-induced state transitions in a superconducting qubit: Beyond the rotating wave approximation

Authors: Daniel Sank, Zijun Chen, Mostafa Khezri, J. Kelly, R. Barends, B. Campbell, Y. Chen, B. Chiaro, A. Dunsworth, A. Fowler, E. Jeffrey, E. Lucero, A. Megrant, J. Mutus, M. Neeley, C. Neill, P. J. J. O'Malley, C. Quintana, P. Roushan, A. Vainsencher, J. Wenner, T. White, Alexander N. Korotkov, John M. Mar

Abstract: Many superconducting qubit systems use the dispersive interaction between the qubit and a coupled harmonic resonator to perform quant state measurement. Previous works have found that such measurements can induce state transitions in the qubit if the number of photons in the reso is too high. We investigate these transitions and find that they can push the qubit out of the two-level subspac... ▽ More

Submitted 15 November, 2016; **v1** submitted 18 June, 2016; **originally announced** June 2016.

Journal ref: Phys. Rev. Lett. 117, 190503 (2016)

99. arXiv:1606.00109 [pdf, ps, other] cond-mat.str-el doi 10.1103/PhysRevB.94.104409

Magnetic-field- and pressure-induced quantum phase transition in $CsFeCl_3$ proved via magnetization measurement

Authors: Nobuyuki Kurita, Hidekazu Tanaka

Abstract: We have performed magnetization measurements of the gapped quantum magnet $CsFeCl_3$ at temperatures (T) down to 0.5\,K at ambient pressure and down to 1.8\,K at hydrostatic pressures (P) of up to 1.5\,GPa. The lower-field (H) phase boundary of the field-induced ordered phase at ambient pressure is found to follow the power-law behavior expressed by the formula $H_{\rm N}(T)$\,$-$\,... ▽ More

Submitted 16 August, 2016; **v1** submitted 1 June, 2016; **originally announced** June 2016.

Comments: 8 pages, 8 figures, to appear in Phys. Rev. B

Journal ref: Phys. Rev. B 94, 104409 (2016)

100. arXiv:1603.03561 [pdf, other] quant-ph cond-mat.stat-mech doi 10.1103/PhysRevA.93.050103

Measurement-Induced Phase Transition in a Quantum Spin System

Authors: Shrabanti Dhar, Subinay Dasgupta

Abstract: Suppose a quantum system starts to evolve under a Hamiltonian from some initial state. When for the first time, will an observable attain a preassigned value? To answer this question, one method often adopted is to make instantaneous measurements periodically and note down the seria number for which the desired result is obtained for the first time. We apply this protocol to an interacting spin s... ▽ More

Submitted 10 May, 2016; **v1** submitted 11 March, 2016; **originally announced** March 2016.

Comments: 5 pages, 3 figures, Accepted for publication as a Rapid Communication in Physical Review A

Journal ref: Phys. Rev. A 93, 050103 (2016)

97. arXiv:1808.05953 [pdf, other] cond-mat.stat-mech cond-mat.dis-nn cond-mat.str-el hep-th quant-ph

doi 10.1103/PhysRevX.9.031009

Measurement-Induced Phase Transitions in the Dynamics of Entanglement

Authors: Brian Skinner, Jonathan Ruhman, Adam Nahum

Abstract: We define dynamical universality classes for many-body systems whose unitary evolution is punctuated by projective measurements. In cases where such measurements occur randomly at a finite rate p for each degree of freedom, we show that the system has two dynamical phases: `entangling' and `disentangling'. The former occurs for p smaller than a critical rate p_c, and is characterized by volu... ▽ More

Submitted 26 July, 2019; **v1** submitted 17 August, 2018; **originally announced** August 2018.

Comments: 17+4 pages, 16 figures; updated discussion and results for mutual information; graphics error fixed

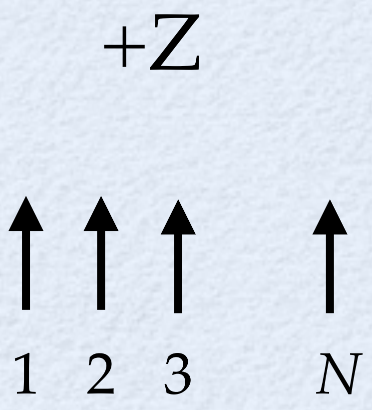
Journal ref: Phys. Rev. X 9, 031009 (2019)

850 citations

1. Chen and Fisher PRB 2018
2. Chen and Fisher PRB 2019
3. Koh, Sun, Motta and Minnich, Nature Physics 2019
4. … > 100 more

Protocol

Chain of N Ising spins ±1



→ Evolve for time τ under $\mathcal{H}$ → Perform measurement

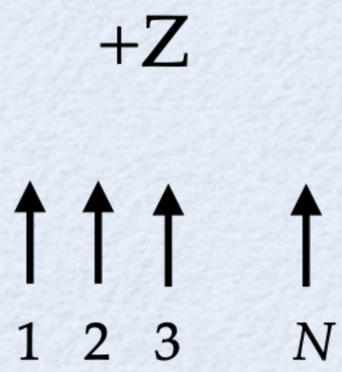
Repeat

Transverse Ising Hamiltonian $$\mathcal{H} = -\sum_{i=1}^{N} s_i^x s_{i+1}^x - h \sum_{i=1}^{N} s_i^z$$

Measurement: ``Are all spins up?'' Yes/No

Perform measurement at each time-step with certainty

τ and h are two parameters of our study



Unitary evolution under $\mathcal{H}$ for time τ

Measurement: ``Are all spins up?"

Yes: Probability p_1

No: Probability 1-p_1

Unitary evolution under $\mathcal{H}$ for time τ

Measurement: Are all spins up?

Yes: Probability p_2

No: Probability 1-p_1-p_2

Repeat

After *n* steps:

Probability that the answer is *no* for *n* steps

Survival probability $R_n = 1\text{-}p_1\text{-}p_2 \ldots \text{-}p_\mathrm{n}$

Start with $|\mathcal{I}\rangle = |++\cdots+\rangle_z$

First time step:
Evolve for time τ to get $e^{-i\mathcal{H}\tau}|\mathcal{I}\rangle$
Perform measurement: "Are all spins up?"
Yes with probability $p_1 = |\langle\mathcal{I}|e^{-i\mathcal{H}\tau}|\mathcal{I}\rangle|^2$
Wave function for *No* : $|\psi_1\rangle = e^{-i\mathcal{H}\tau}|\mathcal{I}\rangle - \langle\mathcal{I}|e^{-i\mathcal{H}\tau}|\mathcal{I}\rangle|\mathcal{I}\rangle$

Second time step:
Evolve for time τ and perform measurement.
Yes with probability $p_2 = |\langle\mathcal{I}|e^{-i\mathcal{H}\tau}|\psi_1\rangle|^2$
Wave function for *No* : $|\psi_2\rangle = e^{-i\mathcal{H}\tau}|\psi_1\rangle - \langle\mathcal{I}|e^{-i\mathcal{H}\tau}|\psi_1\rangle|\mathcal{I}\rangle$

***n*-th time step:**
Evolve for time τ and perform measurement.
Wave function for *No* : $|\psi_n\rangle = e^{-i\mathcal{H}\tau}|\psi_{n-1}\rangle - \langle\mathcal{I}|e^{-i\mathcal{H}\tau}|\psi_{n-1}\rangle|\mathcal{I}\rangle$

Survival probability = Probability of getting *No* for n time-steps = $R_n = |\langle\psi_n|\psi_n\rangle|^2$

Recursion relation for survival probability

We start with $|\mathcal{I}\rangle = |++\cdots+\rangle_z$

Define $|\phi_n\rangle = e^{-i\mathcal{H}\tau n}|\mathcal{I}\rangle,\ f_n = \langle\mathcal{I}|e^{-i\mathcal{H}\tau n}|\mathcal{I}\rangle,\ n = 0, 1, 2, \cdots$

Then, $|\phi_0\rangle = |\mathcal{I}\rangle$ and $\langle\phi_m|\phi_n\rangle = f_{n-m}$

After the first time-step, $|\psi_1\rangle = |\phi_1\rangle - f_1|\mathcal{I}\rangle$

After the second time-step, $|\psi_2\rangle = |\phi_2\rangle - f_1|\phi_1\rangle + (f_1^2 - f_2)|\mathcal{I}\rangle$

Basic idea: for $n = 0, 1, 2$, the wave-function can be expressed as

$$|\psi_n\rangle = \sum_{m=0}^{n} C_m^{(n)}|\phi_m\rangle$$

with $C_0^{(0)} = C_1^{(1)} = C_2^{(2)} = 1, \quad C_0^{(1)} = -f_1, \quad C_0^{(2)} = f_1^2 - f_2 \quad C_1^{(2)} = -f_1$

Then, the wave-function after the next step

$$|\psi_{n+1}\rangle = e^{-i\mathcal{H}\tau}|\psi_n\rangle - \langle\mathcal{I}|e^{-i\mathcal{H}\tau}|\psi_n\rangle|\mathcal{I}\rangle$$

can also be expressed as

$$|\psi_{n+1}\rangle = \sum_{m=0}^{n+1} C_m^{(n+1)}|\phi_m\rangle$$

with the recursion relations

$$C_0^{(n+1)} = -\sum_{m=0}^{n} C_m^{(n)} f_{m+1}, \quad C_m^{(n+1)} = C_{m-1}^{(n)}$$

for $0 < m \leq n$. Using the expansion of the wave function in terms of the $|\phi\rangle$ functions, one can now express the survival probability after n measurements as

$$R_n = \langle\psi_n|\psi_n\rangle = \sum_{m_1,m_2=0}^{(n)} \left(C_{m_1}^{(n)}\right)^* C_{m_2}^{(n)} f_{m_2-m_1}$$

Note that $C_n^{(n)} = 1$ and $C_m^{(n)} = C_{m-1}^{(n-1)} = C_0^{(n-m)}$.

If f_n are known, $C_m^{(n)}$ are also known.

These expressions are valid for any Hamiltonian and any initial state, provided f_n are known

For the specific case of transverse Ising Hamiltonian, one can derive an expression for $f_n = \langle \mathcal{I} | e^{-i\mathcal{H}\tau n} | \mathcal{I} \rangle$ in closed form for any even value of N, by using the known exact solution. (Damski and Rams 2013)

$$f_n = \prod_{k=0}^{\pi} \left[\cos(\lambda_k n\tau) + i \sin(\lambda_k n\tau) \cos(2\theta_k)\right]$$

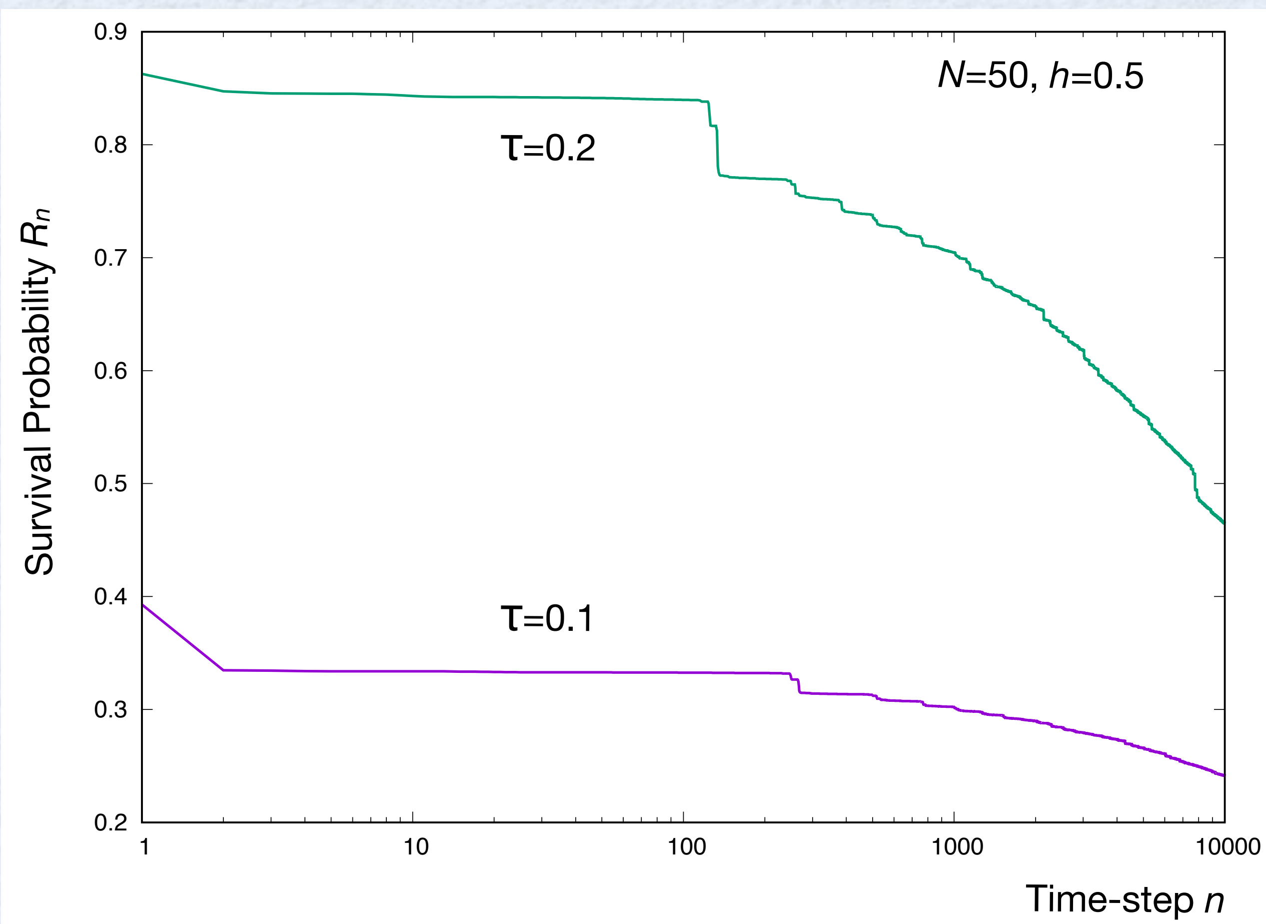
$$\lambda_k = 2\sqrt{h^2 + 1 + 2h\cos k} \quad \cos\left(2\theta_k\right) = \frac{2(h + \cos k)}{\lambda_k}$$

$$k = (2n+1)\frac{\pi}{N} \quad \text{with} \quad n = 0, 1, 2, \cdots, \frac{N}{2} - 1$$

