Entanglement transitions in integrable non-Hermitian Floquet systems

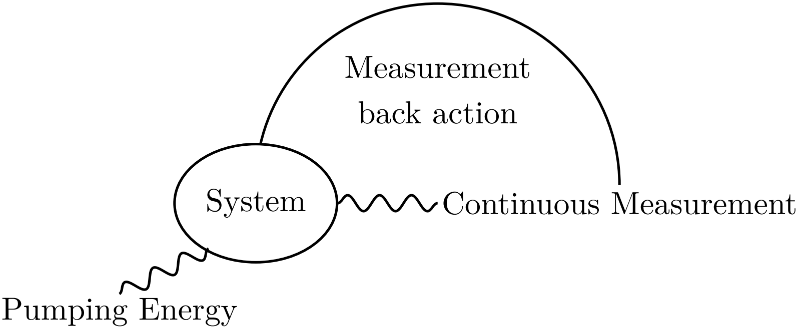
Tista Banerjee

Indian Association for the Cultivation of Science
(In collaboration with Prof. Krishnendu Sengupta)

January 21, 2025

Quantum Trajectories
ICTS, Bangalore, India.

Physical Setup

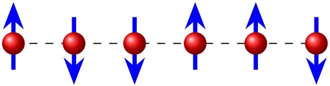


Result → *non*-unitary evolution for the system fields

One Possible description: Effective non-Hermitian Hamiltonian

More General: Open Quantum System, Lindblad Master Equation

Model Hamiltonian - 1D Quantum TFIM (complex field)



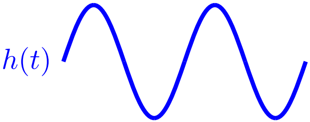
$$\hat{\mathcal{H}}_{\text{Ising}} = -J\left(\sum_{\langle ij\rangle}\hat{\sigma}_i^x\hat{\sigma}_j^x + (h+i\gamma)\sum_j \hat{\sigma}_j^z\right)$$

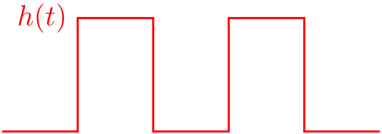
X.Turkeshi, M.Schiro (PRB 2023)

T.E.Lee, C.K.Chan (PRX 2014)

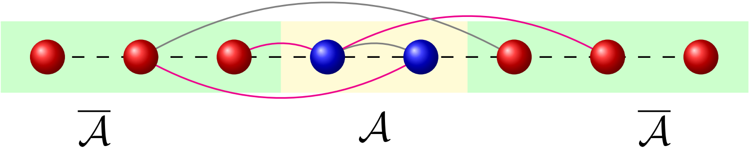
$$\hat{\mathcal{H}}_{\text{Ising}}(t) = -J\left(\sum_{\langle ij\rangle}\hat{\sigma}_i^x\hat{\sigma}_j^x + (h(t)+i\gamma)\sum_j \hat{\sigma}_j^z\right)$$

Phys. Rev. B 109, 094306





Bipartite (von-Neumann) Entanglement Entropy



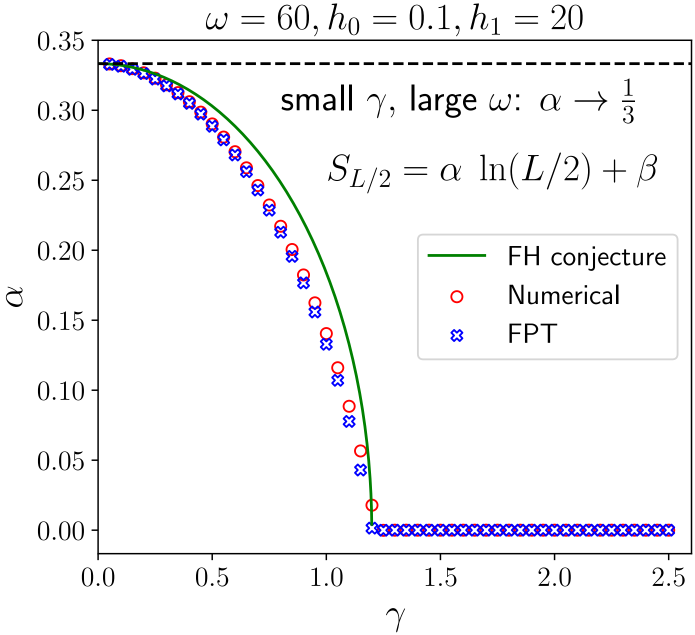
$$\mathscr{H} = \mathscr{H}_{\mathcal{A}} \bigotimes \mathscr{H}_{\overline{\mathcal{A}}}$$

