Completely positive trace-preserving maps for higher-order unraveling of Lindblad master equations

N. Wonglakhon, H. M. Wiseman and A. Chantasri, Phys. Rev. A **110**, 062207 (2024)

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ICTS Program “Quantum Trajectories"

International Centre for Theoretical Sciences (ICTS) Bengaluru, India

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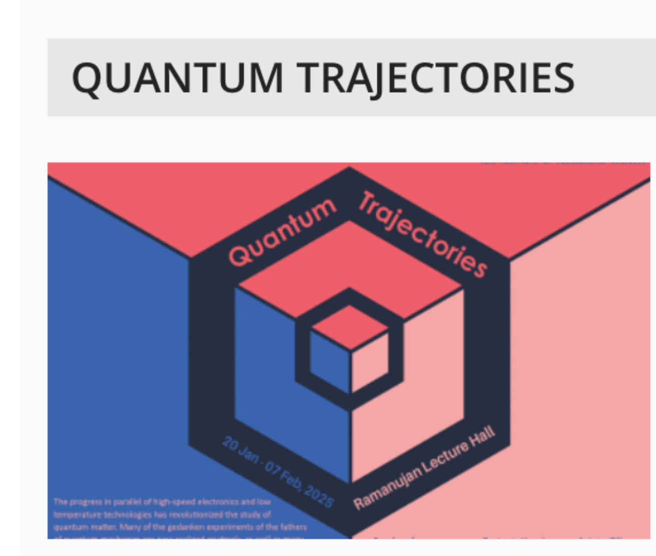
NRCT

"Quantum Trajectories"

- The stochastic master equation (SME) for the quantum state:



$$\mathrm{d}\rho_t = -i\mathrm{d}t[\hat{H}, \rho_t] + \mathrm{d}t\mathscr{D}[\hat{c}]\rho_t + \mathrm{d}W_t \mathscr{H}[\hat{c}]\rho_t$$

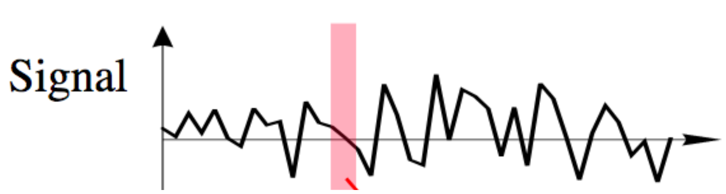
where $$\mathscr{D}[\hat{c}] \bullet = \hat{c} \bullet \hat{c}^\dagger - \frac{1}{2}\{\hat{c}^\dagger \hat{c}, \bullet\}$$

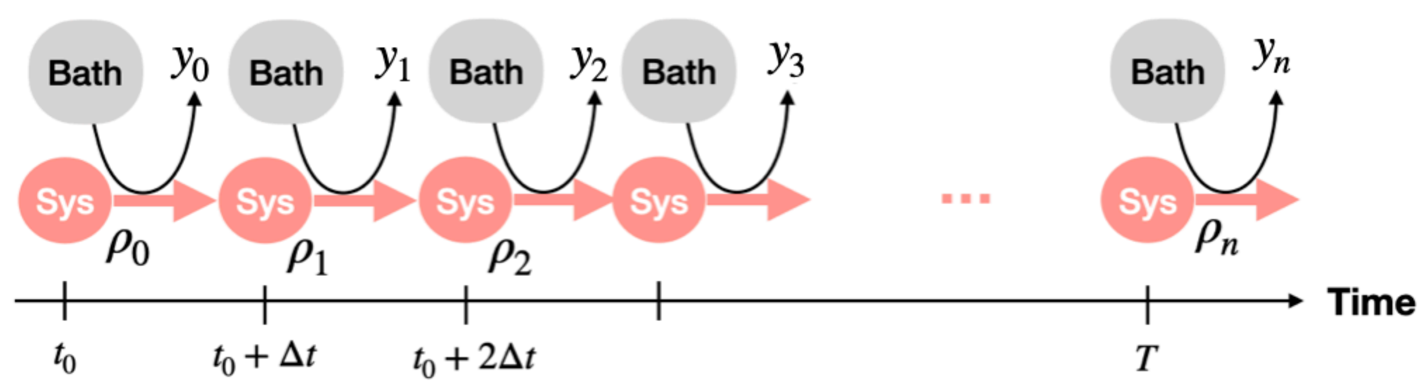
$$\mathscr{H}[\hat{c}] \bullet = \hat{c} \bullet + \bullet \hat{c}^\dagger - \mathrm{Tr}(\hat{c} \bullet + \bullet \hat{c}^\dagger) \bullet$$

$$\mathrm{d}W_t = y_t \mathrm{d}t - \mathrm{Tr}[(\hat{c} + \hat{c}^\dagger)\rho_t]\mathrm{d}t$$ (Wiener increment)

- For continuous measurement : Diffusive-type measurements

*strong Markov assumption

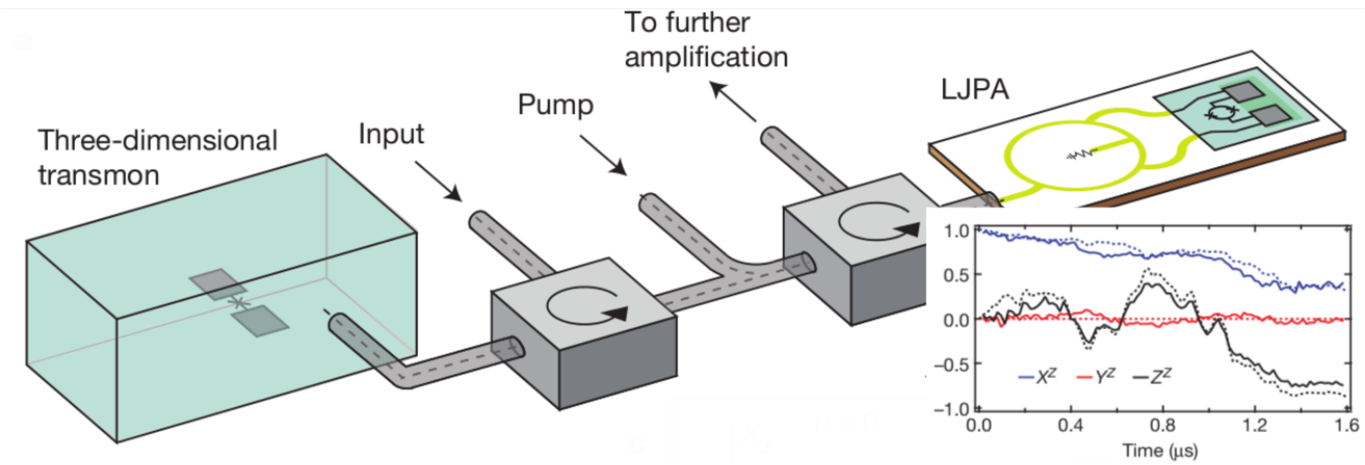




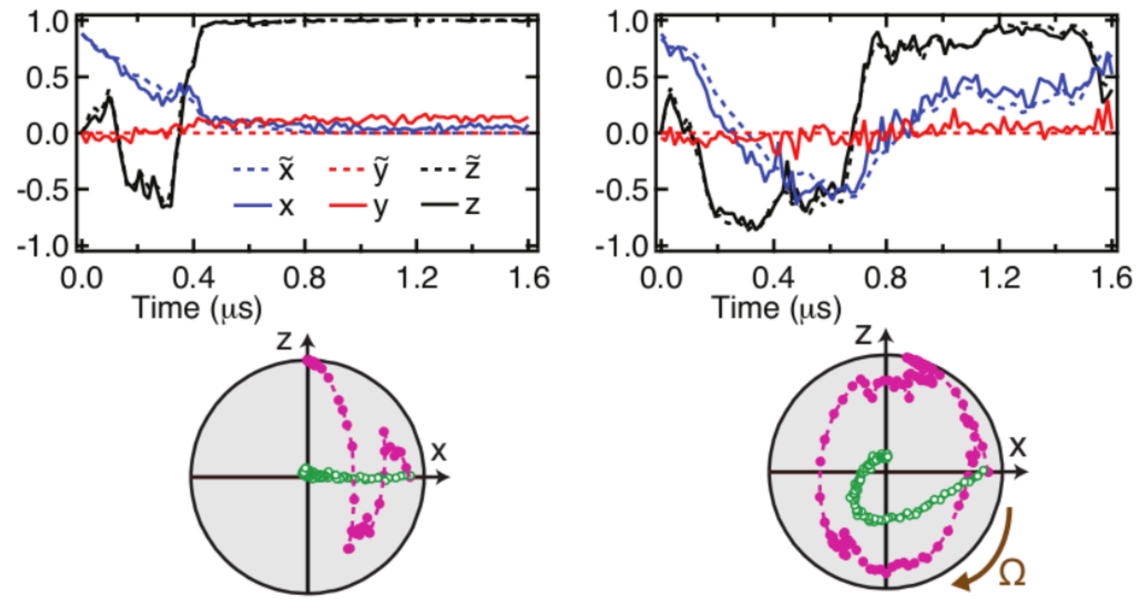
By averaging all possible quantum trajectories, one gets the **Lindblad master equation.**

Quantum Trajectories in experiments

- **Example: qubit z-measurement**

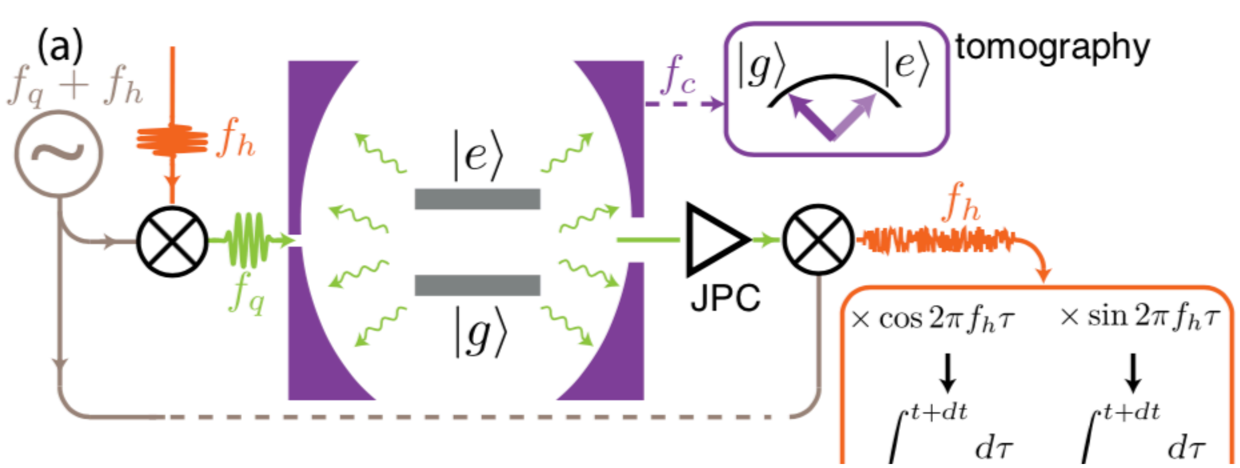


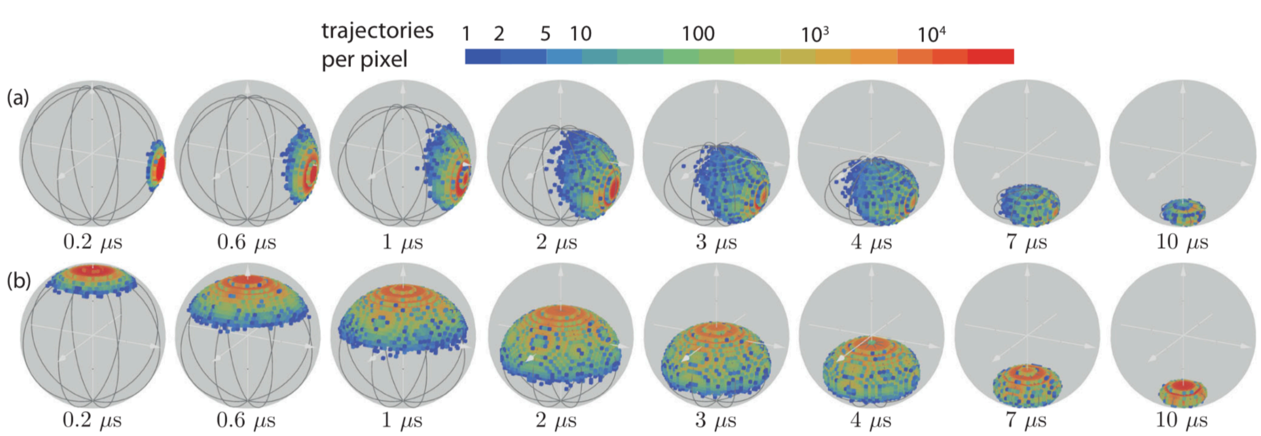
K. Murch et al., Nature **502**, 211 (2013)