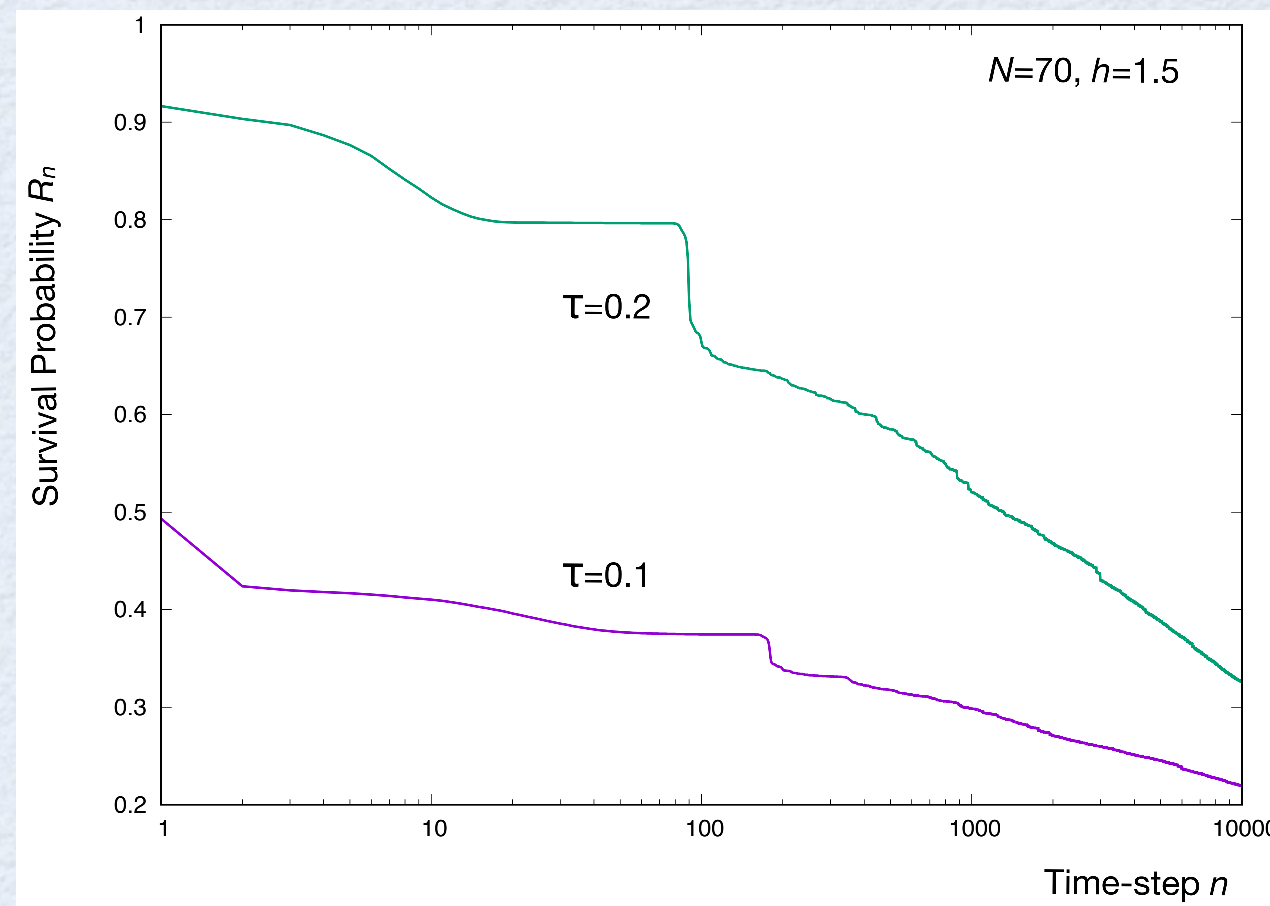
In principle, one should be able to calculate the survival probability for any system-size and time-step.