$$\mathcal{S}^{\mathsf{vN}}_{\mathcal{A},\overline{\mathcal{A}}} = -\mathsf{Tr}\left(\rho_{\mathcal{A}} \ln \rho_{\mathcal{A}}\right) \qquad \rho_{\mathcal{A}} = \mathsf{Tr}_{\overline{\mathcal{A}}}\left(|\Psi\rangle\langle\Psi|\right)$$

Generally: Need to diagonalize $\rho_{\mathcal{A}}$ $\left(2^{\ell} \times 2^{\ell}\right)$ matrix numerically

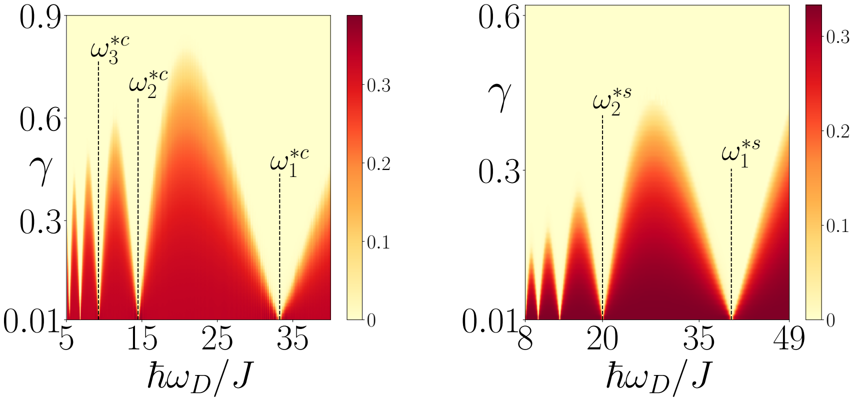
Integrable Systems: Special tools exist to find spectrum of $\rho_{\mathcal{A}}$

Area law to Logarithmic scaling transition



Entanglement "phase diagram"

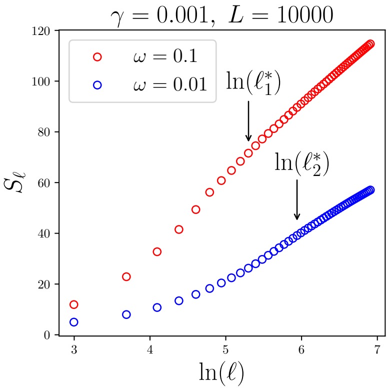
$$\mathcal{S}_{L/2} = \alpha \log(\ell) + \beta$$



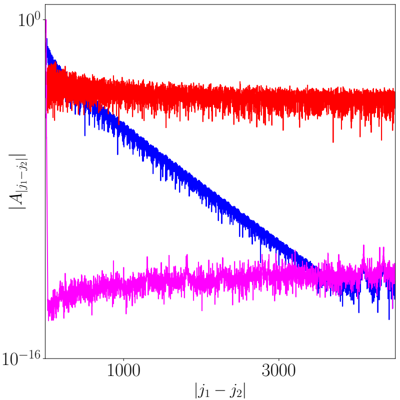
Cosine wave protocol

Square wave protocol

Volume Law in "small" subsystem size—Low frequency, Low measurement rate feature



$\omega_D = 9$, $\omega_D = 1$, $\omega_D = 0.1$



Entaglement "crossover"

Effective Long-range System

$$J_{ij} \sim \frac{J_0}{|i-j|^\alpha}$$

Dynamical signatures see [Phys. Rev. B 107, 15517 (2023)]