S. J. Weber, ACH, et al., Nature **511**, 570 (2014)

- **Example: qubit fluorescence measurement**





Campagne-Ibarcq, Six, Bretheau, Sarlette, Mirrahimi, Rouchon, and Huard, **PRX 6**, 011002 (2016)

Quantum Trajectories in experiments

Common problems related to implementing theories to experimental data

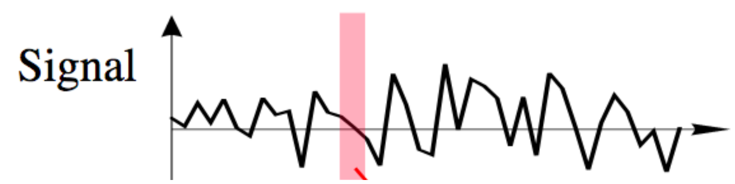
- Using SMEs requires that the time increment $\mathrm{d}t \to 0$ is infinitesimal.

Theory

Ito rule: $\mathrm{d}W^2 \approx \mathrm{d}t$

Equations with first order in $\mathrm{d}t$

Experiment



Time resolution: $\Delta t \gg \mathrm{d}t$

E.g., Cavity decay time is finite

Quantum states being unnormalized or not positive!

For qubits, a state can be outside of the Bloch sphere

- Markovian assumption

Strong Markovian

Non-Markovian processes

- White (no colored) noises

Wiener processes, Gaussian white noises

Colored noises

Can be ignored for now

Quantum Trajectories in experiments

SME • The stochastic master equation (SME) (Too much error using SMEs)

Measurement backaction can be described by the state update in the Kraus' form:

$$\rho_r(t+\Delta t) = \frac{\hat{K}(r)\rho(t)\hat{K}^\dagger(r)}{\mathrm{Tr}[\hat{K}(r)\rho(t)\hat{K}^\dagger(r)]}$$

$\hat{K}_{\mathrm{I}}(y_t)$ • Measurement operator: Conventional Itô approach

H. J. Carmichael (1993)

$\hat{K}_{\mathrm{R}}(y_t)$ • Measurement operator: Rouchon-Ralph approach

P. Rouchon, Annu. Rev. Control 54, 252 (2022).

P. Rouchon and J. F. Ralph, Phys. Rev. A 91, 012118 (2015).

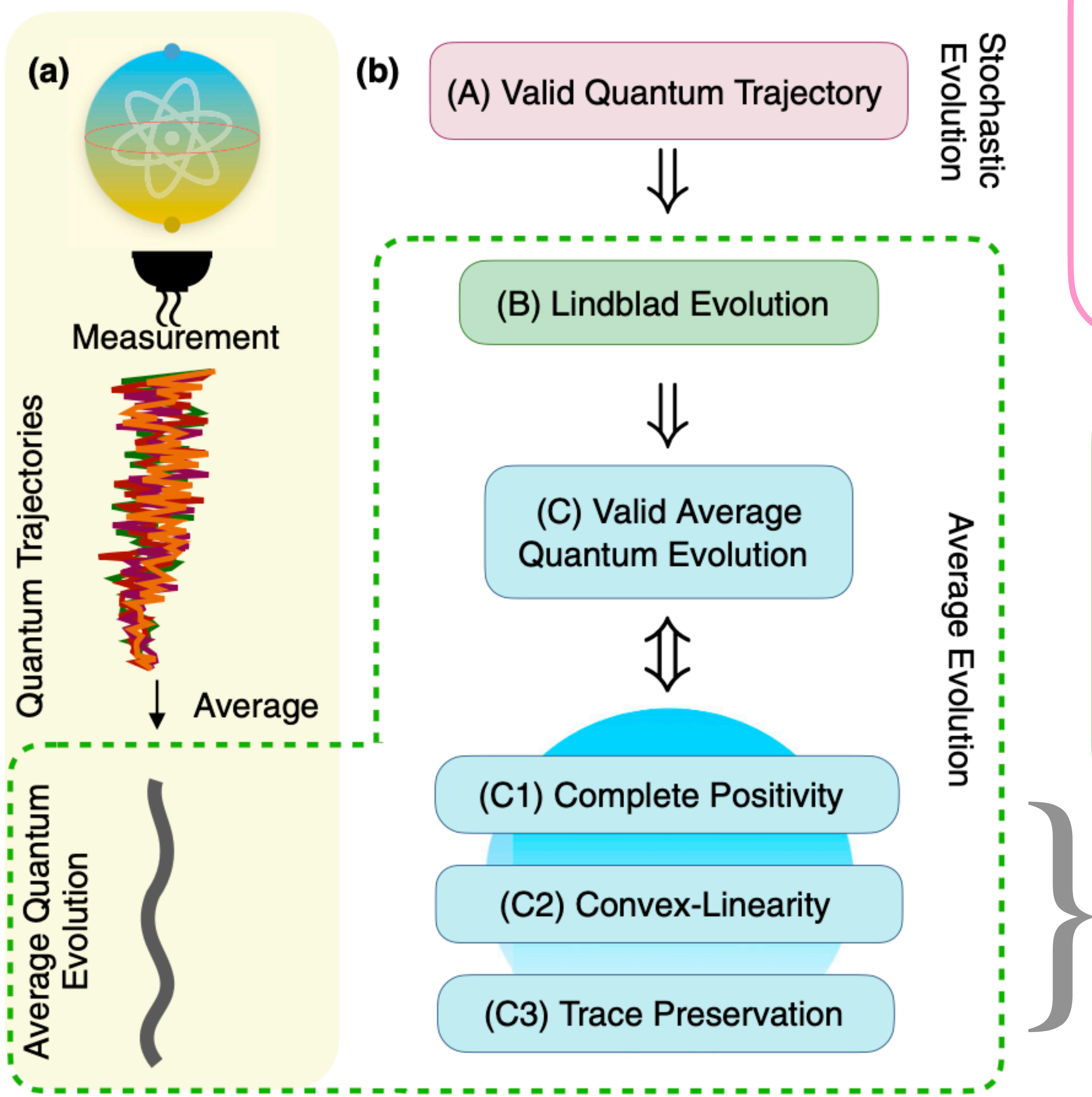
$\hat{K}_{\mathrm{G}}(y_t)$ • Measurement operator: Guevara-Wiseman approach

I. Guevara and H. M. Wiseman, Phys. Rev. A 102, 052217 (2020)

$\hat{K}_{\mathrm{W}}(y_t)$ • Measurement operator: “High-order” completely positive map

N. Wonglakhon, H. M. Wiseman and A. Chantasri, Phys. Rev. A 110, 062207 (2024)

Hierarchy criteria for Quantum Trajectories



Wonglakhon, Wiseman and ACH, PRA (2024)

(A) The strongest condition:

If the Δt-map:

$$\rho_r(t+\Delta t) = \frac{\hat{K}(r)\rho(t)\hat{K}^\dagger(r)}{\mathrm{Tr}[\hat{K}(r)\rho(t)\hat{K}^\dagger(r)]}$$

reproduces **exact/true** trajectories with $\mathrm{d}t$.

(B) A stronger condition:

$$\rho(t+\Delta t) = \int \mathrm{d}r\, \hat{K}(r)\rho(t)\hat{K}^\dagger(r) = e^{\Delta t \mathcal{L}\bullet}\rho(t),$$

where $\mathcal{L}\bullet = -i[\hat{H},\bullet] + \sum_{j=1}^{N} \mathcal{D}[\hat{c}_j]\bullet$

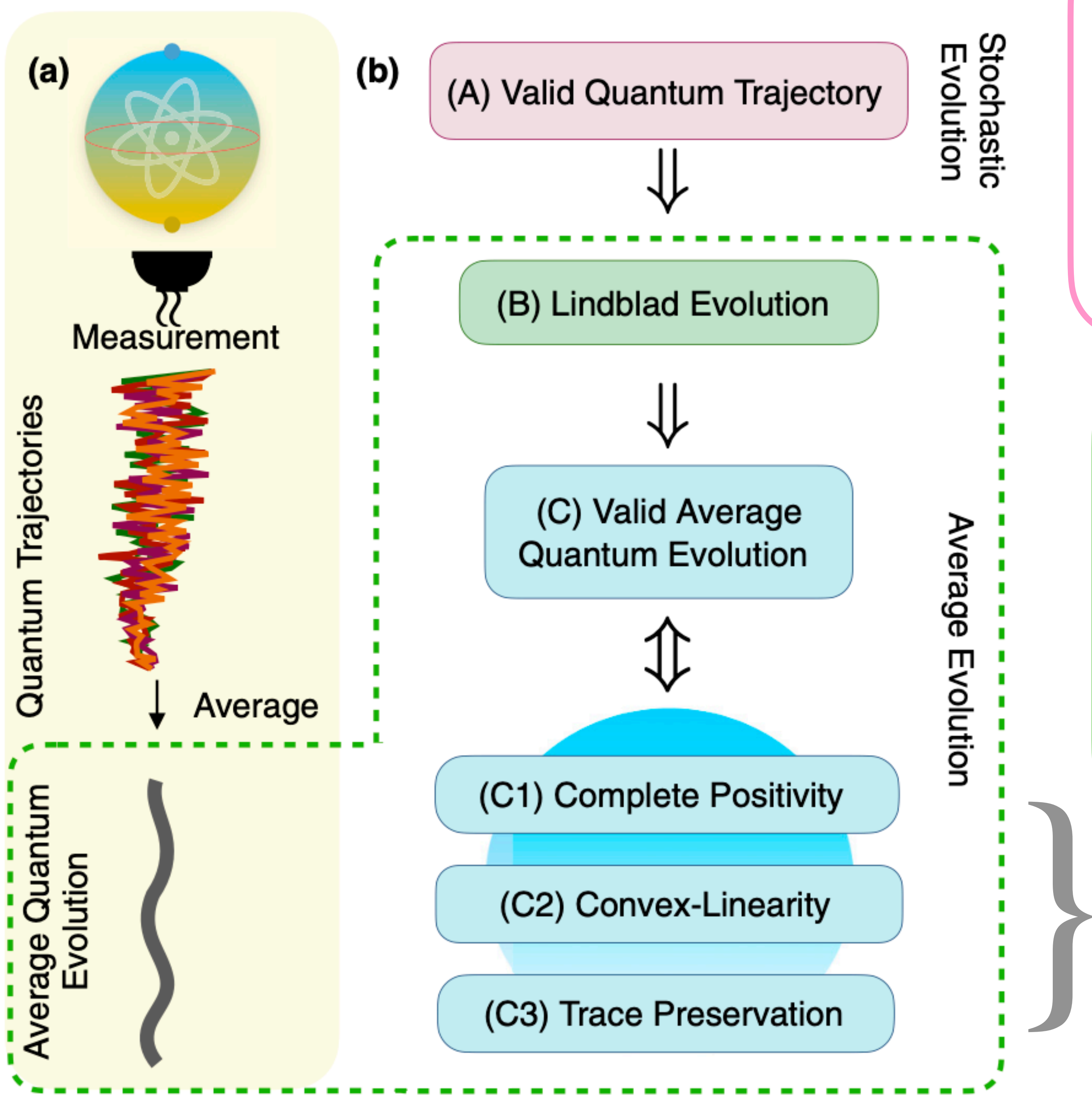
(C) A weaker condition:

Time resolution: $\Delta t \gg \mathrm{d}t$

SME $\quad \hat{K}_{\mathrm{I}}(y_t)$

$\hat{K}_{\mathrm{R}}(y_t)$ $\quad \hat{K}_{\mathrm{G}}(y_t)$ $\quad \hat{K}_{\mathrm{W}}(y_t)$

Hierarchy criteria for Quantum Trajectories



Wonglakhon, Wiseman and ACH, PRA (2024)

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(B) A stronger condition:

$$\rho(t+\Delta t) = \int \mathrm{d}r\, \hat{K}(r)\rho(t)\hat{K}^\dagger(r)$$

$$= e^{\Delta t \mathcal{L}\bullet}\rho(t), \qquad \hat{H} = 0$$

$$= (\hat{1} + \Delta t \mathcal{L} + \tfrac{1}{2}\Delta t^2 \mathcal{L}^2)\rho(t)$$

$$= \rho(t) + \Delta t \mathcal{D}[\hat{c}]\rho(t) + \tfrac{1}{2}\Delta t^2 \mathcal{D}^2[\hat{c}]\rho(t)$$

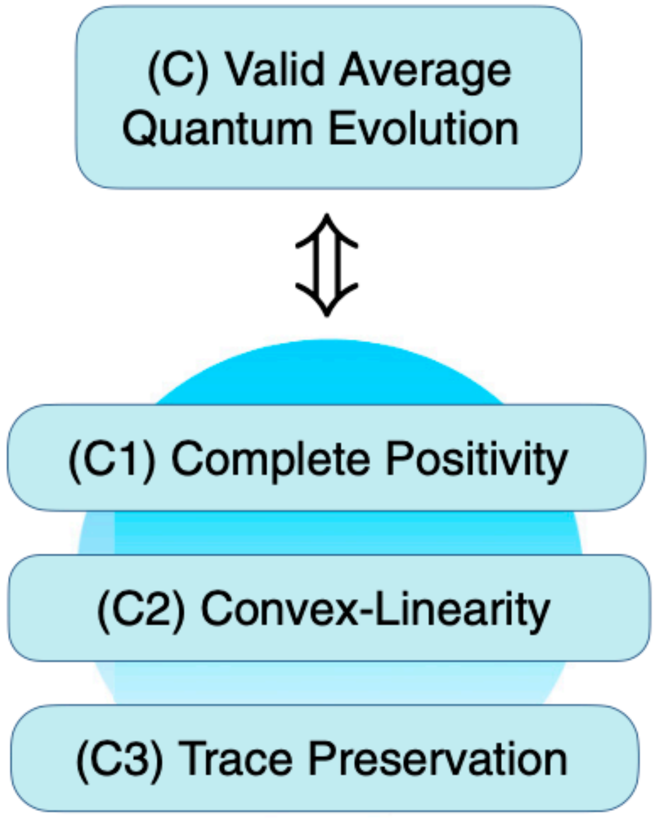
(C) A weaker condition:

Time resolution: $\Delta t \gg \mathrm{d}t$

$$\text{SME} \quad \hat{K}_{\mathrm{I}}(y_t)$$

$$\hat{K}_{\mathrm{R}}(y_t) \quad \hat{K}_{\mathrm{G}}(y_t) \quad \hat{K}_{\mathrm{W}}(y_t)$$

Hierarchy criteria: Valid Average Quantum Evolution



(C1) Complete Positivity

"A map is completely positive when its acting on part of a bipartite quantum state is positive, that is, it maps a positive state to a positive state."

(C2) Convex-Linearity

"A map is convex linear if a quantum state ρ is mapped to a quantum state which is convex linear in ρ, that is, a weighted mixture of two states is mapped to the same-weight mixture of the individually mapped states."

(C3) Trace Preservation

"A map is trace pre- serving if the trace of the mapped state is the same as that of the initial state."

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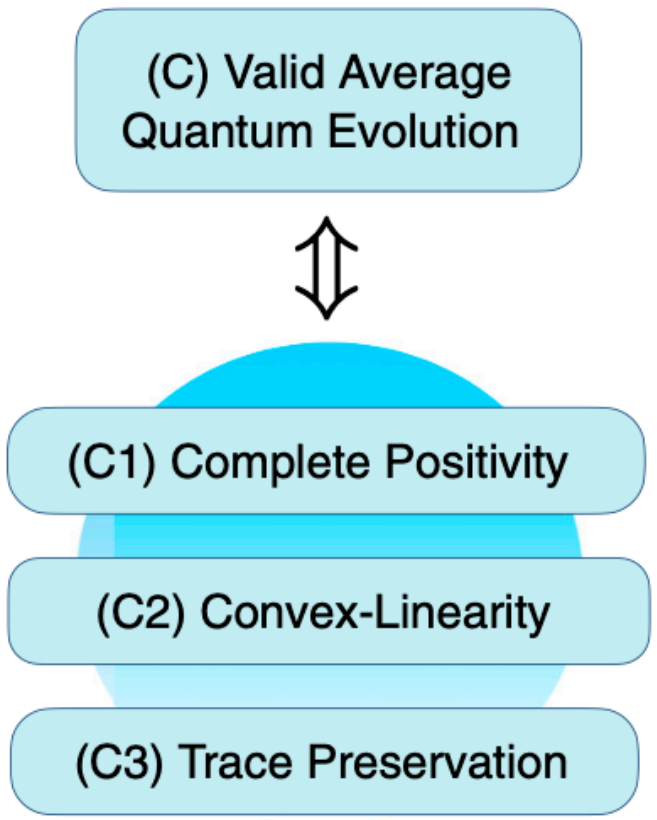
Hierarchy criteria: Valid Average Quantum Evolution



We first need to define how quantum trajectories are generated and averaged.

Method I

- Generate trajectories using **the readout PDF derived from Kraus operators**

$$\wp_{\mathrm{K}}(r|\rho(t)) = \mathrm{Tr}[\hat{K}(r)\rho(t)\hat{K}^{\dagger}(r)]$$

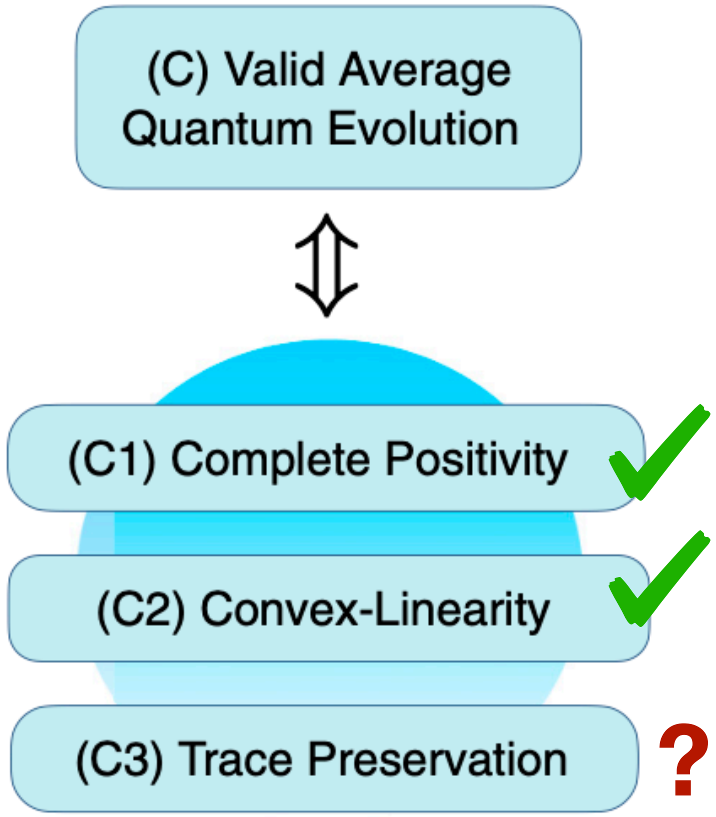
- Calculate average quantum state evolution:

$$\rho(t+\Delta t) = \int \mathrm{d}r\, \wp_{\mathrm{K}}(r|\rho(t)) \frac{\hat{K}(r)\rho(t)\hat{K}^{\dagger}(r)}{\mathrm{Tr}[\hat{K}(r)\rho(t)\hat{K}^{\dagger}(r)]}$$
$$= \int \mathrm{d}r \hat{K}(r)\rho(t)\hat{K}^{\dagger}(r),$$

The state mapping is in the Kraus' form $\Rightarrow$ (C1) Complete Positivity

The average is in the linear form $\Rightarrow$ (C2) Convex-Linearity

Hierarchy criteria: Valid Average Quantum Evolution



If and only if…

$$\int \mathrm{d}r\, \hat{K}^\dagger(r)\hat{K}(r) = \hat{1}$$

We first need to define how quantum trajectories are generated and averaged.