But due to precision problems, size ≤ 1000, time-steps ≤ 10000

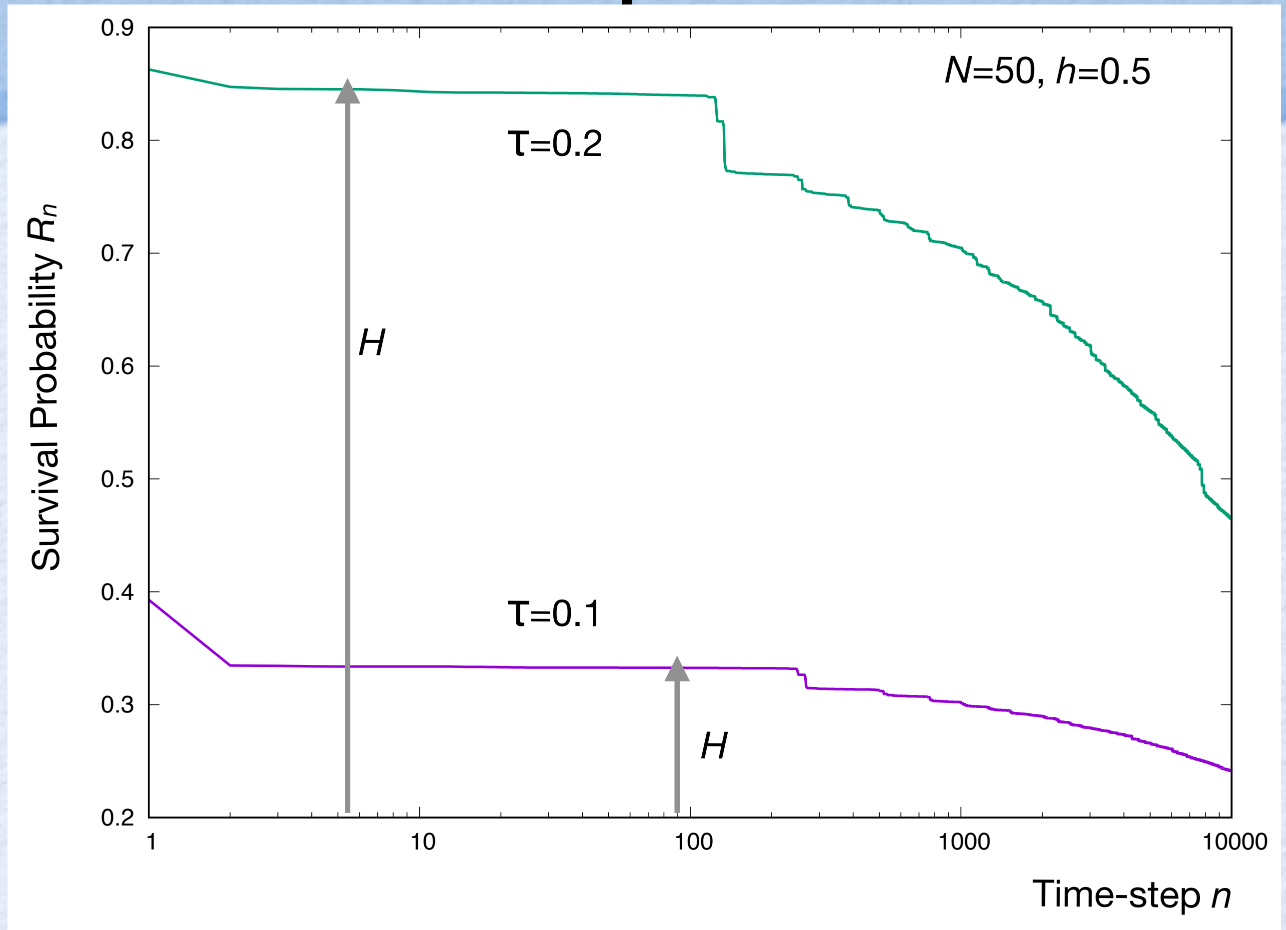
N=50, h=0.5



N=70, h=1.5



Characteristics: h=0.5, ordered phase



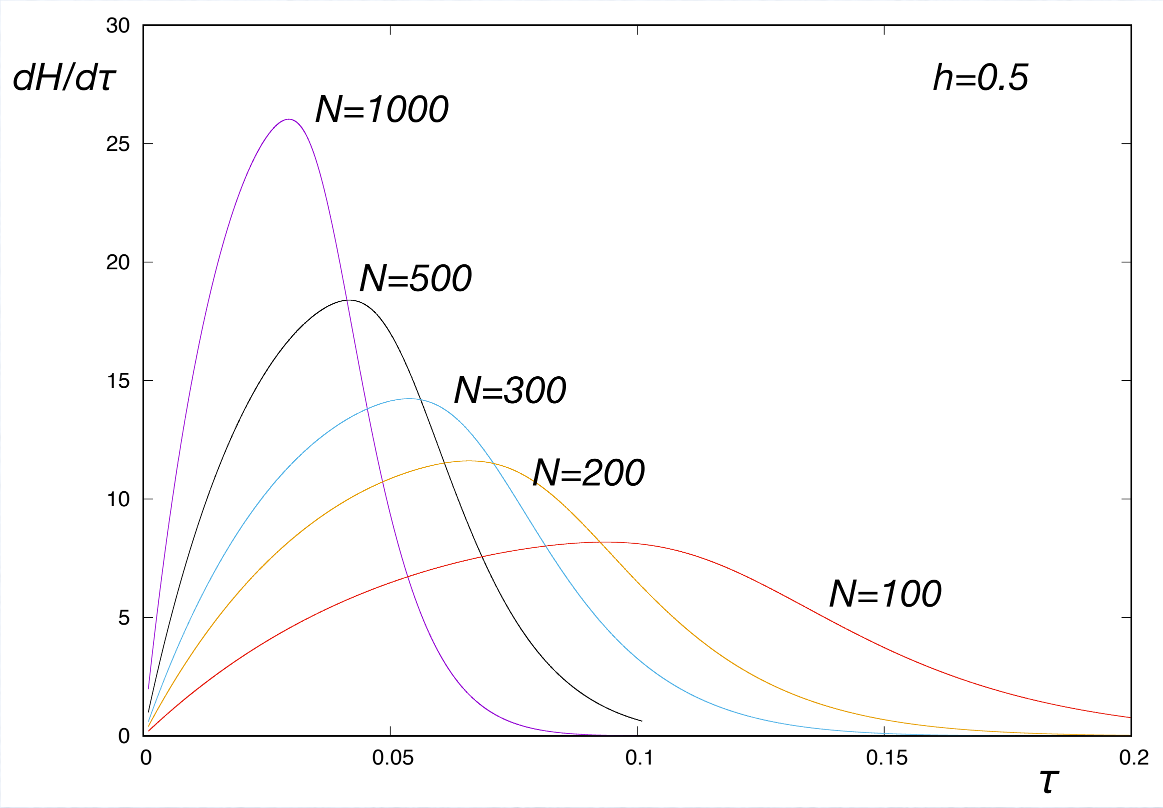
There is a plateau region

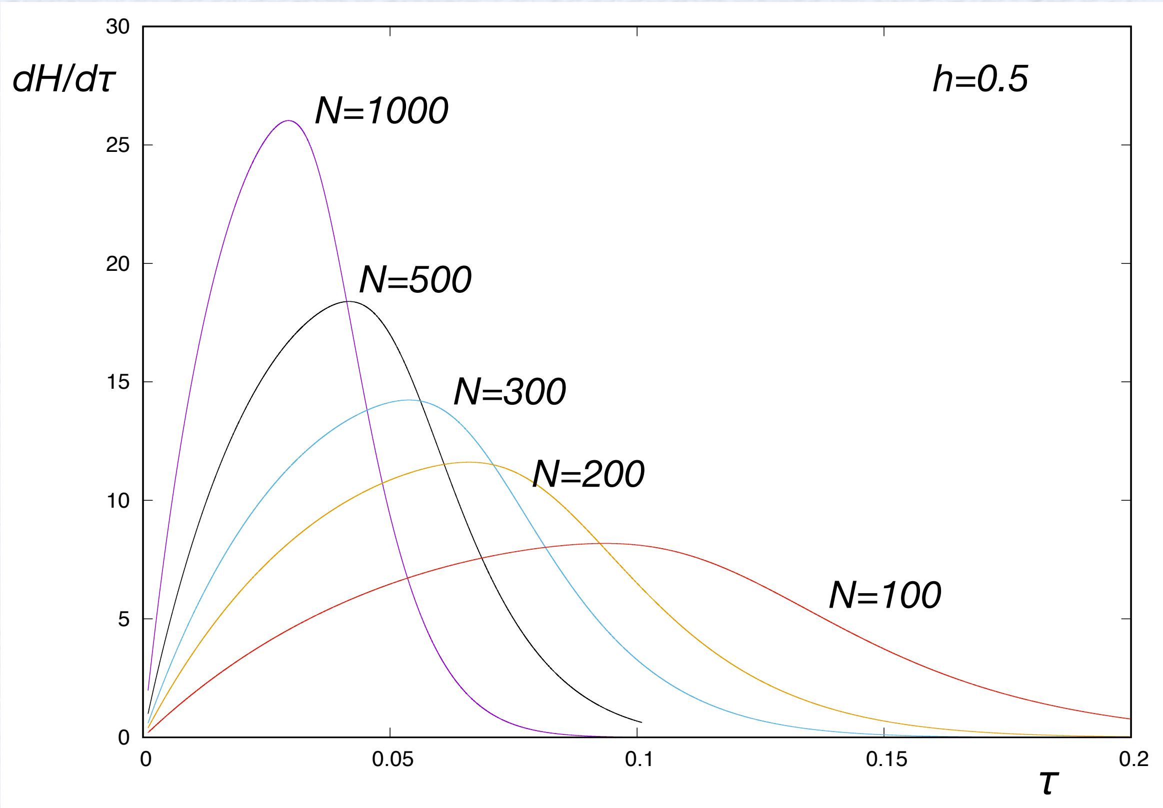
This region increases in size as τ decreases and as system size increases

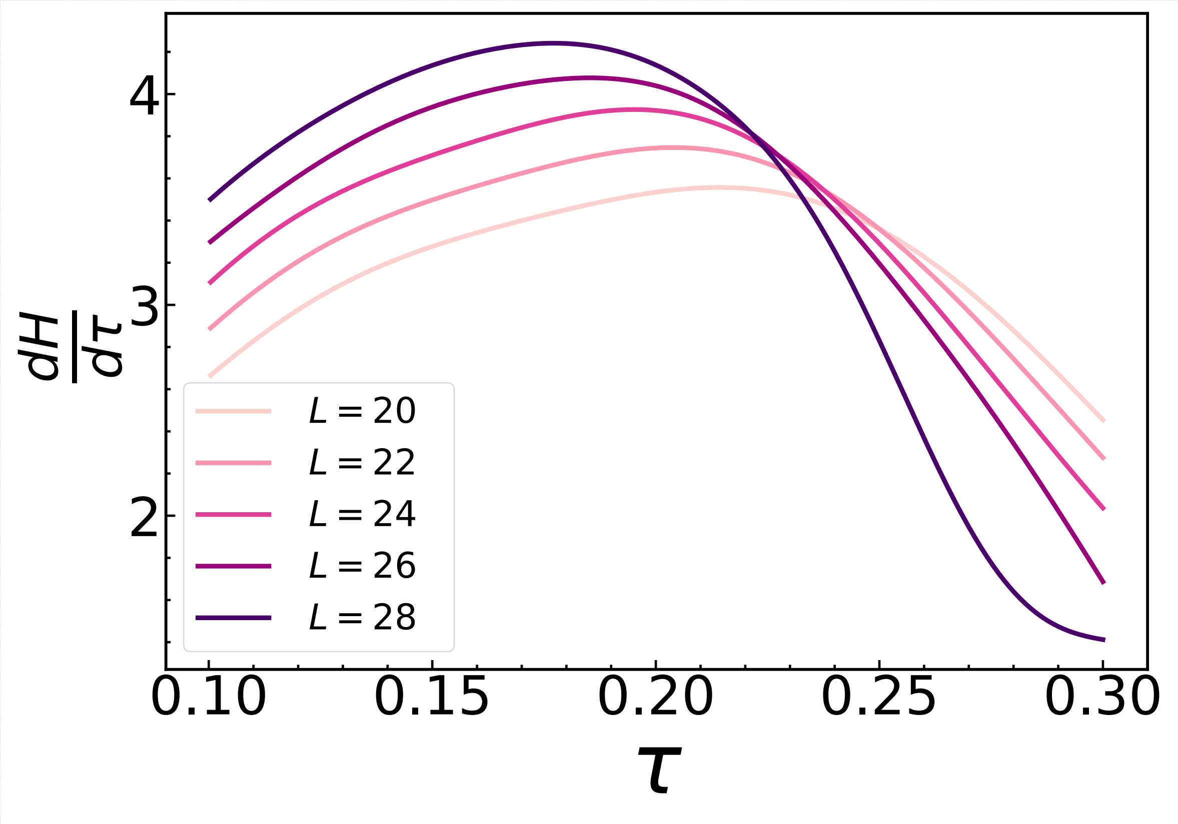
How the height *H* of the plateau region varies with τ?

H increases *monotonically* with τ.

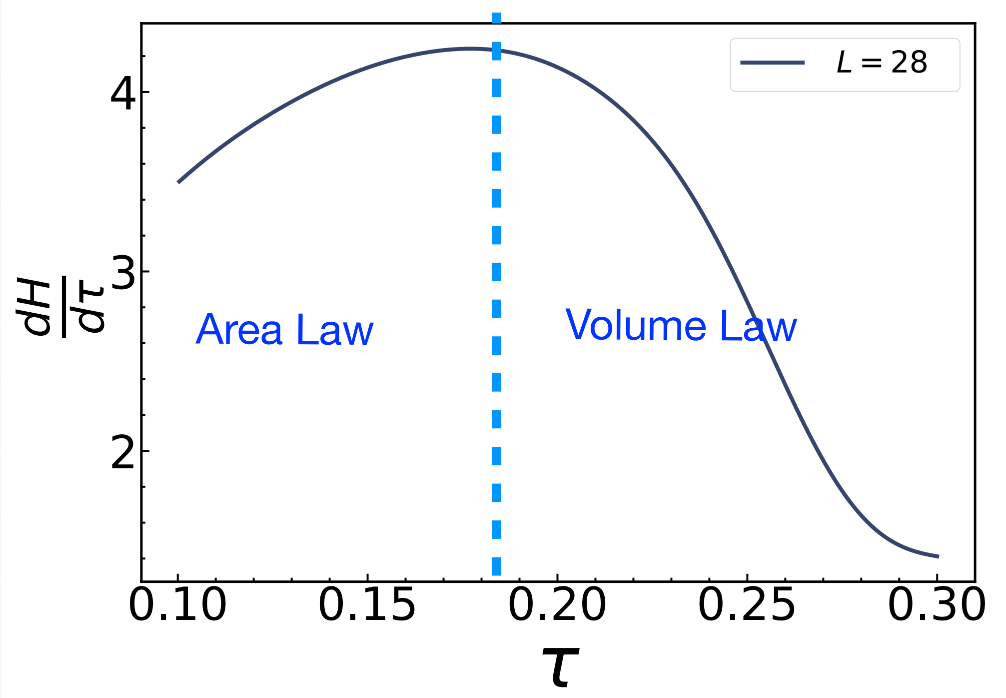
The curve of *dH*/*d*τ vs τ shows a peak. Sharpness and position of the peak depends on system-size.





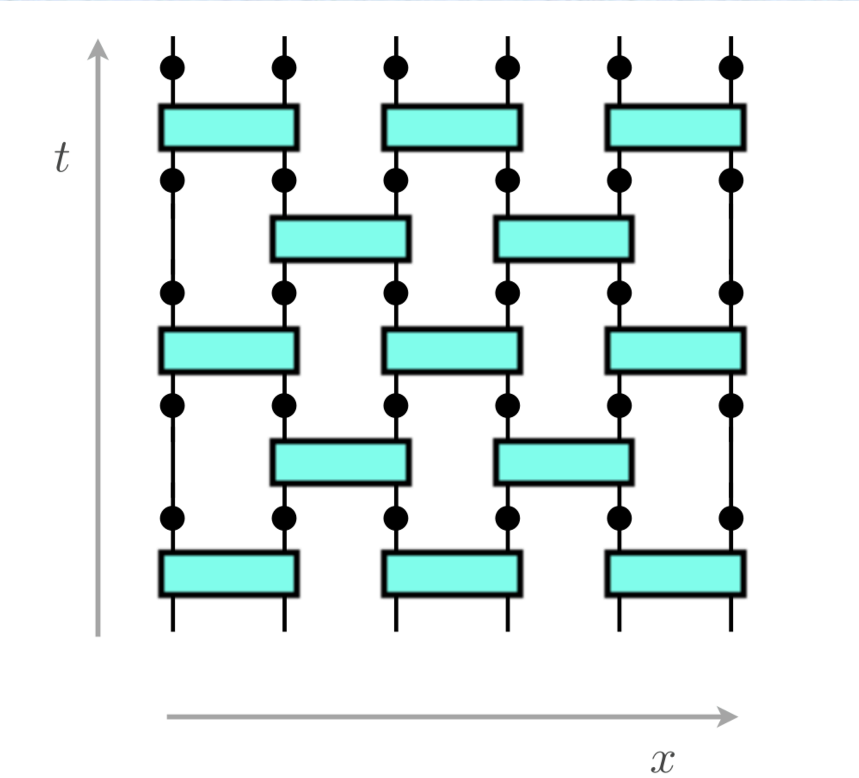


1. As system-size increases, the peak becomes sharper and higher.
2. At the peak there is an area-law to volume-law transition in entanglement



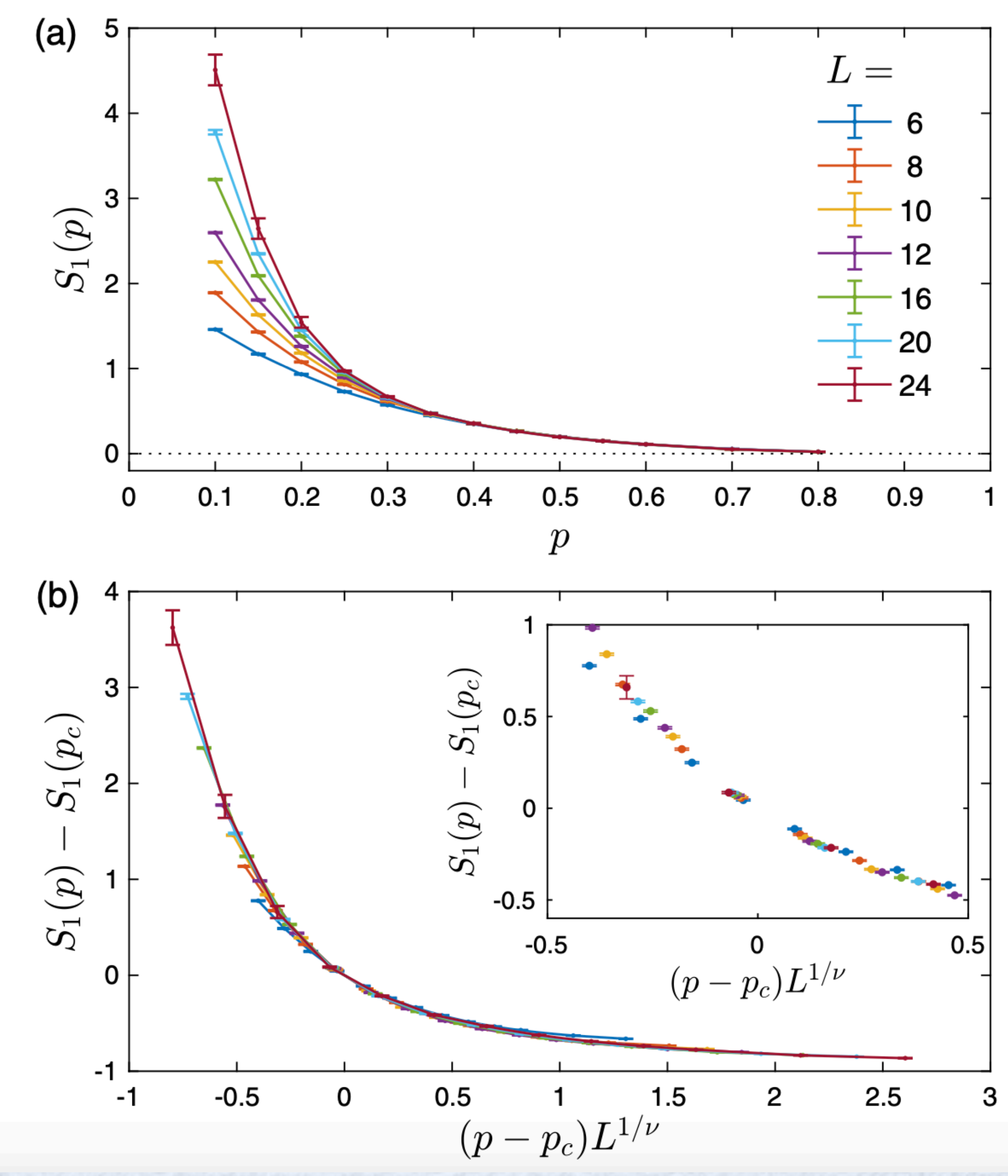
Hence we identify the peak with a transition. Location of the peak is the critical value of τ, namely τ_c