Continuous drive protocol: $h(t) = h_0 + h_1 \cos(\omega_D t)$

$\omega_D =$ Drive frequency , $T = \frac{2\pi}{\omega_D}$ Time period

Initial state: $|\Psi(t=0)\rangle$

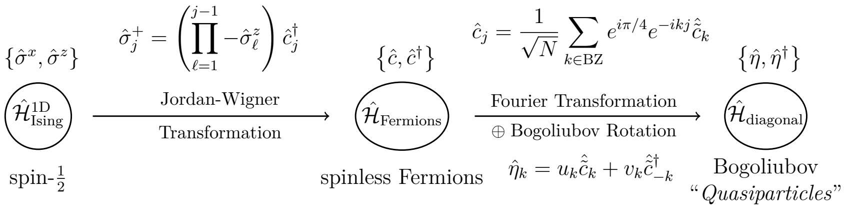
$h(t+T) = h(t)$

Application of Floquet theory

Time evolution operator $\hat{U}(T,0) = \mathcal{T}\left(\exp\left[-\frac{i}{\hbar} \int_0^T \mathcal{H}(t')dt' \right] \right)$

$$|\Psi(nT)\rangle = \hat{U}(nT,0)|\Psi(t=0)\rangle \longrightarrow |\tilde{\Psi}(nT)\rangle = \frac{\hat{U}(nT,0)|\Psi(t=0)\rangle}{\left|\left|\hat{U}(nT,0)|\Psi(t=0)\rangle\right|\right|}$$

Exact Evolution Operator see [SciPost Phys. Lect. Notes 82 (2024)]



$$\psi_k = \left(\hat{\tilde{c}}_k \;\; \hat{\tilde{c}}^\dagger_{-k}\right)^T, \; \hat{H} = 2 \sum_{k \in \mathrm{BZ}/2} \psi_k^\dagger \hat{h}_k \psi_k$$

$$h_k = \tau_z \left(h(t) - \cos k - i\gamma/2\right) + \left(\tau^+ \sin k + \mathrm{h.c.}\right)$$

$$\hat{U}(T,0) = \prod_k \hat{\mathcal{U}}_k(T,0) = \prod_{k>0} \exp\left(-\frac{i}{\hbar} \hat{\mathcal{H}}_k^{\mathrm{F}} T\right)$$

Note: *All* momentum modes are decoupled

Floquet Perturbation Theory (FPT)

FPT Regime

$$\boxed{g_1 \gg g_0, |\Delta_k|, |a_{3k}|}$$

$$g(t) = g_0 + g_1 \cos(\omega_D t)$$
$$= 2h_0 + 2h_1 \cos(\omega_D t)$$

$$\hat{H}_{\text{eff}} = \underbrace{g_1 \cos(\omega_D t)\ \tau_3}_{\hat{\mathcal{H}}_{0k}} + \underbrace{(g_0 - 2\cos(k) + i\gamma)\ \tau_3 + 2\sin(k)\ \tau_1}_{\hat{\mathcal{H}}_{1k}\text{(perturbative part)}}$$

0$^{\text{th}}$order : $\quad U_{0k}(t,0) = \exp\left(-\frac{i}{\hbar}\int_0^t H_{0k}dt'\right) = \exp\left(-i\tau_3 \frac{g_1 \sin\omega_D t}{\hbar\omega_D}\right)$

Time-dependent perturbation theory in Interaction picture:

$$\hat{U}_I(t,0) = \hat{1} - \frac{i}{\hbar}\int_0^t dt'\ \hat{V}_I(t') + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt''\ \hat{V}_I(t')\hat{V}_I(t'') + \ldots$$

Floquet Hamiltonian - 1st order (Emergent Conservation Law)

$$\hat{V}_I(t',0) = \hat{U}_0^\dagger(t',0)\hat{\mathcal{H}}_1\hat{U}_0(t',0) \qquad \hat{U}(t,0) = \hat{U}_0(t,0)\hat{U}_I(t,0)$$

$$\hat{\mathcal{H}}_{\mathsf{F}k} = \frac{i\hbar}{T}\left[\hat{U}_{I,k}^{(1)}(T,0) + \left(\hat{U}_{I,k}^{(2)}(T,0) - \frac{1}{2}\left(\hat{U}_{I,k}^{(1)}(T,0)\right)^2\right) + \ldots\right]$$