Method I

- Generate trajectories using **the readout PDF derived from Kraus operators**

$$\wp_{\mathrm{K}}(r|\rho(t)) = \mathrm{Tr}[\hat{K}(r)\rho(t)\hat{K}^\dagger(r)]$$

- Calculate average quantum state evolution:

$$\begin{aligned}\rho(t+\Delta t) &= \int \mathrm{d}r\, \wp_{\mathrm{K}}(r|\rho(t)) \frac{\hat{K}(r)\rho(t)\hat{K}^\dagger(r)}{\mathrm{Tr}[\hat{K}(r)\rho(t)\hat{K}^\dagger(r)]} \\ &= \int \mathrm{d}r \hat{K}(r)\rho(t)\hat{K}^\dagger(r),\end{aligned}$$

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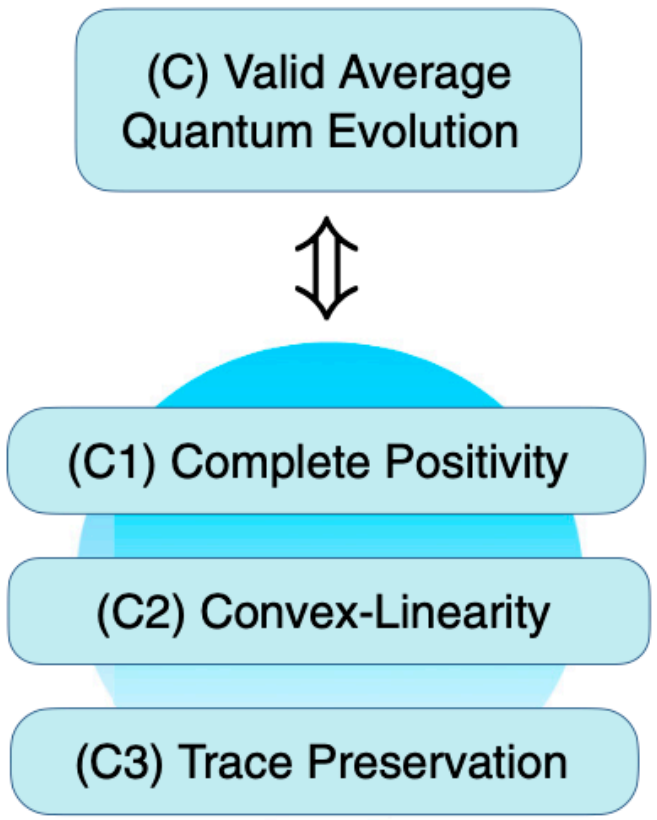
The average is in the linear form $\Rightarrow$ (C2) Convex-Linearity

Hierarchy criteria: Valid Average Quantum Evolution



We first need to define how quantum trajectories are generated and averaged.

<u>Method II</u>

- Generate trajectories using **simple guessed readout PDF (e.g., Gaussian distribution)**

  Guessed readout PDF: $\wp_{\mathrm{g}}(r|\rho(t))$

- Calculate average quantum state evolution:

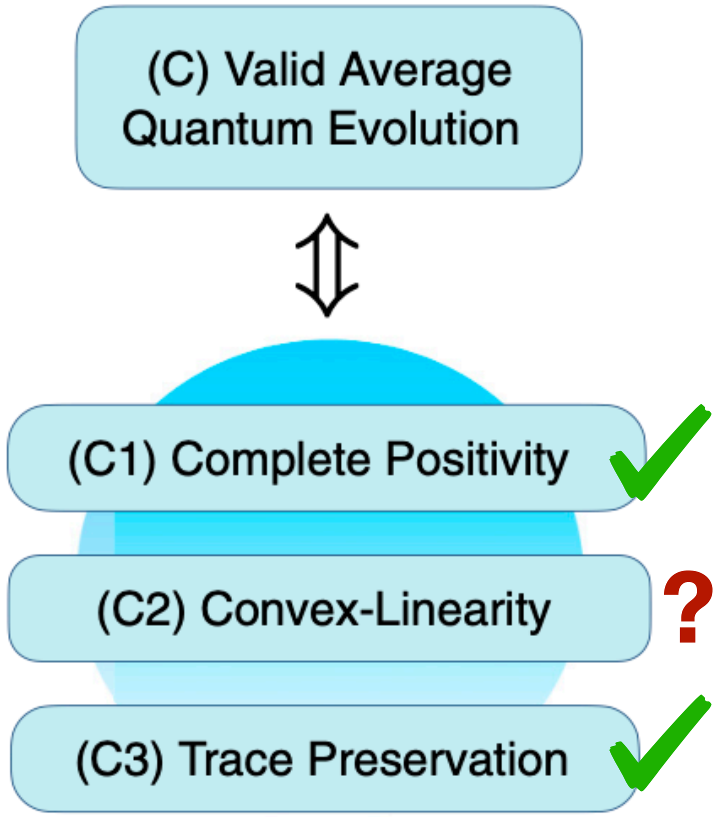
$$\rho(t+\Delta t) = \int \mathrm{d}r\, \wp_{\mathrm{g}}(r|\rho(t)) \frac{\hat{K}(r)\rho(t)\hat{K}^{\dagger}(r)}{\mathrm{Tr}[\hat{K}(r)\rho(t)\hat{K}^{\dagger}(r)]}$$

(Non-linear map)

The state mapping is in the Kraus' form $\Rightarrow$ (C1) Complete Positivity

The map is always normalized by its trace $\Rightarrow$ (C3) Trace Preservation

Hierarchy criteria: Valid Average Quantum Evolution



If and only if…

$$\wp_{\mathrm{g}}(r|\rho(t)) \propto \mathrm{Tr}[\hat{K}(r)\rho(t)\hat{K}^{\dagger}(r)]$$

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- Generate trajectories using **simple guessed readout PDF (e.g., Gaussian distribution)**

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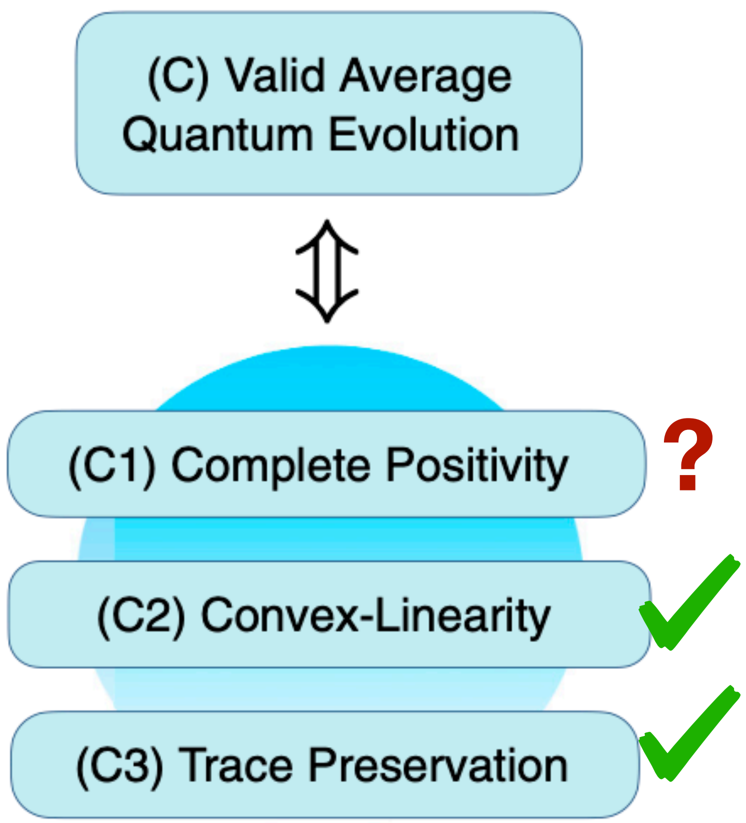


We first need to define how quantum trajectories are generated and averaged.

Method III

- Generate trajectories using **the Stochastic Master Equations (SMEs):**

$$\rho(t+\Delta t) = \rho(t) + \Delta t \mathcal{D}[\hat{c}]\rho(t) + \Delta W \mathcal{H}[\hat{c}]\rho(t)$$

- Calculate average quantum state evolution:

$$\rho(t+\Delta t) = \rho(t) + \Delta t \mathcal{D}[\hat{c}]\rho(t) + \Delta W \mathcal{H}[\hat{c}]\rho(t)$$

0

The state mapping is NOT in the Kraus' form $\overset{?}{\not\Rightarrow}$ (C1) Complete Positivity

The SME is linear in the state $\Rightarrow$ (C2) Convex-Linearity

The trace is preserved because $\mathrm{Tr}(\mathcal{D}[\hat{c}]\rho) = 0$ $\Rightarrow$ (C3) Trace Preservation

Testing Hierarchy criteria

We will test the hierarchy criteria for the following maps:

$\hat{K}_{\mathrm{I}}(y_t)$ • Measurement operator: Conventional Itô approach

$\hat{K}_{\mathrm{R}}(y_t)$ • Measurement operator: Rouchon-Ralph approach

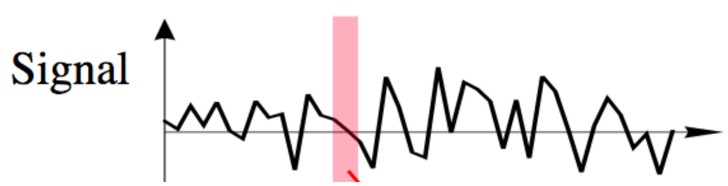
$\hat{K}_{\mathrm{G}}(y_t)$ • Measurement operator: Guevara-Wiseman approach

$\hat{K}_{\mathrm{W}}(y_t)$ • Measurement operator: “High-order” completely positive map

Considering the finite time resolution: $\Delta t \gg \mathrm{d}t$

The **coarse-grained** measurement record:

$$Y_t = \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathrm{d}s \, y_s$$



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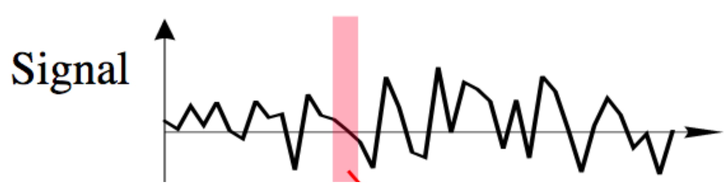


Method I $\int \mathrm{d}r\, \hat{K}^\dagger(r)\hat{K}(r) = \hat{1}$. **?**

(C3) Trace Preservation

Method II $\int \mathrm{d}r\, \wp_{\mathrm{g}}(r|\rho(t)) \frac{\hat{K}(r)\rho(t)\hat{K}^\dagger(r)}{\mathrm{Tr}[\hat{K}(r)\rho(t)\hat{K}^\dagger(r)]}$ **?**

(C2) Convex-Linearity

Testing Hierarchy criteria

$\hat{M}_\mathrm{I}(y_t)$ • Measurement operator: Conventional **Itô approach**

$$\hat{K}_\mathrm{I}(Y_t) = \sqrt{\wp_\mathrm{ost}(Y_t)}\hat{M}_\mathrm{I}(Y_t),$$

$$\hat{M}_\mathrm{I}(Y_t) = \hat{1} - \tfrac{1}{2}\hat{c}^\dagger\hat{c}\Delta t + \hat{c}Y_t\Delta t$$

with $\wp_\mathrm{ost}(Y_t) = \left(\dfrac{\Delta t}{2\pi}\right)^{1/2} \exp\left(-Y_t^2\Delta t/2\right)$

Method I

(C3) Trace Preservation

$$\int \mathrm{d}Y_t \wp_\mathrm{ost}(Y_t)\hat{M}_\mathrm{I}^\dagger(Y_t)\hat{M}_\mathrm{I}(Y_t) = \hat{1} + \frac{1}{4}(\hat{c}^\dagger\hat{c})^2\Delta t^2$$

Satisfies (C3) condition at: $\mathcal{O}(\Delta t)$

Method II

(C2) Convex-Linearity

$$\wp_\mathrm{I,g}(Y_t|\rho(t)) = \left(\frac{\Delta t}{2\pi}\right)^{1/2} \exp[-(Y_t - \langle\hat{c}+\hat{c}^\dagger\rangle)^2\Delta t/2].$$

$$\mu_\mathrm{I} = \langle\hat{c}+\hat{c}^\dagger\rangle + O(\Delta t)$$

$$\sigma_\mathrm{I}^2 = 1/\Delta t + O(\Delta t^0)$$

Mean and variance of Gaussian PDF

Satisfies (C2) condition at: $\mathcal{O}(\Delta t)$

Satisfies (C) condition at: $\mathcal{O}(\Delta t)$ $\Rightarrow$ Satisfies (B) condition at: $\mathcal{O}(\Delta t)$