Skinner, Ruthman & Nahum, PRX, 2019

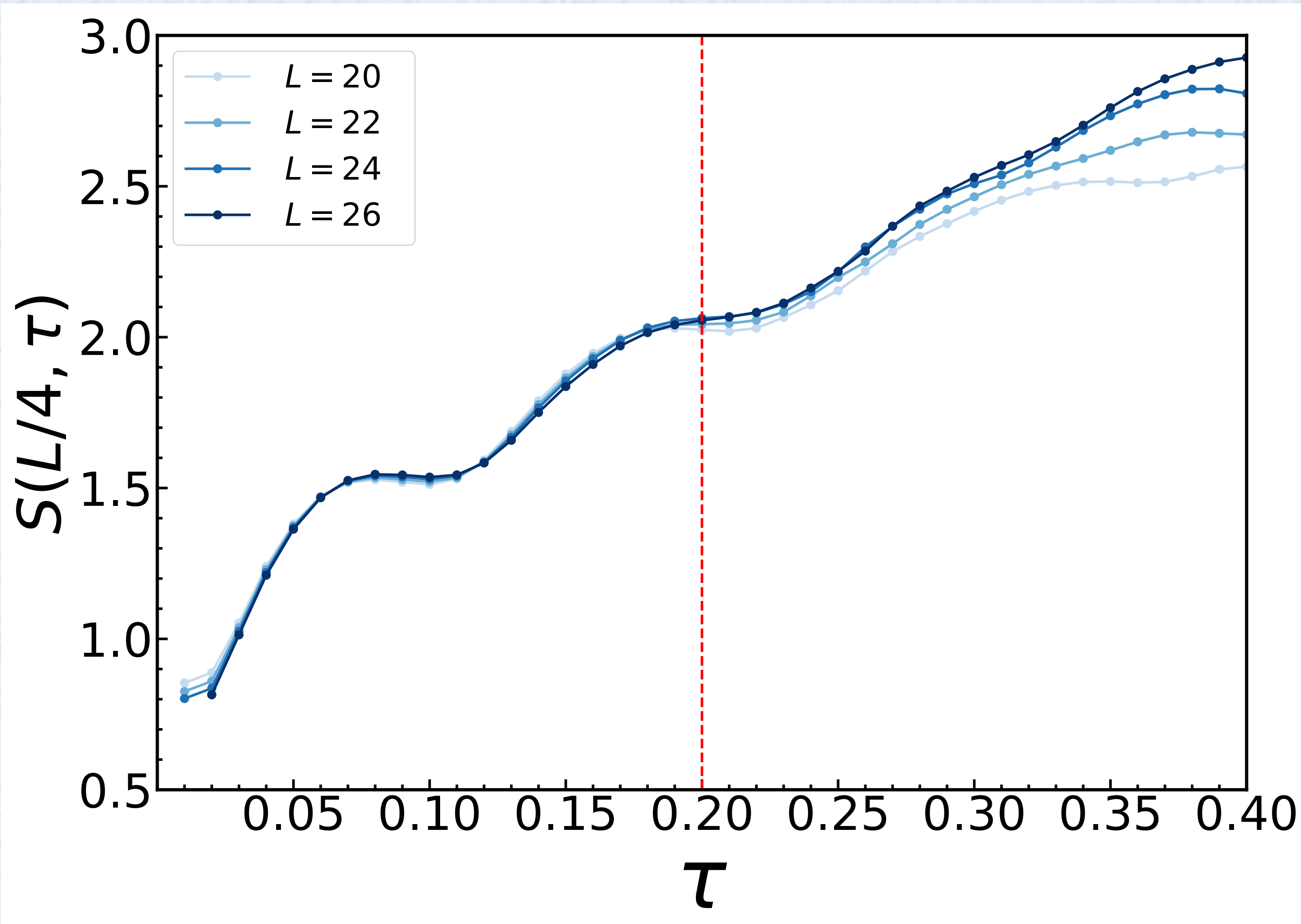


Quantum circuit with random unitary dynamics,
At every time step z-component of spin is measured with probability p.
Volume-law for $p < p_c$ Area-law for $p > p_c$

von-Neumann entanglement entropy between the two halves of the system shows a collapse for $p > p_c$ when $S(p)$ - $S(p_c)$ is plotted against $(p - p_c)L^{1/\nu}$

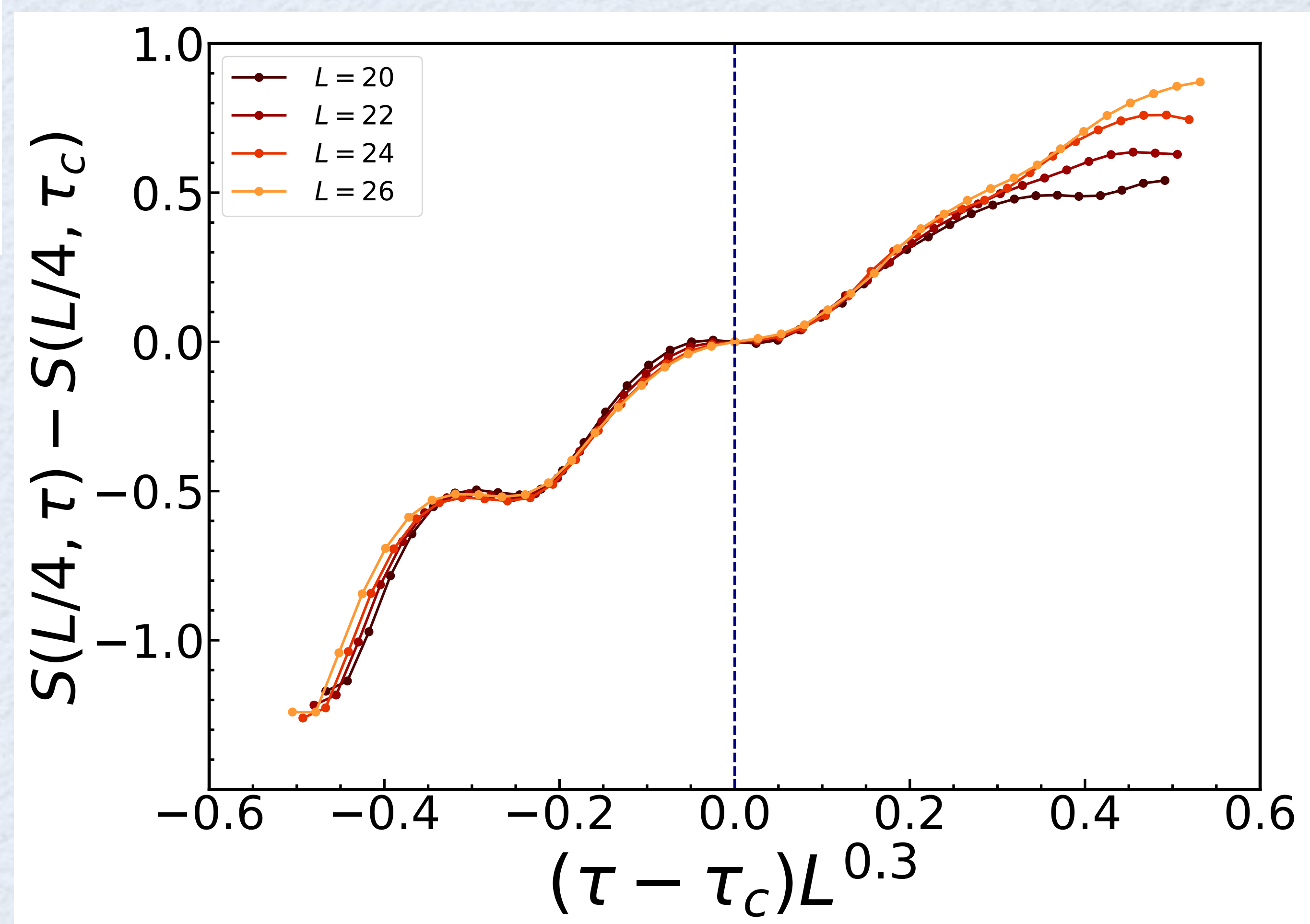


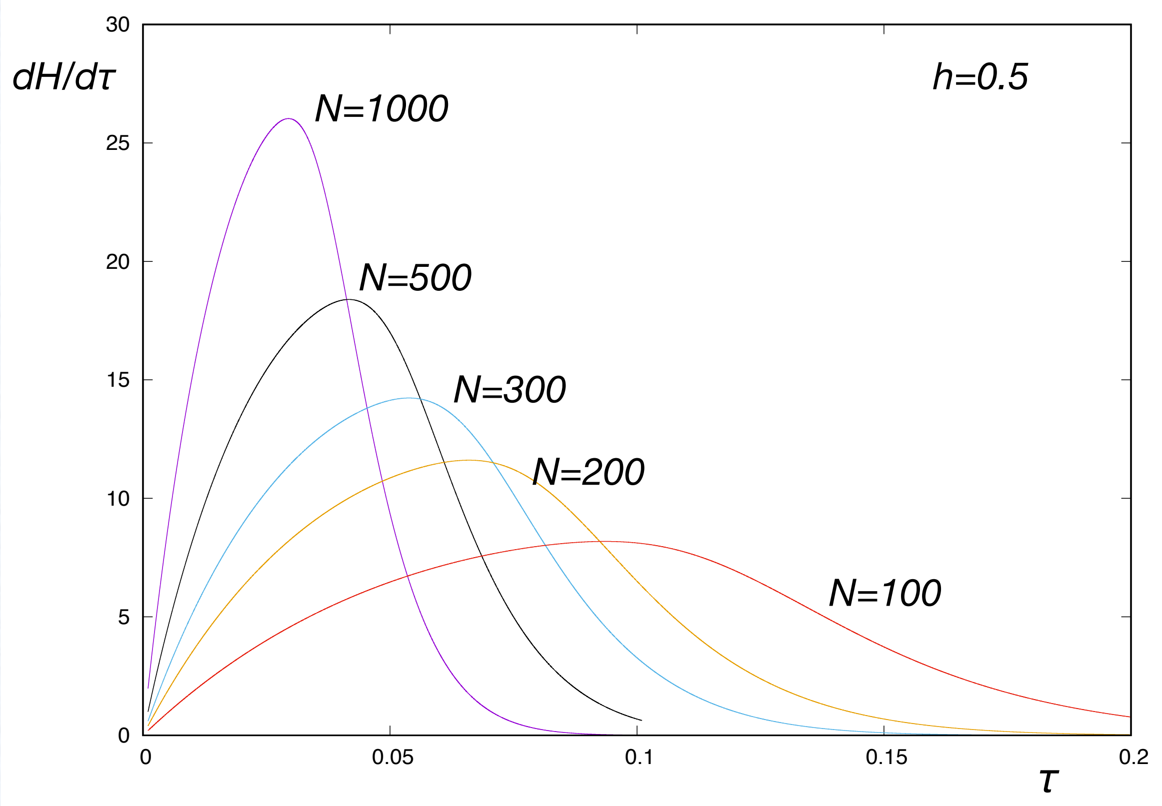
For our system we compute von-Neumann entanglement entropy between the subsystems of sizes $(L/4)$ and $(3L/4)$ for system size 20 to 26, and find a collapse for $\tau < \tau_c$ when $S(\tau)$ - $S(\tau_c)$ is plotted against $(\tau - \tau_c)L^{0.3}$ τ_c=0.2