$$\mathcal{H}_{\mathsf{F}k}^{(1)} = \begin{bmatrix} \alpha_k + i\gamma & \Delta_k \mathcal{J}_0\left(\frac{2g_1}{\omega_D}\right) \\ \Delta_k \mathcal{J}_0\left(\frac{2g_1}{\omega_D}\right) & -(\alpha_k + i\gamma) \end{bmatrix}$$

$$\alpha_k = g_0 - 2\cos(k)$$
$$\Delta_k = 2\sin(k)$$

At special frequencies $\omega_D = \omega_m^*$

$$\mathcal{J}_0\left(\frac{2g_1}{\omega_m^*}\right) = 0 \rightarrow [\mathcal{H}_{\mathsf{F}k}^{(1)}, \hat{\tau}_3] = 0$$

$\hat{\tau}_3$ is a conserved quantity at ω_m^*

Floquet Hamiltonian - 2nd order (*Approximate* Emergent Conservation Law)

$$\hat{\mathcal{H}}^{(2)}_{\mathsf{F}k} = \left(\alpha_k - 2\Delta_k^2 \sum_{n=0}^{\infty} \frac{\mathcal{J}_0\left(\frac{2g_1}{\hbar\omega_D}\right)\mathcal{J}_{2n+1}\left(\frac{2g_1}{\hbar\omega_D}\right)}{(n+1/2)\hbar\omega_D} + i\gamma\right)\hat{\tau}_3$$

$$+ \left(\mathcal{J}_0\left(\frac{2g_1}{\hbar\omega_D}\right) + 2\alpha_k \sum_{n=0}^{\infty} \frac{\mathcal{J}_{2n+1}\left(\frac{2g_1}{\hbar\omega_D}\right)}{(n+1/2)\hbar\omega_D}\right)\hat{\tau}_1 = S_{1k}\hat{\tau}_1 + S_{2k}\hat{\tau}_3$$

At $\omega_D = \omega_m^*$ $\quad [\hat{\mathcal{H}}^{(2)}_{\mathsf{F}k}, \hat{\tau}_3] \approx 0 \rightarrow$ $\hat{\tau}_3$ approximately conserved

Steady State

$$|\psi^k_{\text{steady}}\rangle = \begin{pmatrix} u^k_{\text{steady}} \\ v^k_{\text{steady}} \end{pmatrix}, \quad C^2 = |u^k_{\text{steady}}|^2 + |v^k_{\text{steady}}|^2$$

with $u^k_{\text{steady}} = \dfrac{\text{sign}(\Gamma_k)\ E_k + S_{1k}}{C}, \quad v^k_{\text{steady}} = \dfrac{S_{2k}}{C}, \quad \text{sign}(\Gamma_k) = \dfrac{\Gamma_k}{|\Gamma_k|}$

Steady State Entanglement

$$\Pi^{\text{steady}}_{xk} = \langle \hat{\tilde{c}}^\dagger_k \hat{\tilde{c}}^\dagger_{-k} + \text{h.c.} \rangle \quad \Pi^{\text{steady}}_{zk} = \langle \hat{\tilde{c}}_{-k} \hat{\tilde{c}}_k - \hat{\tilde{c}}^\dagger_k \hat{\tilde{c}}^\dagger_{-k} \rangle \quad \Pi^{\text{steady}}_{yk} = \langle 2 \hat{\tilde{c}}^\dagger_k \hat{\tilde{c}}_k - 1 \rangle$$

$$= 2\mathbf{Re}(u^*_k(nT) v_k(nT)) \quad = 2\mathbf{Im}(u^*_k(nT) v_k(nT)) \quad = |v_k|^2 - |u_k|^2$$

<u>Generator/Symbol</u> $\quad \Pi_\ell = \int_{-\pi}^{+\pi} \frac{dk}{2\pi}\, e^{-ik\ell} \underbrace{\hat{\Pi}_{\text{steady}}(k) \cdot \hat{\sigma}_k}_{\hat{\tilde{\Pi}}_{\text{steady}}(k)}$