Testing Hierarchy criteria

$\hat{M}_{\mathrm{R}}(y_t)$ • Measurement operator: **Rouchon-Ralph approach**

$$\hat{K}_{\mathrm{R}}(Y_t) = \sqrt{\wp_{\mathrm{ost}}(Y_t)}\hat{M}_{\mathrm{R}}(Y_t)$$

$$\hat{M}_{\mathrm{R}}(Y_t) = \hat{1} - \tfrac{1}{2}\hat{c}^{\dagger}\hat{c}\Delta t + \hat{c}Y_t\Delta t - \tfrac{1}{2}\hat{c}^2\left(\Delta t - Y_t^2\Delta t^2\right)$$

Method I

(C3) Trace Preservation

$$\int \mathrm{d}Y_t \wp_{\mathrm{ost}}(Y_t)\hat{M}_{\mathrm{R}}^{\dagger}(Y_t)\hat{M}_{\mathrm{R}}(Y_t)$$
$$= \hat{1} + \left[\frac{1}{4}(\hat{c}^{\dagger}\hat{c})^2 + \frac{1}{2}(\hat{c}^{\dagger})^2\hat{c}^2\right]\Delta t^2 + O(\Delta t^3).$$

Satisfies (C3) condition at: $\mathcal{O}(\Delta t)$

Method II

(C2) Convex-Linearity

Similar to the Ito case

Satisfies (C2) condition at: $\mathcal{O}(\Delta t)$

Satisfies (C) condition at: $\mathcal{O}(\Delta t)$ $\Rightarrow$ Satisfies (B) condition at: $\mathcal{O}(\Delta t)$

Testing Hierarchy criteria

$\hat{M}_{\rm G}(y_t)$ • Measurement operator: **Guevara-Wiseman approach**

$$\hat{K}_{\rm G}(Y_t) = \sqrt{\wp_{\rm ost}(Y_t)}\hat{M}_{\rm G}(Y_t).$$

$$\hat{M}_{\rm G}(Y_t) = \hat{1} + \left(Y_t\hat{c} - \tfrac{1}{2}\hat{c}^\dagger\hat{c}\right)\Delta t - \tfrac{1}{8}(\hat{c}^\dagger\hat{c})^2\Delta t^2$$

Method I

(C3) Trace Preservation

$$\int \mathrm{d}Y_t \wp_{\rm ost}(Y_t)\hat{M}_{\rm G}^\dagger(Y_t)\hat{M}_{\rm G}(Y_t) = \hat{1} + O(\Delta t^3)$$

Satisfies (C3) condition at: $\mathcal{O}(\Delta t^2)$

Method II

(C2) Convex-Linearity

$$\mu_{\rm G} = \langle \hat{c} + \hat{c}^\dagger \rangle - \frac{1}{2}\langle \hat{c}^\dagger\hat{c}^2 + \hat{c}(\hat{c}^\dagger)^2\rangle\Delta t + O(\Delta t^2)$$

$$\sigma_{\rm G}^2 = \frac{1}{\Delta t} + [2\langle \hat{c}^\dagger\hat{c}\rangle - \langle \hat{c}^\dagger + \hat{c}\rangle^2] + O(\Delta t)$$

These came from: $\mathrm{Tr}[\wp_{\rm ost}(Y_t)\hat{M}_{\rm G}(Y_t)\rho(t)\hat{M}_{\rm G}^\dagger(Y_t)]$

Guevara and Wiseman, PRA **102**, 052217 (2020)

Satisfies (C2) condition at: $\mathcal{O}(\Delta t^2)$

Satisfies (C) condition at: $\mathcal{O}(\Delta t^2)$ BUT Satisfies (B) condition at: $\mathcal{O}(\Delta t)$

Proposed High-order Completely Positive map

$\hat{M}_{\mathrm{W}}(y_t)$ • Measurement operator: **“High-order” completely positive map**

N. Wonglakhon, H. M. Wiseman and A. Chantasri, Phys. Rev. A 110, 062207 (2024)

- A physical model of a quantum system coupled to a Markovian Bosonic field and the bath’s state is then observed via a homodyne measurement:

$$\hat{K}(y_s) = \langle y_s|\, \hat{U}_{t+\mathrm{d}t,t}\, |0\rangle = \langle y_s|\exp(\hat{c}\mathrm{d}\hat{B}_t^\dagger - \hat{c}^\dagger \mathrm{d}\hat{B}_t)|0\rangle$$

Homodyne measurement: $\hat{Q}_t = \mathrm{d}\hat{B}_t + \mathrm{d}\hat{B}_t^\dagger$ with $\hat{Q}_s|y_s\rangle = y_s|y_s\rangle$ and $[\mathrm{d}\hat{B}_t, \mathrm{d}\hat{B}_t^\dagger] = \mathrm{d}t$

This gives:
$$\begin{aligned}\hat{K}(y_s) &= \langle y_s|\hat{1} + \hat{c}\mathrm{d}\hat{B}_t^\dagger - \tfrac{1}{2}\hat{c}^\dagger\hat{c}\mathrm{d}\hat{B}_t\mathrm{d}\hat{B}_t^\dagger + \tfrac{1}{2}\hat{c}^2(\mathrm{d}\hat{B}_t^\dagger)^2 \\ &\quad + O(|\mathrm{d}\hat{B}_t|^3)|0\rangle \\ &= \langle y_s|0\rangle\big[\hat{1} - \tfrac{1}{2}\hat{c}^\dagger\hat{c}\mathrm{d}t + \hat{c}y_s\mathrm{d}t + \tfrac{1}{2}\hat{c}^2\big(y_s^2\mathrm{d}t^2 - \mathrm{d}t\big) \\ &\quad + O(\mathrm{d}t^{3/2})\big]\end{aligned}$$

Some interesting features here!

where $\langle y_s|n\rangle = \dfrac{(\alpha/\pi)^{1/4}}{\sqrt{2^n n!}}\exp\big(-\alpha y_s^2/2\big)H_n(\sqrt{\alpha}y_s)$ is the wave function of a harmonic oscillator in terms of the Hermite polynomials.

Proposed High-order Completely Positive map

$\hat{M}_{\rm W}(y_t)$ • Measurement operator: **"High-order" completely positive map**

N. Wonglakhon, H. M. Wiseman and A. Chantasri, Phys. Rev. A 110, 062207 (2024)

- We construct a high-order measurement operator for the coarse-grained record with the time resolution $\Delta t \gg {\rm d}t$:

$$\hat{M}_{\rm W}(Y_t) \approx \lim_{m\to\infty} \hat{M}_{{\rm d}t}(y_{t+(m-1){\rm d}t})\cdots\hat{M}_{{\rm d}t}(y_{t+{\rm d}t})\hat{M}_{{\rm d}t}(y_t) \quad \text{when} \quad m\,{\rm d}t = \Delta t$$

and expand terms to Δt^2 to obtain:

$$\hat{M}_{\rm W}(Y_t) = \hat{1} - \tfrac{1}{2}(\hat{c}^2 + \hat{c}^\dagger\hat{c})\Delta t + \tfrac{1}{8}(\hat{c}^\dagger\hat{c})^2\Delta t^2 + \left[\hat{c}\Delta t - \tfrac{1}{4}(\hat{c}^\dagger\hat{c}^2 + \hat{c}\hat{c}^\dagger\hat{c})\Delta t^2\right]Y_t + \tfrac{1}{2}\hat{c}^2\Delta t^2 Y_t^2$$

This gives the Kraus operator: $\hat{K}_{\rm W}(Y_t) = \sqrt{\wp_{\rm ost}(Y_t)}\hat{M}_{\rm W}(Y_t)$

with the ostensible probability: $\wp_{\rm ost}(Y_t) = \left(\dfrac{\Delta t}{2\pi}\right)^{1/2}\exp\left(-Y_t^2\Delta t/2\right)$ coming from an infinite product of the vacuum state wavefunctions.

Proposed High-order Completely Positive map

$\hat{M}_{\mathrm{W}}(y_t)$ • Measurement operator: **"High-order" completely positive map**

N. Wonglakhon, H. M. Wiseman and A. Chantasri, Phys. Rev. A 110, 062207 (2024)

$$\hat{K}_{\mathrm{W}}(Y_t) = \sqrt{\wp_{\mathrm{ost}}(Y_t)}\hat{M}_{\mathrm{W}}(Y_t)$$

$$\hat{M}_{\mathrm{W}}(Y_t) = \hat{1} - \tfrac{1}{2}(\hat{c}^2 + \hat{c}^\dagger\hat{c})\Delta t + \tfrac{1}{8}(\hat{c}^\dagger\hat{c})^2\Delta t^2 + \left[\hat{c}\Delta t - \tfrac{1}{4}(\hat{c}^\dagger\hat{c}^2 + \hat{c}\hat{c}^\dagger\hat{c})\Delta t^2\right]Y_t + \tfrac{1}{2}\hat{c}^2\Delta t^2 Y_t^2$$

Method I

(C3) Trace Preservation

$$\int \mathrm{d}Y_t \wp_{\mathrm{ost}}(Y_t)\hat{M}_{\mathrm{W}}^\dagger(Y_t)\hat{M}_{\mathrm{W}}(Y_t) = \hat{1} + O(\Delta t^3)$$

Satisfies (C3) condition at: $\mathcal{O}(\Delta t^2)$

Method II

(C2) Convex-Linearity

$$\rho(t+\Delta t) = \rho(t) + \mathcal{D}[\hat{c}]\rho(t)\Delta t + \tfrac{1}{2}\mathcal{D}^2[\hat{c}]\rho(t)\Delta t^2 + O(\Delta t^3),$$

Satisfies (C2) condition at: $\mathcal{O}(\Delta t^2)$

Bonus!

Satisfies (C) condition at: $\mathcal{O}(\Delta t^2)$ AND Satisfies (B) condition at: $\mathcal{O}(\Delta t^2)$

Testing Hierarchy criteria

- $\hat{K}_{\mathrm{I}}(y_t)$ • Measurement operator: Conventional Itô approach
- $\hat{K}_{\mathrm{R}}(y_t)$ • Measurement operator: Rouchon-Ralph approach
- $\hat{K}_{\mathrm{G}}(y_t)$ • Measurement operator: Guevara-Wiseman approach
- $\hat{K}_{\mathrm{W}}(y_t)$ • Measurement operator: "High-order" completely positive map

Condition	$\hat{K}_{\mathrm{I}}$	$\hat{K}_{\mathrm{R}}$	$\hat{K}_{\mathrm{G}}$	$\hat{K}_{\mathrm{W}}$
(A) VQT (two examples)	$O(\Delta t)$	$O(\Delta t)$	$O(\Delta t)$	$O(\Delta t)$
(B) Lindblad solution	$O(\Delta t)$	$O(\Delta t)$	$O(\Delta t)$	$O(\Delta t^2)$
(C) VAQE (methods I & II)	$O(\Delta t)$	$O(\Delta t)$	$O(\Delta t^2)$	$O(\Delta t^2)$

N. Wonglakhon, H. M. Wiseman and A. Chantasri, Phys. Rev. A 110, 062207 (2024)

Valid Quantum Trajectory criterion

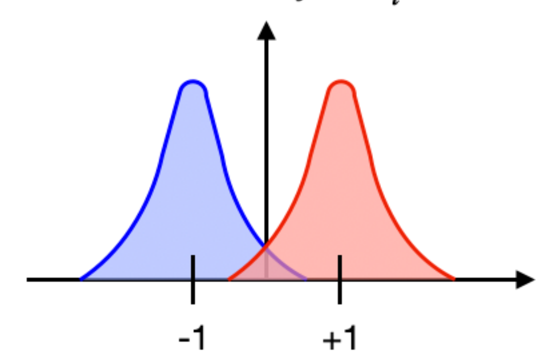
Using two qubit examples to show the map's accuracy at the quantum trajectory level.