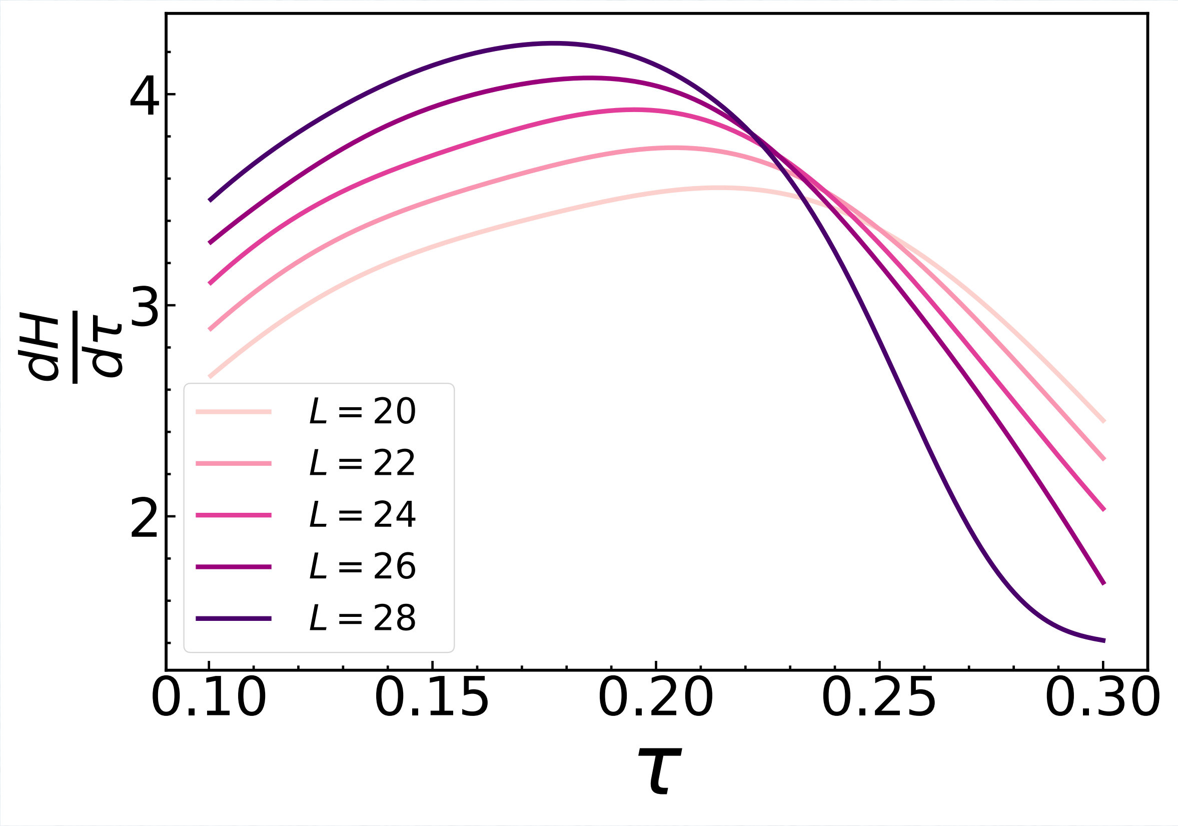


Area Law : $\tau < \tau_c$
Volume Law : $\tau > \tau_c$

$\tau_c = 0.2$

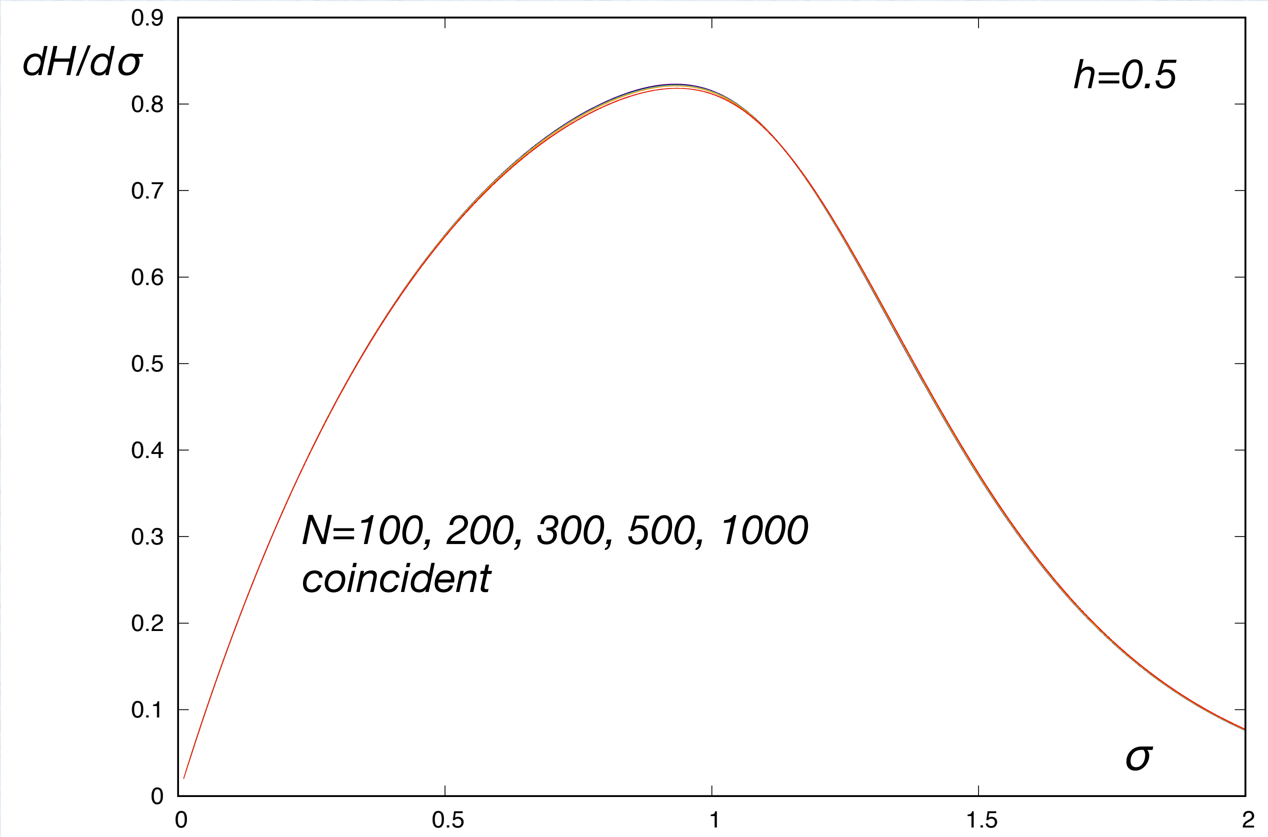






The peak moves to the left as N increases.

Scaling: $\tau \propto 1/\sqrt{N}$



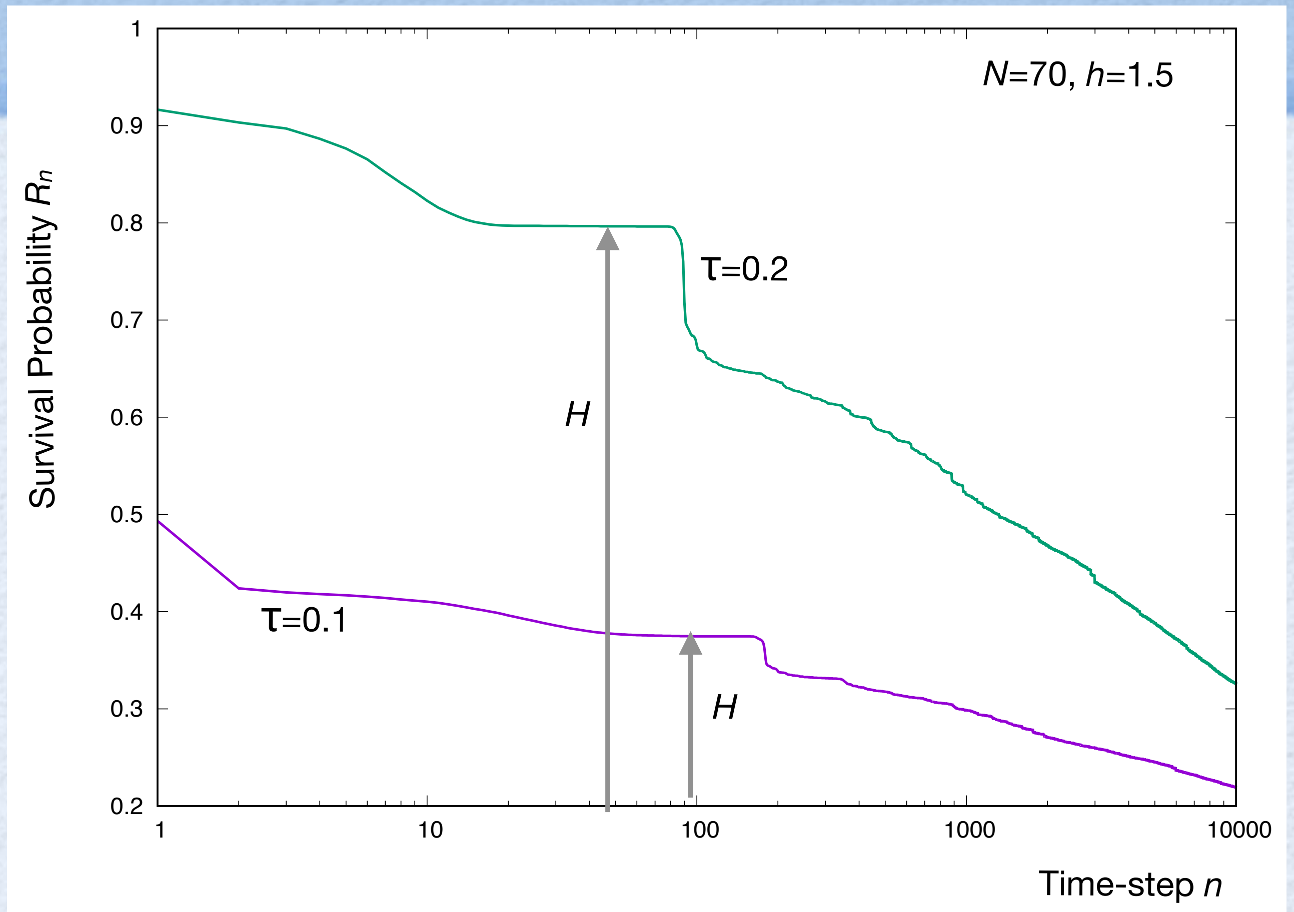
Curves for different N coincide when $\sigma = \tau \sqrt{N}$ is used instead of τ.

$\tau_c \propto 1/\sqrt{N}$

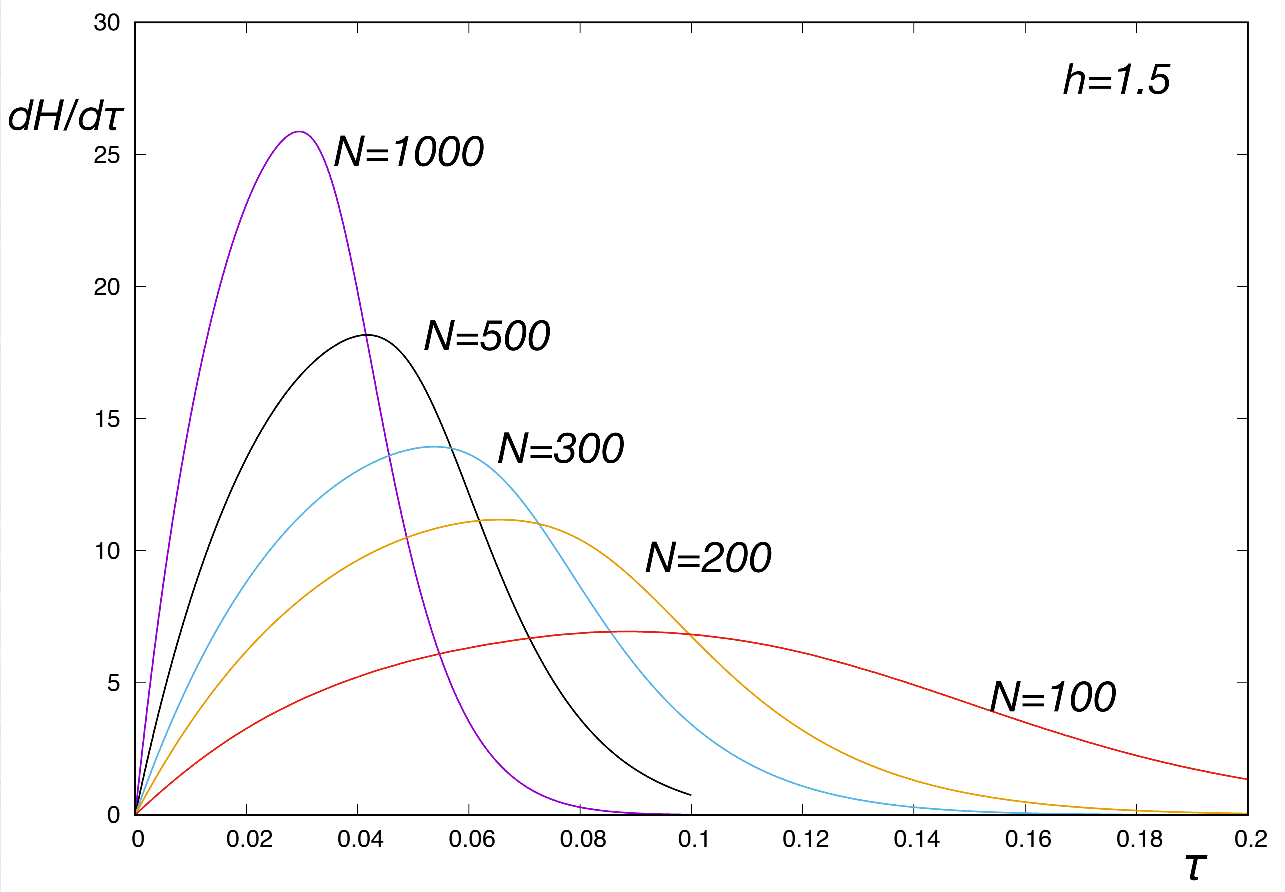
At $N \to \infty$, $\tau_c = 0$.