Block Toeplitz Matrix $\quad \Gamma^\ell = \begin{pmatrix} \Pi_0 & \Pi_{-1} & \ldots & \Pi_{1-\ell} \\ \Pi_1 & \Pi_0 & \ldots & \Pi_{2-\ell} \\ \ldots & \ldots & \ldots & \ldots \\ \Pi_{\ell-1} & \Pi_{\ell-2} & \ldots & \Pi_0 \end{pmatrix}$

$$\mathcal{S}_\ell = -\mathbf{Tr}\left(\frac{\mathbb{I}_\ell - \Gamma_\ell}{2} \ln \frac{\mathbb{I}_\ell - \Gamma_\ell}{2} \right), \; \mathbf{spec}\,(\Gamma_\ell) = \{\nu_j, j = 1, 2, .., \ell\}$$

$$\mathbf{spec}\,(\rho_\ell) \to \lambda_{x_1, x_2, \ldots, x_\ell} = \prod_{j=1}^{\ell} \frac{1 + (-1)^{x_j} \nu_j}{2}, \quad x_j = 0, 1 \quad \forall j$$

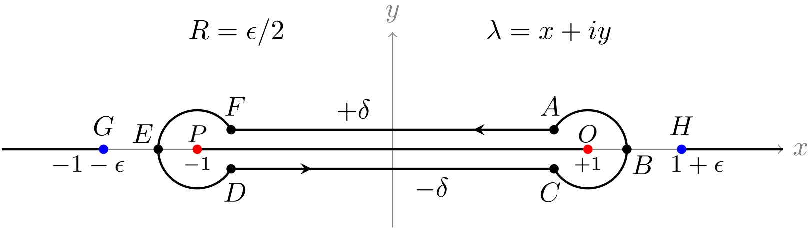
The bipartite (von-Neumann) entanglement entropy of a sub-system of size ℓ with an infinite chain then given by

$$\mathcal{S}_t(\ell) = -\sum_{m=1}^{\ell}\left(\frac{1-\nu_m}{2}\right)\ln\left(\frac{1-\nu_m}{2}\right) - \sum_{m=1}^{\ell}\left(\frac{1+\nu_m}{2}\right)\ln\left(\frac{1+\nu_m}{2}\right)$$

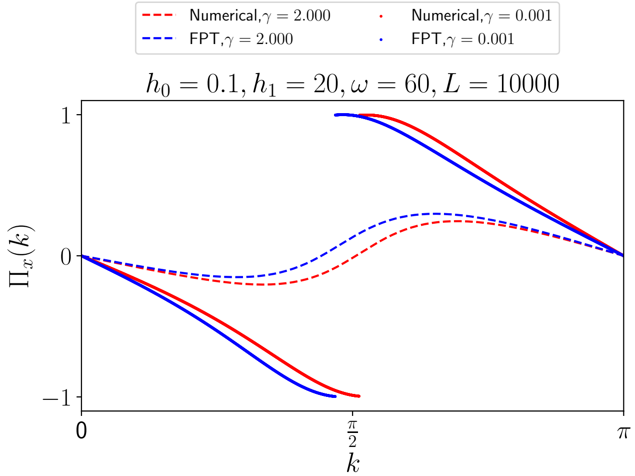
Using Cauchy's Residue Theorem

$$\mathcal{S}_t(\ell) = \frac{1}{4\pi i}\oint_{\mathcal{C}} d\lambda\, e(1,\lambda)\,\frac{d}{d\lambda}\ln\left[\det\left(\mathcal{D}_\ell(\lambda)\right)\right]$$

$$\begin{aligned}\mathcal{D}_\ell(\lambda) &:= \lambda\mathbb{I}_{2\ell\times 2\ell} - \Gamma_\ell \\ \mathcal{A}(k) &:= \lambda\mathbb{I}_{2\ell\times 2\ell} - \tilde{\Pi}_{\text{steady}}(k)\end{aligned} \qquad e(1,\lambda) \equiv -\frac{1+\lambda}{2}\ln\frac{1+\lambda}{2} - \frac{1-\lambda}{2}\ln\frac{1-\lambda}{2}$$