- **Example: qubit z-measurement (Exact)**

(Or any measurements with Hermitian Lindblad operators)

$$\hat{K}_{\mathrm{H}}(Y_t) = \sum_j \sqrt{\wp(Y_t|a_j)}|a_j\rangle\langle a_j|$$

where $\wp(Y_t|a_j) = \left(\frac{\gamma\Delta t}{\pi}\right)^{1/2} \exp\left[-\frac{\gamma}{2}\left(\frac{Y_t}{\sqrt{2\gamma}} - a_j\right)^2 \Delta t\right]$



Qubit's z-measurement

- **Example: qubit fluorescence measurement (Nearly exact)**

$$\hat{M}_{\mathrm{F,ex}}(X_t) = \begin{pmatrix} e^{-\gamma\Delta t/2} & 0 \\ \sqrt{\gamma}X_t & 1 \end{pmatrix}$$

where $X_t \equiv \int_t^{t+\Delta t} e^{-\gamma s/2} y_s \mathrm{d}s \approx \left[1 - \frac{\gamma}{2}\left(t + \frac{\Delta t}{2}\right)\right] Y_t \Delta t - \frac{\gamma}{2} Z_t$

Approximation to Δt^2

Estimated by two types of coarse-grained records

$$Y_t = \frac{1}{\Delta t}\int_t^{t+\Delta t} \mathrm{d}s\, y_s$$

$$Z_t \equiv \int_t^{t+\Delta t} \left[s - \left(t + \frac{\Delta t}{2}\right)\right] y_s \mathrm{d}s$$

Valid Quantum Trajectory criterion

Using the exact maps, we can now compute the **average trace distance** for our maps.

Coarse-grained trajectories

$$\rho_{\rm A}(Y_t, \tilde{\rho}) = \frac{\hat{K}_{\rm A}(Y_t)\tilde{\rho}\hat{K}_{\rm A}^\dagger(Y_t)}{{\rm Tr}[\hat{K}_{\rm A}(Y_t)\tilde{\rho}\hat{K}_{\rm A}^\dagger(Y_t)]}$$

Exact quantum trajectories

$$\rho_{\rm ex}(\tilde{Y}_t, \tilde{\rho}) = \frac{\hat{K}_{\rm ex}(\tilde{Y}_t)\tilde{\rho}\hat{K}_{\rm ex}^\dagger(\tilde{Y}_t)}{{\rm Tr}[\hat{K}_{\rm ex}(\tilde{Y}_t)\tilde{\rho}\hat{K}_{\rm ex}^\dagger(\tilde{Y}_t)]}$$

$$D_{\rm A} = \frac{1}{2}\int {\rm d}\mu_{\rm H}(\tilde{\rho})\int {\rm d}\tilde{Y}_t \wp_{\rm ex}(\tilde{Y}_t|\tilde{\rho}){\rm Tr}|\rho_{\rm A} - \rho_{\rm ex}|$$

For Q fluorescence

$$\tilde{Y}_t = \{Y_t, Z_t\}$$

The Kraus operators: $\hat{K}_{\rm A}(Y_t) \in \left\{\hat{K}_{\rm I}(y_t),\ \hat{K}_{\rm R}(y_t),\ \hat{K}_{\rm G}(y_t),\ \hat{K}_{\rm W}(y_t)\right\}$

	Method for Qubit measurement examples	
Kraus operator	Average trace distance compared to $\hat{K}_{\rm H}(Y_t)$	Average trace distance compared to $\hat{K}_{\rm F}(Y_t, Z_t)$
$\hat{K}_{\rm I}$	$\frac{7\sqrt{\pi}}{48}(\gamma\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.2585(\gamma\Delta t)^{3/2}$	$\frac{1}{2\sqrt{6\pi}}(\gamma\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.1152(\gamma\Delta t)^{3/2}$
$\hat{K}_{\rm R}$	$\frac{(1+e^3)\sqrt{\pi}}{12e^3}(\gamma\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.1551(\gamma\Delta t)^{3/2}$	$\frac{1}{2\sqrt{6\pi}}(\gamma\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.1152(\gamma\Delta t)^{3/2}$
$\hat{K}_{\rm G}$	$\frac{7\sqrt{\pi}}{48}(\gamma\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.2585(\gamma\Delta t)^{3/2}$	$\frac{1}{2\sqrt{6\pi}}(\gamma\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.1152(\gamma\Delta t)^{3/2}$
$\hat{K}_{\rm W}$	$\frac{(4+e^{3/2})\sqrt{\pi}}{48e^{3/2}}(\gamma\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.0699(\gamma\Delta t)^{3/2}$	$\frac{1}{4\sqrt{6\pi}}(\gamma\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.0576(\gamma\Delta t)^{3/2}$

Wonglakhon, Wiseman and ACH, PRA 110, 062207 (2024)

Valid Quantum Trajectory criterion

Using the exact maps, we can now compute the **average trace distance** for our maps.

Coarse-grained trajectories

$$\rho_{\mathrm{A}}(Y_t, \tilde{\rho}) = \frac{\hat{K}_{\mathrm{A}}(Y_t)\tilde{\rho}\hat{K}_{\mathrm{A}}^{\dagger}(Y_t)}{\mathrm{Tr}[\hat{K}_{\mathrm{A}}(Y_t)\tilde{\rho}\hat{K}_{\mathrm{A}}^{\dagger}(Y_t)]}$$

Exact quantum trajectories

$$\rho_{\mathrm{ex}}(\tilde{Y}_t, \tilde{\rho}) = \frac{\hat{K}_{\mathrm{ex}}(\tilde{Y}_t)\tilde{\rho}\hat{K}_{\mathrm{ex}}^{\dagger}(\tilde{Y}_t)}{\mathrm{Tr}[\hat{K}_{\mathrm{ex}}(\tilde{Y}_t)\tilde{\rho}\hat{K}_{\mathrm{ex}}^{\dagger}(\tilde{Y}_t)]}$$

$$D_{\mathrm{A}} = \frac{1}{2}\int \mathrm{d}\mu_{\mathrm{H}}(\tilde{\rho}) \int \mathrm{d}\tilde{Y}_t \wp_{\mathrm{ex}}(\tilde{Y}_t|\tilde{\rho})\mathrm{Tr}|\rho_{\mathrm{A}} - \rho_{\mathrm{ex}}|$$

For Q fluorescence

$$\tilde{Y}_t = \{Y_t, Z_t\}$$

The Kraus operators: $\hat{K}_{\mathrm{A}}(Y_t) \in \left\{ \hat{K}_{\mathrm{I}}(y_t), \hat{K}_{\mathrm{R}}(y_t), \hat{K}_{\mathrm{G}}(y_t), \hat{K}_{\mathrm{W}}(y_t) \right\}$

	Method for Qubit measurement examples	
Kraus operator	Average trace distance compared to $\hat{K}_{\mathrm{H}}(Y_t)$	Average trace distance compared to $\hat{K}_{\mathrm{F}}(Y_t, Z_t)$
$\hat{K}_{\mathrm{I}}$	$\frac{7\sqrt{\pi}}{48}(\gamma\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.2585(\gamma\Delta t)^{3/2}$	$\frac{1}{2\sqrt{6\pi}}(\gamma\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.1152(\gamma\Delta t)^{3/2}$
$\hat{K}_{\mathrm{R}}$	$\frac{(1+e^3)\sqrt{\pi}}{12e^3}(\gamma\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.1551(\gamma\Delta t)^{3/2}$	$\frac{1}{2\sqrt{6\pi}}(\gamma\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.1152(\gamma\Delta t)^{3/2}$
$\hat{K}_{\mathrm{G}}$	$\frac{7\sqrt{\pi}}{48}(\gamma\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.2585(\gamma\Delta t)^{3/2}$	$\frac{1}{2\sqrt{6\pi}}(\gamma\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.1152(\gamma\Delta t)^{3/2}$
$\hat{K}_{\mathrm{W}}$	$\frac{(4+e^{3/2})\sqrt{\pi}}{48e^{3/2}}(\gamma\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.0699(\gamma\Delta t)^{3/2}$	$\frac{1}{4\sqrt{6\pi}}(\gamma\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.0576(\gamma\Delta t)^{3/2}$

Wonglakhon, Wiseman and ACH, PRA 110, 062207 (2024)