Transition occurs for finite size only.

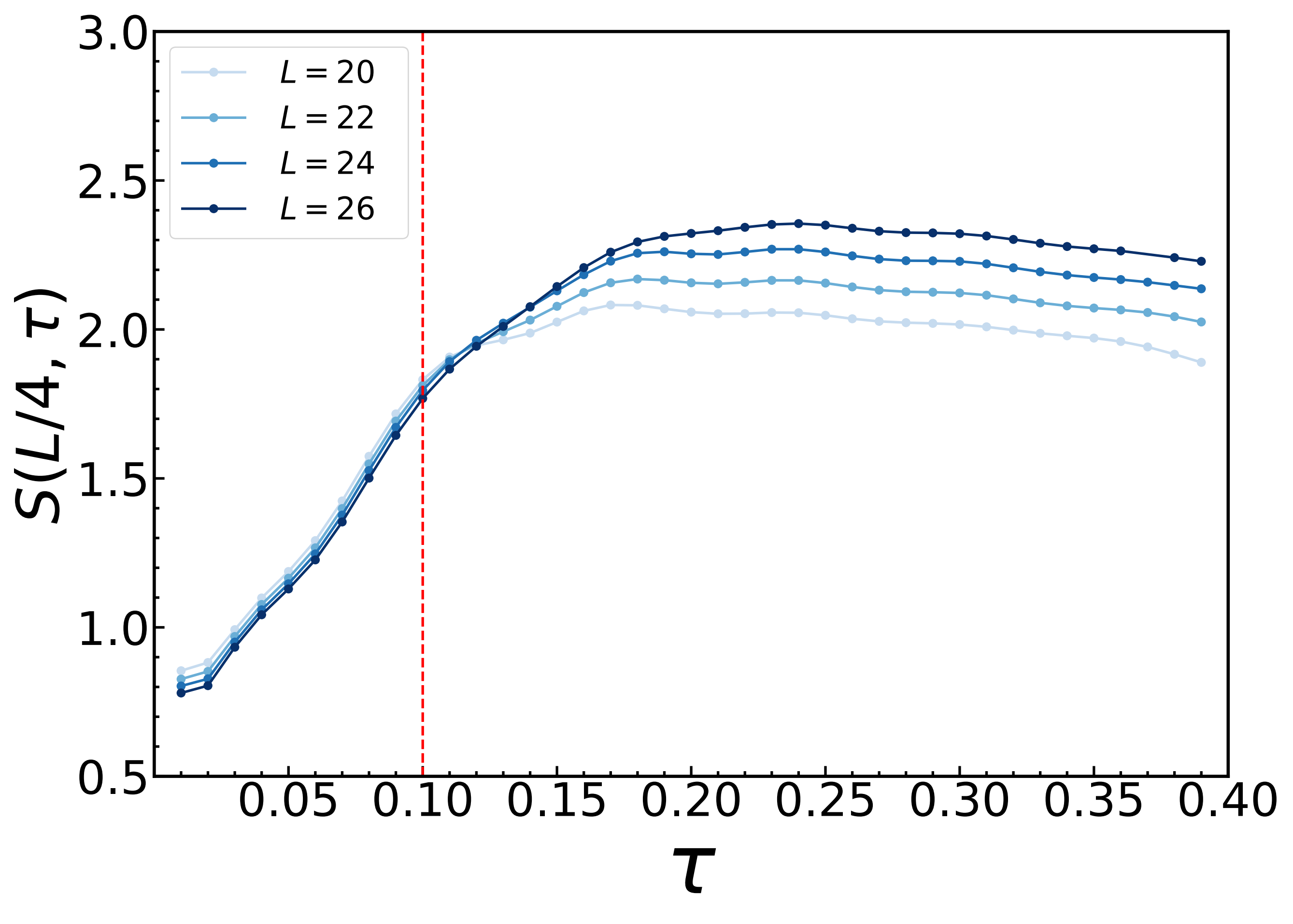
Characteristics: h=1.5, disordered phase



Again, there is a plateau region and this region increases in size as τ decreases and as system size increases

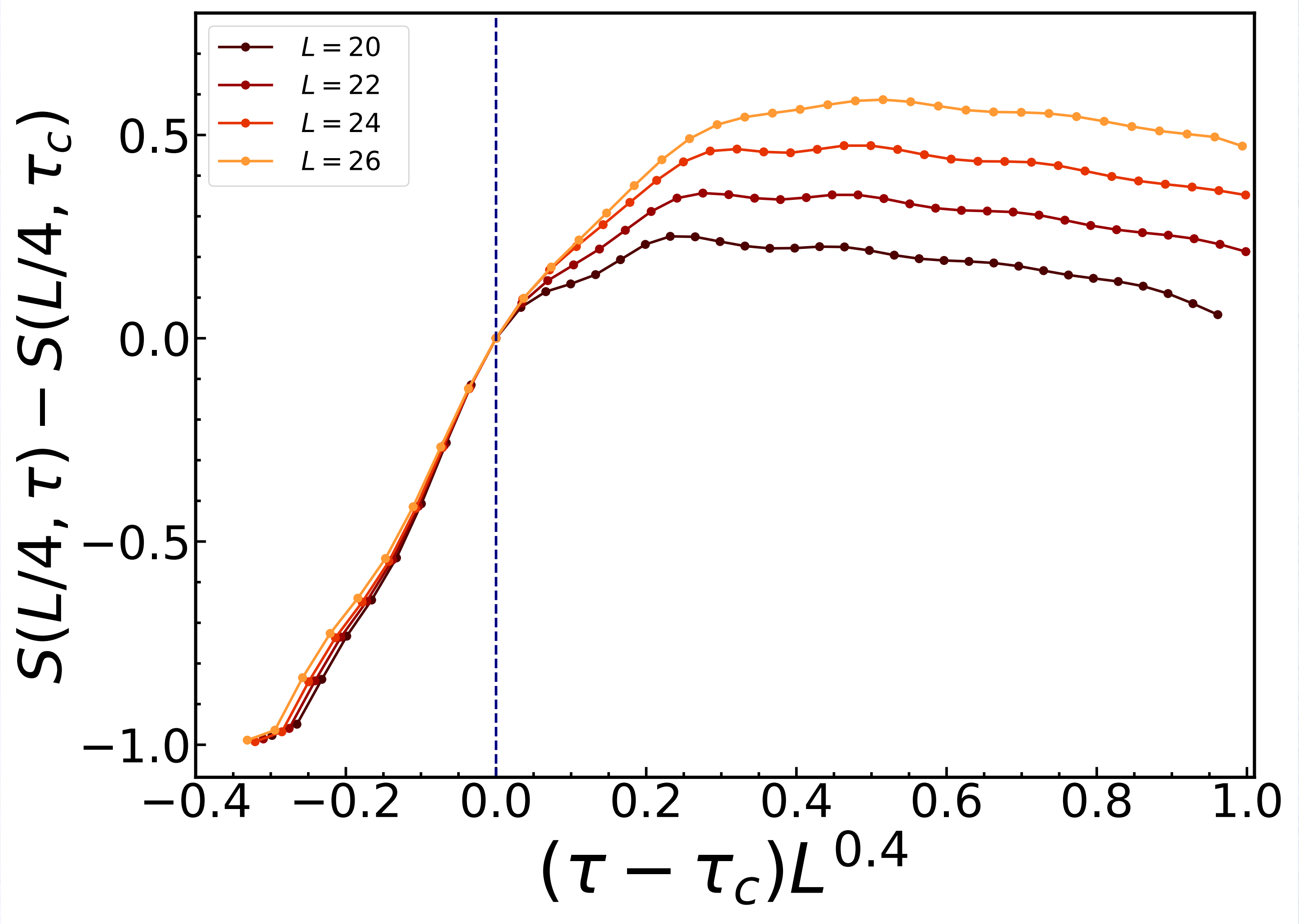


$dH/d\tau$ vs τ shows peak. As N increases, it
becomes sharper
becomes higher
moves to the left