Valid Quantum Trajectory: Numerical results

Two Examples: $\mathrm{d}t = 0.0001, \Delta t = 0.01$

- Qubit z-basis measurement

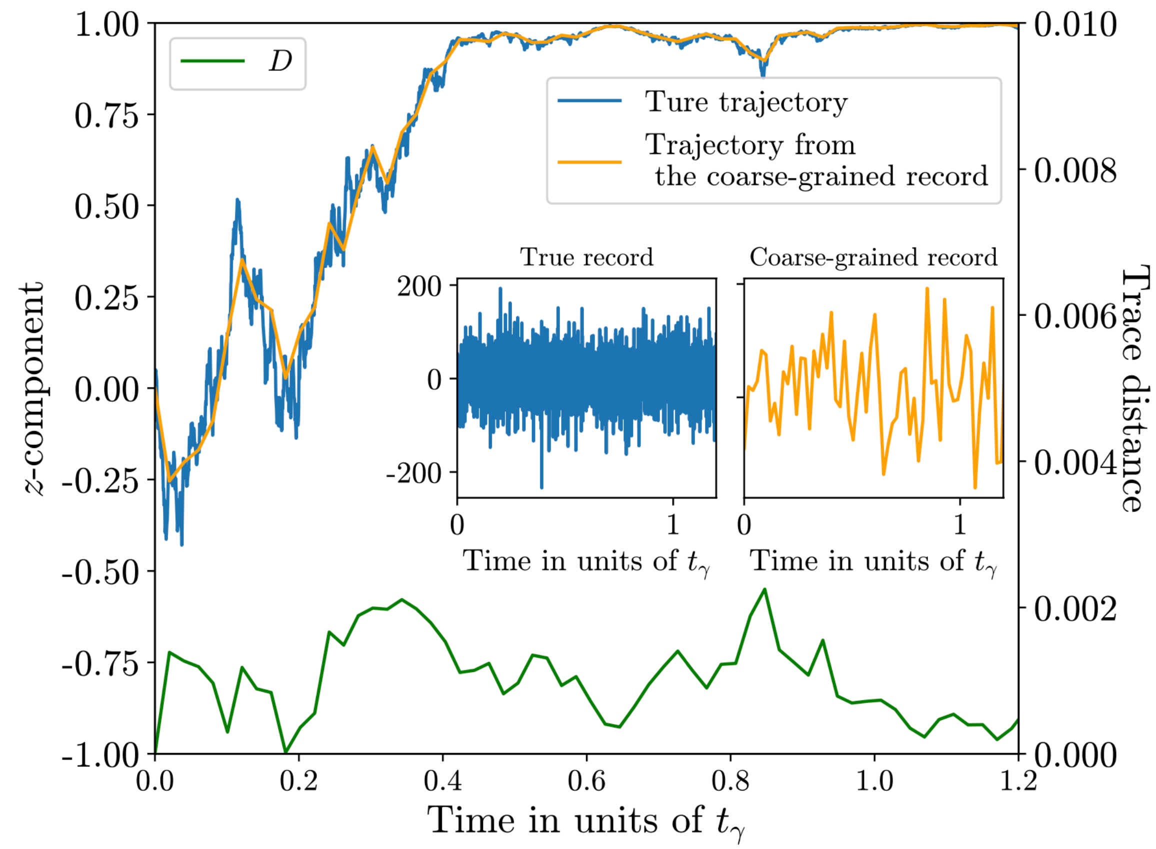
  $\Omega = 0, \eta = 1$

  $\Omega = 2\pi, \eta = 0.4$

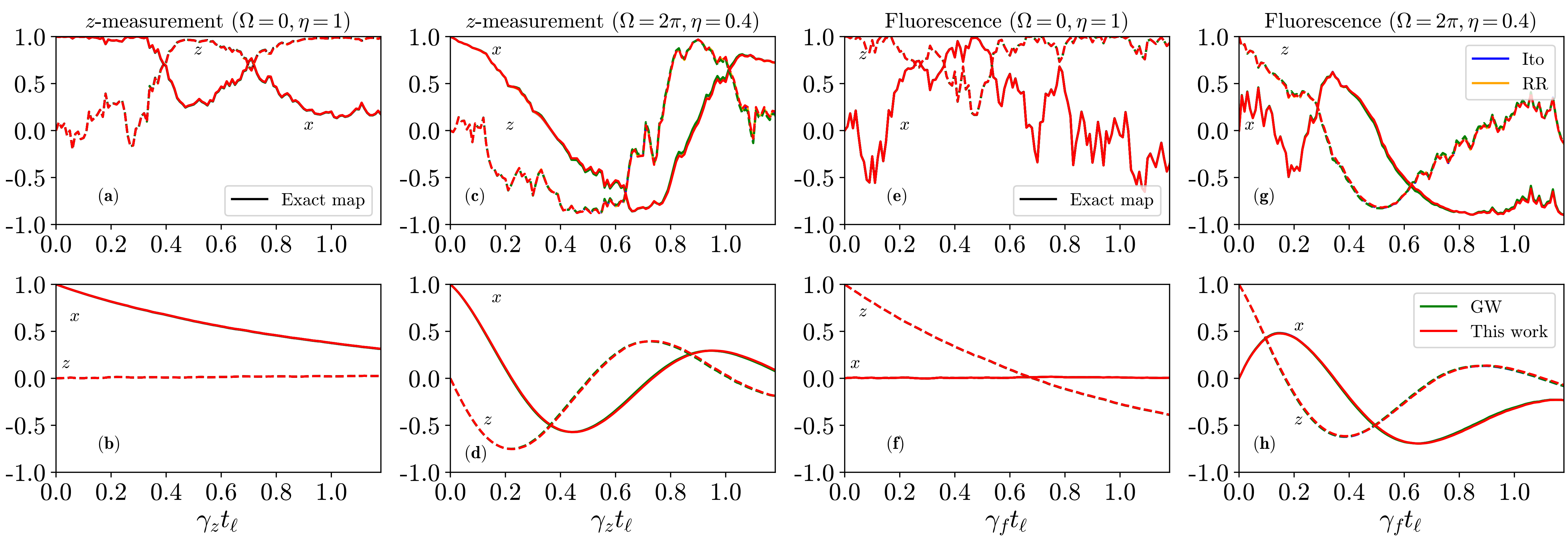
- Qubit Fluorescence measurement

  $\Omega = 0, \eta = 1$

  $\Omega = 2\pi, \eta = 0.4$



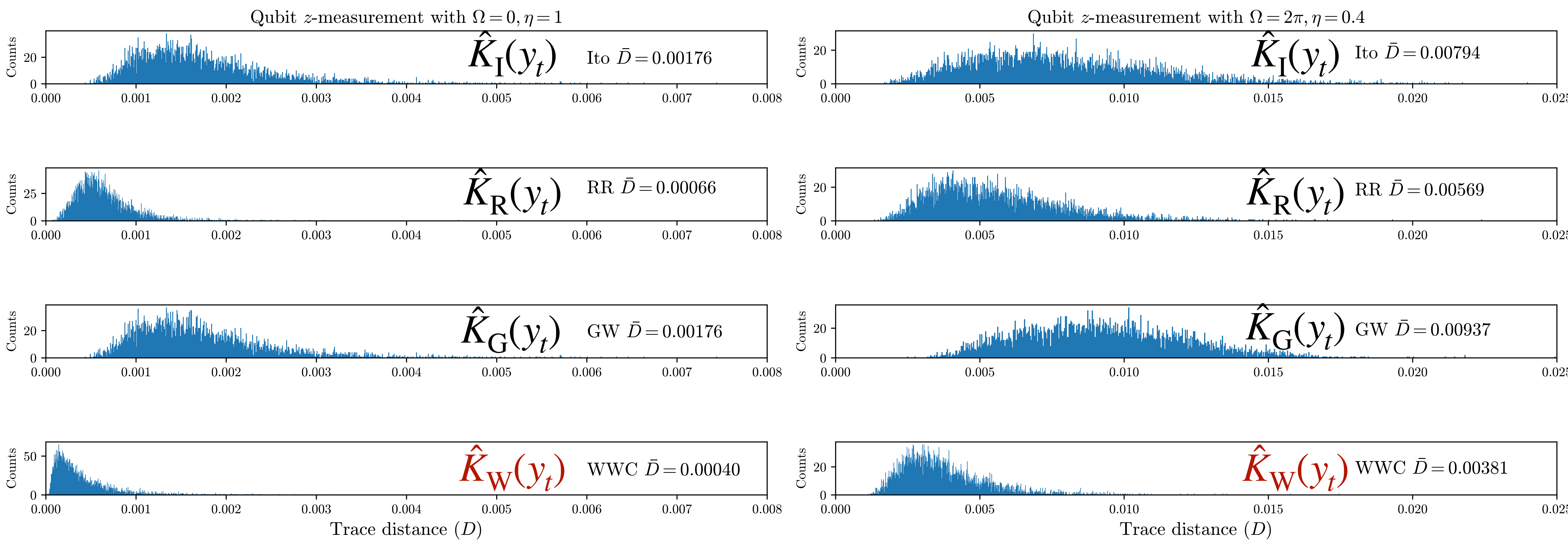
Wonglakhon, Wiseman and ACH, (in preparation)



Valid Quantum Trajectory: Numerical results

The average trace distances for the Qubit z-basis measurement

$\Omega = 0, \eta = 1$ $\Omega = 2\pi, \eta = 0.4$



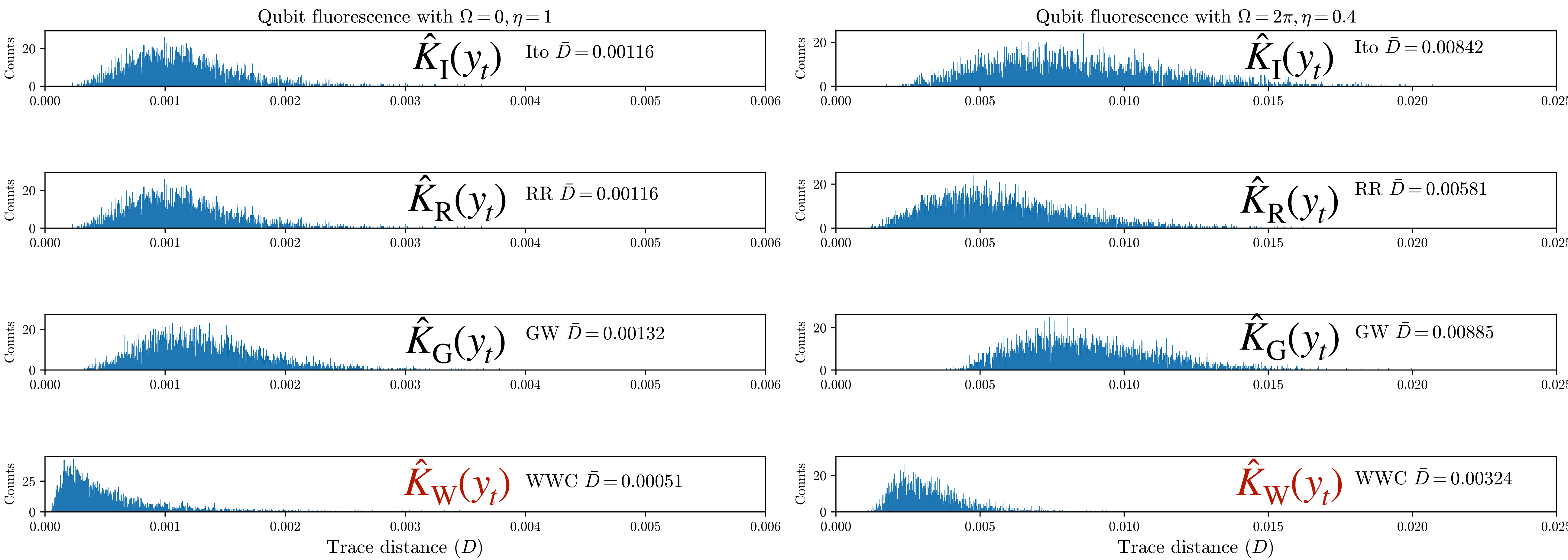
Summary of the order of accuracy for the three hierarchy conditions (A), (B), and (C)

Kraus operator	Average trace distance compared to $\hat{K}_{\mathrm{H}}(Y_t)$
$\hat{K}_{\mathrm{I}}$	$\frac{7\sqrt{\pi}}{48}(\gamma\,\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.2585(\gamma\,\Delta t)^{3/2}$
$\hat{K}_{\mathrm{R}}$	$\frac{(1+e^3)\sqrt{\pi}}{12e^3}(\gamma\,\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.1551(\gamma\,\Delta t)^{3/2}$
$\hat{K}_{\mathrm{G}}$	$\frac{7\sqrt{\pi}}{48}(\gamma\,\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.2585(\gamma\,\Delta t)^{3/2}$
$\hat{K}_{\mathrm{W}}$	$\frac{(4+e^{3/2})\sqrt{\pi}}{48e^{3/2}}(\gamma\,\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.0699(\gamma\,\Delta t)^{3/2}$

Valid Quantum Trajectory: Numerical results

The average trace distances for the Qubit Fluorescence measurement

$\Omega = 0, \eta = 1$ $\Omega = 2\pi, \eta = 0.4$



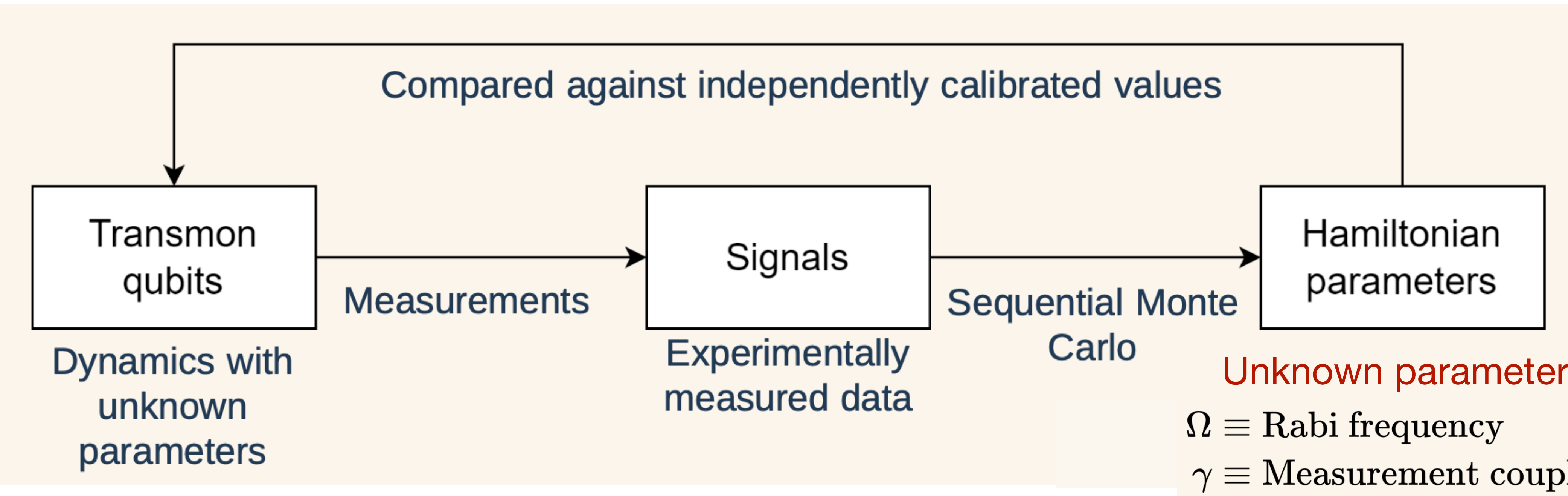
Summary of the order of accuracy for the three hierarchy conditions (A), (B), and (C)

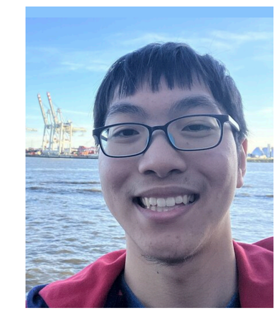
Kraus operator	Average trace distance compared to $\hat{K}_{\mathrm{F}}(Y_t, Z_t)$
$\hat{K}_{\mathrm{I}}$	$\frac{1}{2\sqrt{6\pi}}(\gamma\,\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.1152(\gamma\,\Delta t)^{3/2}$
$\hat{K}_{\mathrm{R}}$	$\frac{1}{2\sqrt{6\pi}}(\gamma\,\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.1152(\gamma\,\Delta t)^{3/2}$
$\hat{K}_{\mathrm{G}}$	$\frac{1}{2\sqrt{6\pi}}(\gamma\,\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.1152(\gamma\,\Delta t)^{3/2}$
$\hat{K}_{\mathrm{W}}$	$\frac{1}{4\sqrt{6\pi}}(\gamma\,\Delta t)^{3/2} + O(\Delta t^{5/2}) \approx 0.0576(\gamma\,\Delta t)^{3/2}$

Other applications of the high-order map

Parameter calibration for Transmon qubit experiments

C. Manoworakul, J. Ralph, H. Wiseman, and ACH, (in preparation)





Chattamas Manoworakul (Mahidol University)

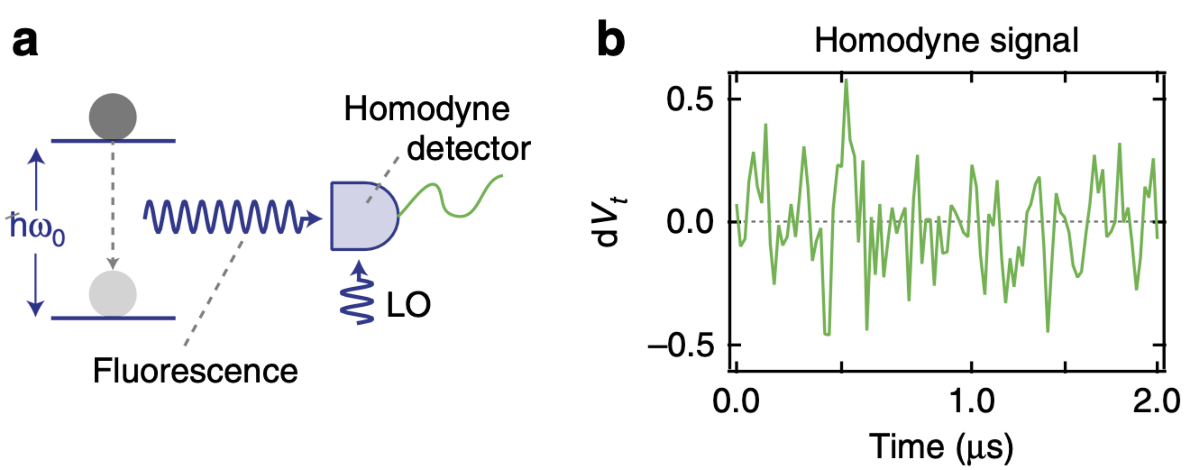
Unknown parameters

$\Omega \equiv$ Rabi frequency

$\gamma \equiv$ Measurement coupling rate

$\eta \equiv$ Measurement efficiency

Using signal data from K. Murch's group:



Naghiloo, Foroozani, Tan, Jadbabaie, Murch, Nat. Commun. 7, 11527 (2016)

Processing data using:

$$\rho_r(t+\Delta t) = \frac{\hat{K}(r)\rho(t)\hat{K}^\dagger(r)}{\mathrm{Tr}[\hat{K}(r)\rho(t)\hat{K}^\dagger(r)]}$$

with $\hat{K}_{\mathrm{W}}(Y_t) = \sqrt{\wp_{\mathrm{ost}}(Y_t)}\hat{M}_{\mathrm{W}}(Y_t)$

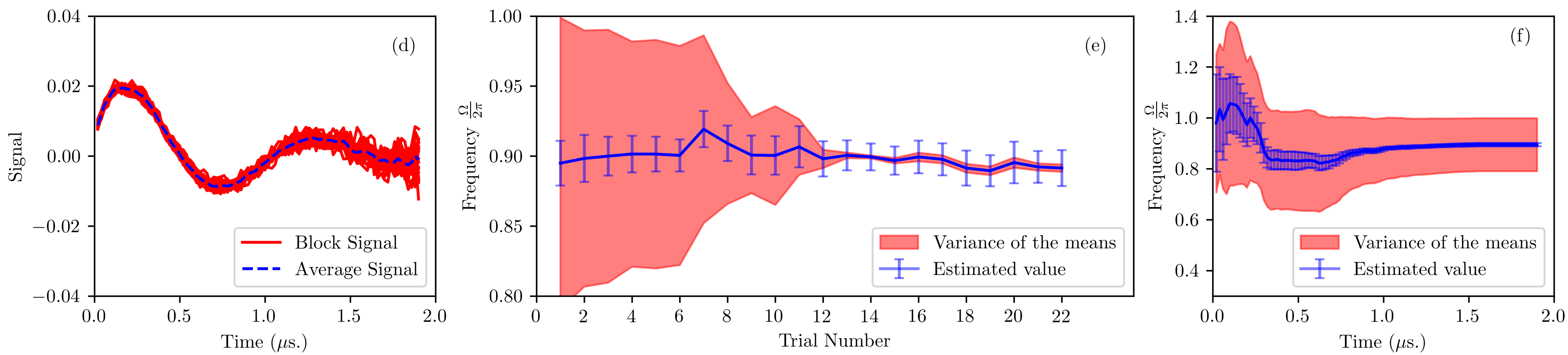
and finding Bayesian estimators using The Sequential Monte Carlo technique.

Other applications of the high-order map

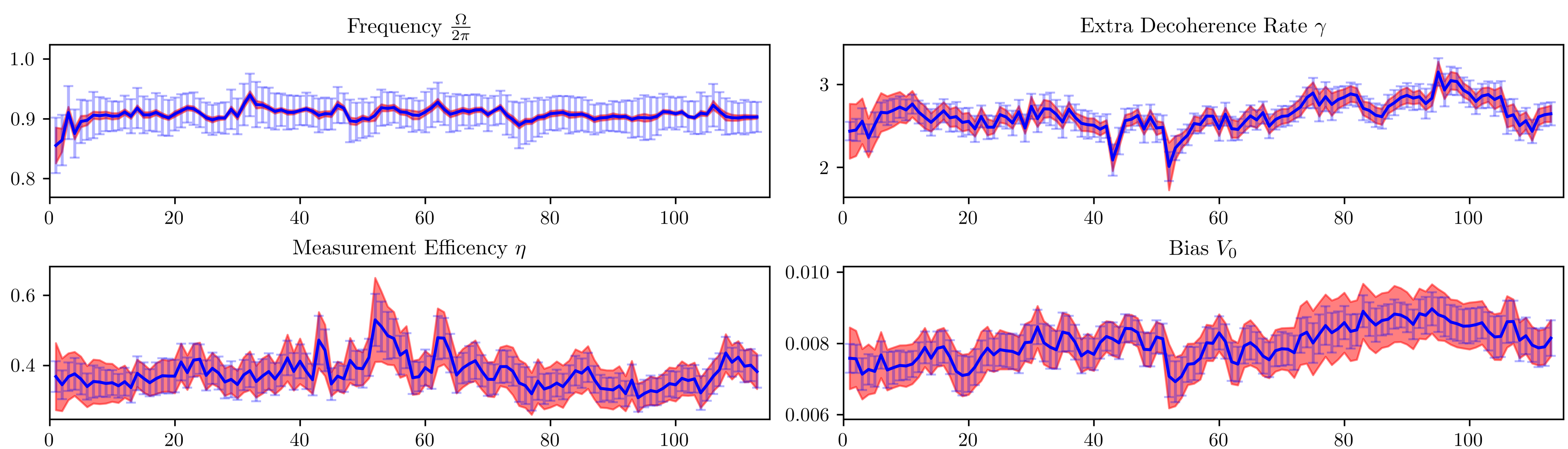
Parameter calibration for Transmon qubit experiments

Manoworakul, Ralph, Wiseman and ACH, (in preparation)

Numerical results showing convergence of the unknown parameters:



Estimated parameters with the error bars:



Take away messages

- Quantum trajectories for real measurement data **require more than just integrating the Stochastic Master Equations**.
- There are issues regarding: the **time resolution** $\Delta t \gg \mathrm{d}t$ (finite time step).
- We have introduced the **Hierarchy criteria** for Quantum Trajectories
    - (A) Valid Quantum Trajectory (strongest)
    - (B) Linblad Evolution (stronger)
    - (C) Valid Average Quantum Evolution (weaker)
        - (C1) Complete Positivity
        - (C2) Convex-Linearity
        - (C3) Trace Preservation
- We considered 3 approaches in the literature and proposed 1 approach:
    - $\hat{K}_{\mathrm{I}}$, $\hat{K}_{\mathrm{R}}$, $\hat{K}_{\mathrm{G}}$, and $\hat{K}_{\mathrm{W}}$: Wonglakorn "high-order" complete positive map
- We showed that $\hat{K}_{\mathrm{W}}$ satisfied all conditions(B) and (C) to $\mathcal{O}(\Delta t^2)$ and gave the best condition (A) among all maps.