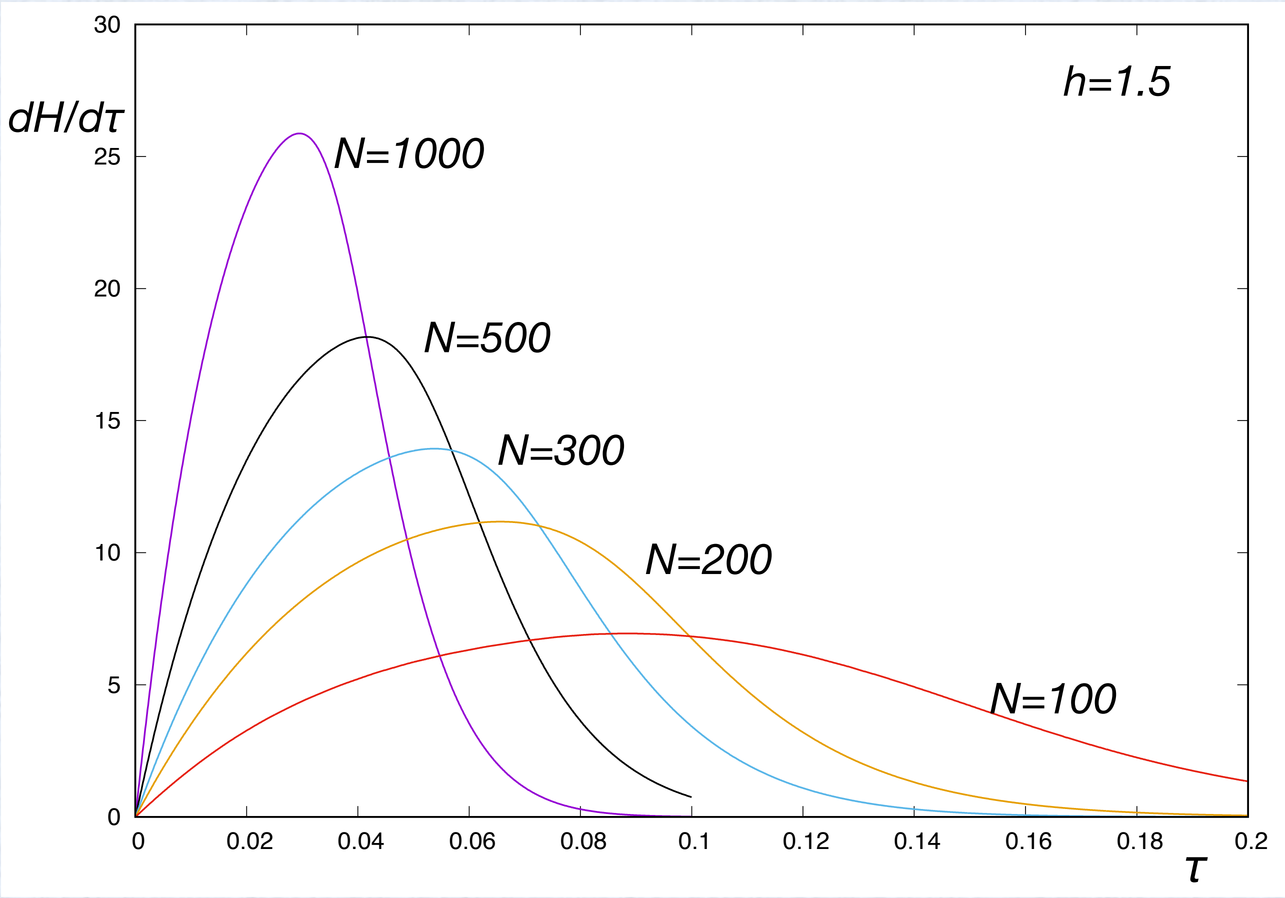


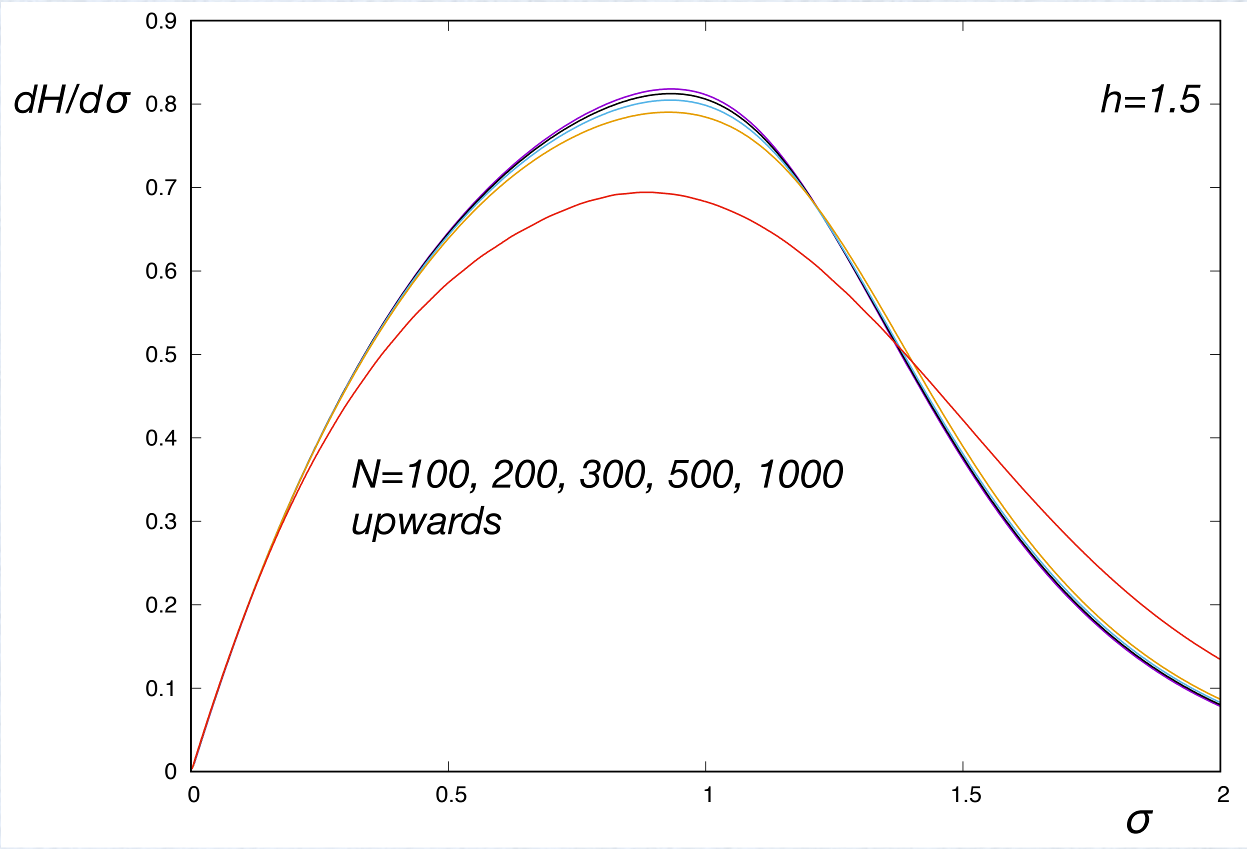
Area Law : $\tau < \tau_c$
Volume Law : $\tau > \tau_c$

$\tau_c = 0.1$



For h=1.5, the scaling $\tau \propto 1/\sqrt{N}$ is good for larger system size only





At $N \to \infty$, $\tau_c = 0$

Scaling Relation $\tau \propto 1/\sqrt{N}$

Survival probability is $R_n = \langle\psi_n|\psi_n\rangle = \sum_{m_1,m_2=0}^{(n)} \left(C_{m_1}^{(n)}\right)^* C_{m_2}^{(n)} f_{m_2-m_1}$
with the recursion relations

$$C_0^{(n+1)} = -\sum_{m=0}^{n} C_m^{(n)} f_{m+1}, \quad C_m^{(n+1)} = C_{m-1}^{(n)} \ (0 < m \leq n) \quad C_n^{(n)} = 1 \quad C_m^{(n)} = C_0^{(n-m)}$$

and the initial values, $C_0^{(1)} = -f_1, \quad C_1^{(2)} = -f_1, \quad C_0^{(2)} = f_1^2 - f_2$

The coefficients are $f_n = \prod_k \left[\cos(\lambda_k n\tau) + i\sin(\lambda_k n\tau)\cos(2\theta_k)\right]$

Write f_n as $f_n = \rho_n e^{i\Phi_n}$, so that

$$\rho_n = \exp\left[\frac{1}{2}\sum_k \log\left[1 - \left(\frac{2\sin k\ \sin(\lambda_k n\tau)}{\lambda_k}\right)^2\right]\right]$$

$$\Phi_n = \sum_k \tan^{-1}\left[\frac{2(h+\cos k)}{\lambda_k}\tan(\lambda_k n\tau)\right]$$

For small values of τ and n, $\Phi_n = \sum_k 2n\tau(h+\cos k) = \mu n$ where $\mu = 2\tau h N$

$$\Rightarrow \mathrm{Arg}\left(C_m^{(n)}\right) = \mu(n-m) \quad \Rightarrow \mathrm{Arg}\left[\left(C_{m_1}^{(n)}\right)^* C_{m_2}^{(n)} f_{m_2-m_1}\right] = 0$$

$$R_n = \langle \psi_n | \psi_n \rangle = \sum_{m_1, m_2 = 0}^{(n)} \left(C_{m_1}^{(n)} \right)^* C_{m_2}^{(n)} f_{m_2 - m_1}$$

$\Rightarrow$ only the modulus of f_n matters. In the limit of small $n\tau$,

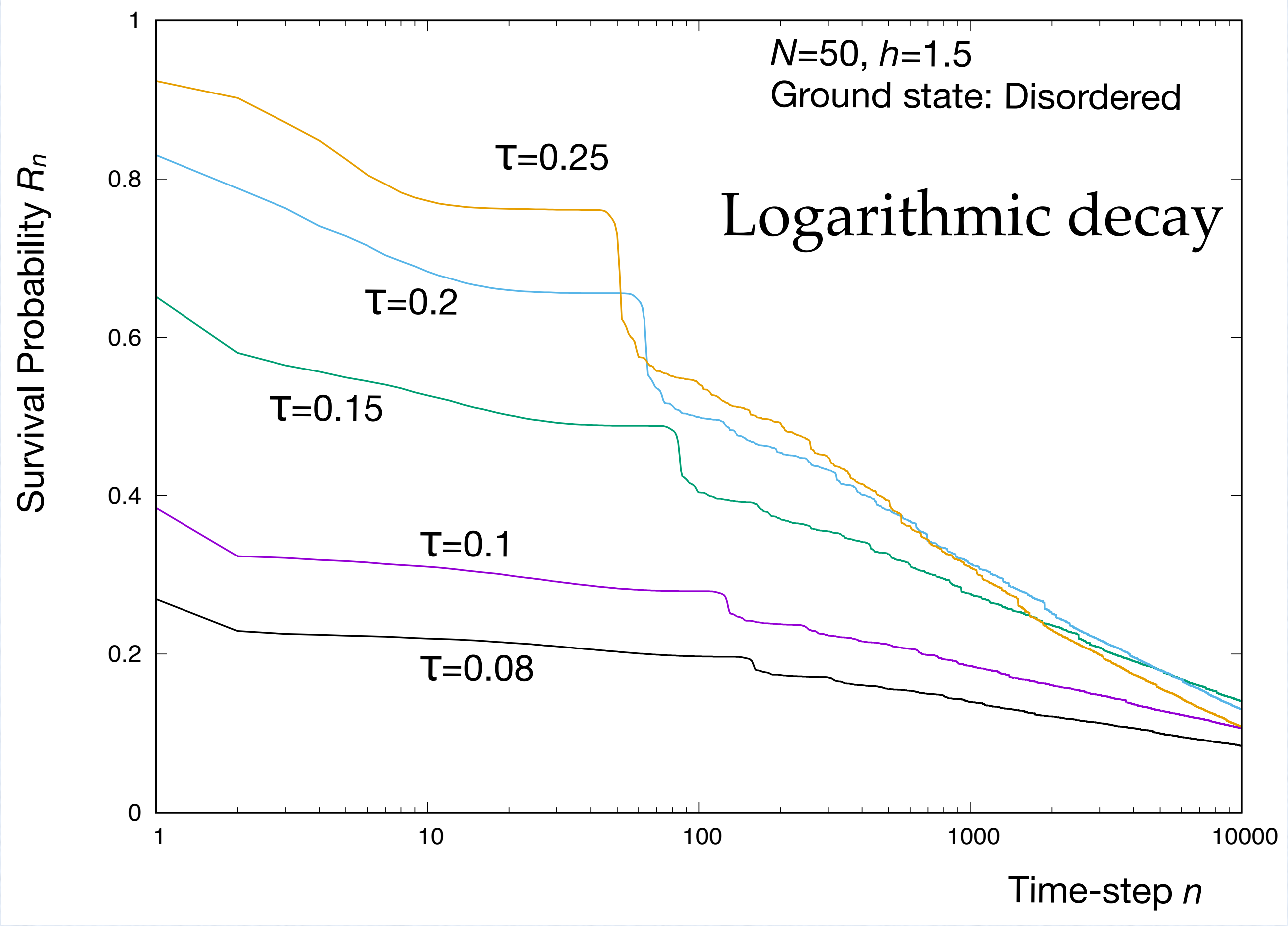
$$\rho_n = |f_n| = \exp\left[-\frac{1}{2} n^2 \tau^2 N \right]$$

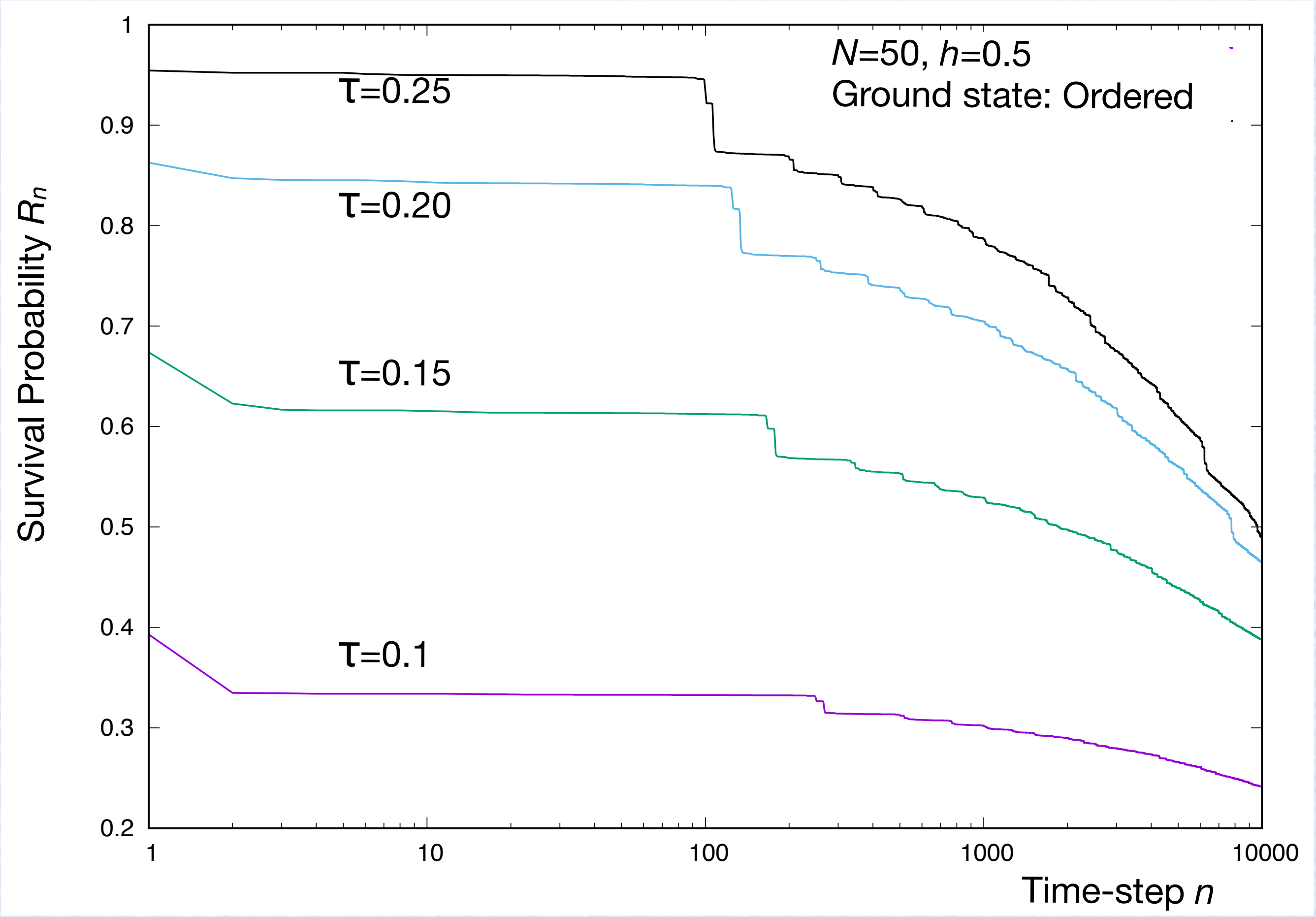
Hence, τ and N occur as the combination $\tau^2 N$ in the value of survival probability.

Quantum Phase Transition and Measurement-Induced Transition

Normally, measurement-induced transitions may not be related to quantum phase transitions.

Here, we find a change in behavior in the pattern of decay of the survival probability





However, when *h* is too high, the logarithmic decay is not there

Another Analytically Tractable Measurement-Induced Transition

Here, at each measurement, we have removed the component of initial state and retained the other states. Something more straightforward happens when we do the reverse, that is, *retain the initial state only.* (S. Dhar & Dasgupta, PRA 2016)

Start with $|\mathcal{I}\rangle = |++\cdots+\rangle_z$

After the first time-step:
Wave function : $|\psi_1\rangle = p|\mathcal{I}\rangle$ where $p = \langle\mathcal{I}|e^{-i\mathcal{H}\tau}|\mathcal{I}\rangle$

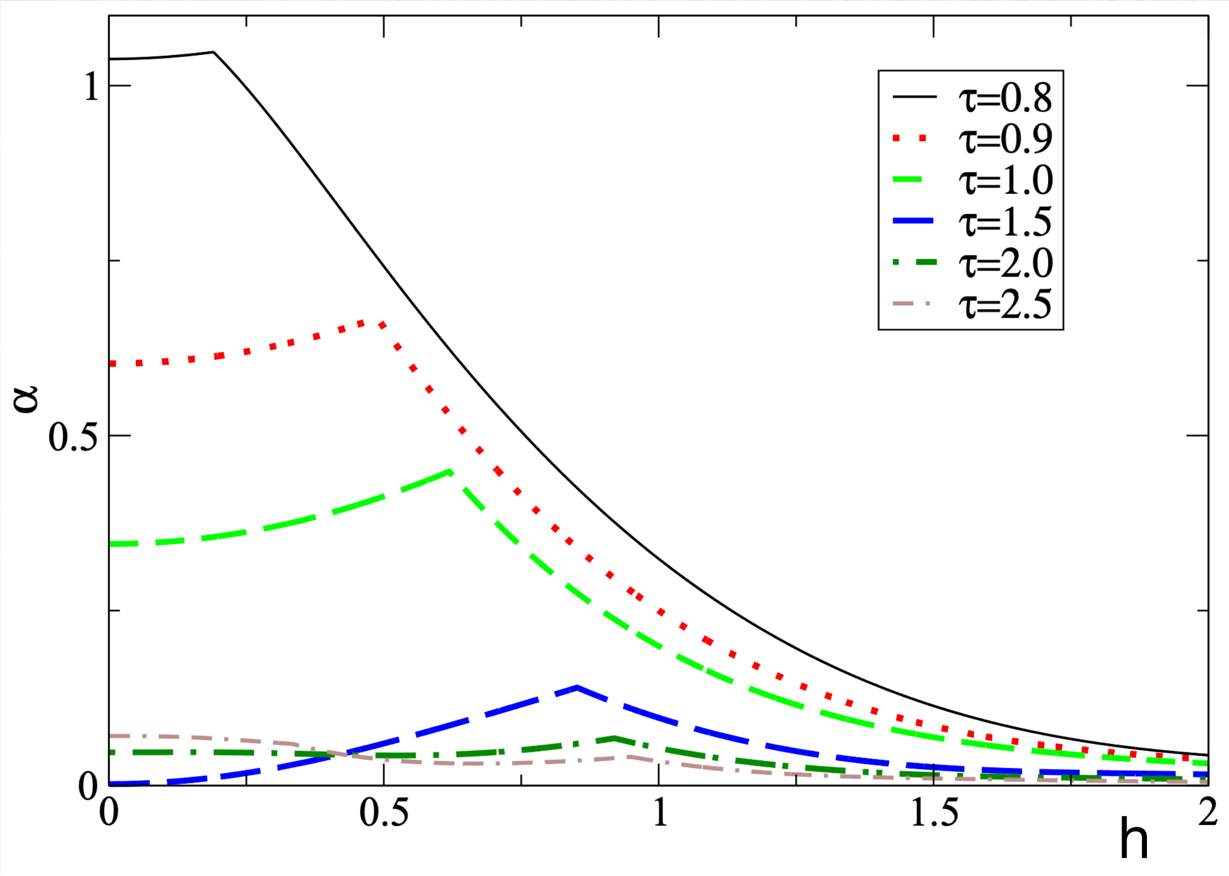
After the second time-step:
Wave function : $|\psi_2\rangle = p|\psi_1\rangle$

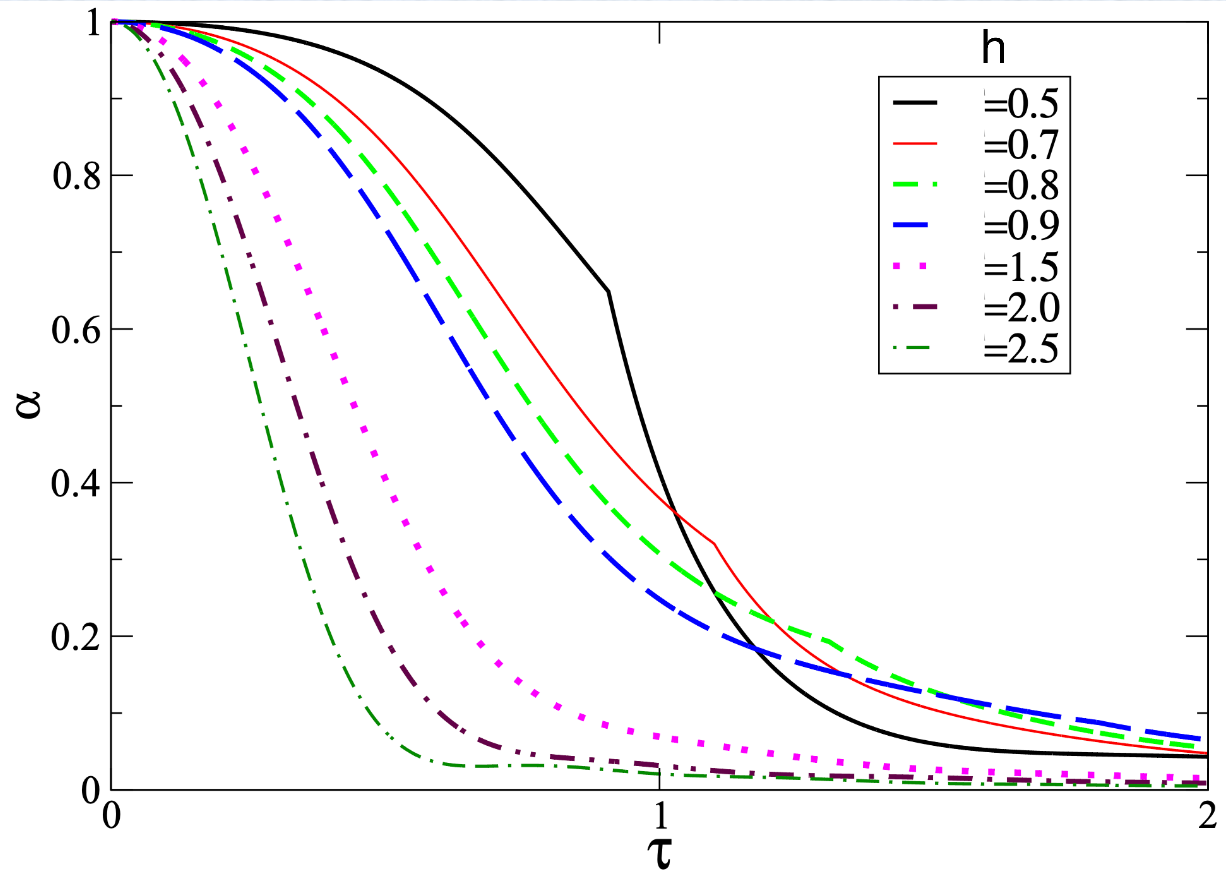
After the n-th time step:
$|\psi_n\rangle = p|\psi_{n-1}\rangle = p^n|\mathcal{I}\rangle$

$$p = e^{-\alpha N\tau^2/2} \quad \text{with} \quad \alpha(h,\tau) = -\frac{1}{2\pi\tau^2}\int_{k=0}^{\pi} dk \, \log\left[1 - \left(\frac{2\sin k \, \sin(\lambda_k\tau)}{\lambda_k}\right)^2\right]$$

α is non-analytic at $\tau_c\sqrt{1-h_c^2} = \frac{\pi}{4}$

$$p = \frac{\langle \psi_{n+1} | \psi_{n+1} \rangle}{\langle \psi_n | \psi_n \rangle} = e^{-\alpha N \tau^2}$$





Kinks exist at h < 1

1. *Phase transition* since there is non-analyticity at thermodynamic limit.
2. *p* vanishes in the thermodynamic limit.
3. At every stage, only the ``all-up'' state is retained. Hence a study of entanglement is not meaningful.

Summary

We present a protocol for measurement-induced transition, where at each time-step we evolve a quantum Ising chain under transverse Ising Hamiltonian for time τ, and then make a global measurement with certainty.

For system size ≤ 28, there is a transition in entanglement at some τ_c. We also calculate survival probability of the initial state and find that a quantity $dH/d\tau$ derived from this probability also shows a peak at τ_c

Using some recursion relation, one can compute survival probability (and hence $dH/d\tau$) in our set-up for size upto 1000. It is found that $\tau_c \sim 1/\sqrt{N}$ Hence, the transition occurs for finite size only. It will be interesting to investigate the size-dependence of the critical point in other measurement-induced transitions.

We also find that the behaviour of survival probability has some relation to the order present in the ground state of the Hamiltonian.

Thank you for your attention

For the specific case of transverse Ising Hamiltonian, one can derive an expression for $f_n = \langle \mathcal{I} | e^{-i\mathcal{H}\tau n} | \mathcal{I} \rangle$ in closed form for any even value of N, by using the known exact solution.

Transform spin variables s_j to fermion variables a_j and perform Fourier transformation to fermion variables a_k, to get $\mathcal{H}$ as a Kronecker sum of commuting operators $\mathcal{H}_k$:

$$\mathcal{H} = \sum_{k=0}^{\pi} \mathcal{H}_k, \quad \mathcal{H}_k = -2i \sin k \left[a_k^\dagger a_{-k}^\dagger + a_k a_{-k} \right] - 2(h + \cos k) \left[a_k^\dagger a_k + a_{-k}^\dagger a_{-k} - 1 \right]$$

where $k = (2n+1)\pi/N$ with $n = 0, 1, 2, \cdots, N/2 - 1$. The ground state and excited state of $\mathcal{H}_k$ are

$$\begin{aligned} |GS\rangle_k &= i \cos\theta_k |11\rangle_{k,-k} - \sin\theta_k |00\rangle_{k,-k} \\ |ES\rangle_k &= i \sin\theta_k |11\rangle_{k,-k} + \cos\theta_k |00\rangle_{k,-k} \end{aligned}$$

with eigenvalues $\mp\lambda_k = \mp 2\sqrt{h^2 + 1 + 2h\cos k}$. Here, $e^{2i\theta_k} = 2(h + e^{-ik})/\lambda_k$.

Since $\sum_j s_j^z = \sum_k (2a_k^\dagger a_k - 1)$, the state $|\mathcal{I}\rangle = | + + + \cdots \rangle_z$ corresponds to one where all k-modes are occupied by fermions. Then

$$e^{-i\mathcal{H}_k n\tau} |11\rangle_{k,-k} = \left[\cos(\lambda_k n\tau) + i \sin(\lambda_k n\tau) \cos(2\theta_k) \right] |11\rangle_{k,-k} - \sin(2\theta_k) \sin(\lambda_k \tau) |00\rangle_{k,-k}$$

Finally, $f_n = \prod_{k=0}^{\pi} \left[\cos(\lambda_k n\tau) + i \sin(\lambda_k n\tau) \cos(2\theta_k) \right]$

Damski & Rams, J. Phys. A 47, 025303 (2013)