Generator structure



$\mathcal{S} = \alpha \log(\ell) + \beta$

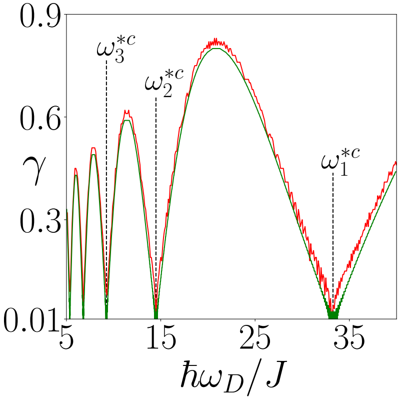
Singular Generator $\longrightarrow$ Logarithmic Scaling

$\mathcal{S} =$ constant

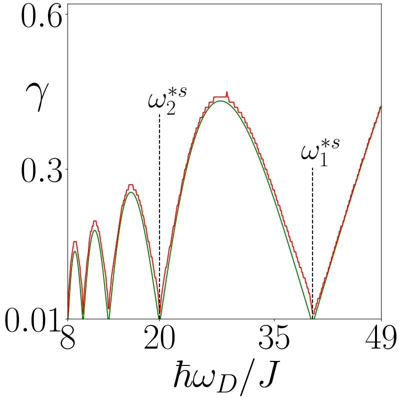
Smooth Generator $\longrightarrow$ Area Law

Entanglement "phase diagram"

Analytical vs Numerical Phase Boundary

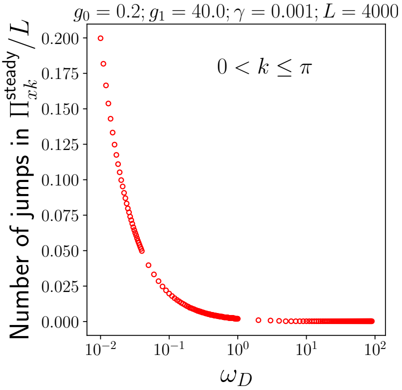


Cosine wave protocol



Square wave protocol

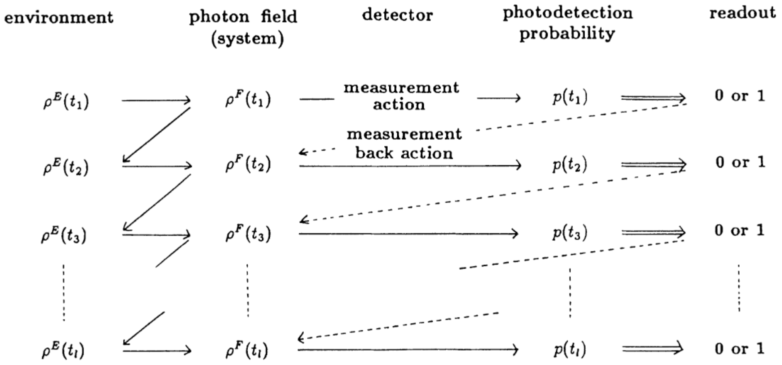
Open questions and future directions



- Late time entanglement properties depend crucially on the structure of 2 point correlation function
- Entanglement scaling transition from logarithmic to area law (intermediate frequency regime – Application of Szegő-Widom Strong Limit theorem)
- Finite subsystem size effect – appearance of volume law in the small frequency regime low measurement rate limit.

Thank you for your attention!!

Physical Setup



Masahito Ueda Phys. Rev. A **41**, 3875 (1990)

Dynamical signatures see [Phys. Rev. B 107, 15517 (2023)]

As this is a 2-level system in each momentum mode

$$\hat{\mathcal{U}}_k(T,0) = \textbf{exp}\left[-\frac{i}{\hbar}\epsilon^{(1)}_{\mathsf{F}k}T\right]|1\rangle\langle 1| + \textbf{exp}\left[-\frac{i}{\hbar}\epsilon^{(2)}_{\mathsf{F}k}T\right]|2\rangle\langle 2|$$

$\epsilon^{(1)}_{\mathsf{F}k}, \epsilon^{(2)}_{\mathsf{F}k} \longrightarrow$ Eigenvalues of $\mathcal{H}^{\mathsf{F}}_k$ (Complex)

$|1\rangle, |2\rangle \longrightarrow$ Eigenstates of $\mathcal{H}^{\mathsf{F}}_k$ corresponding to $\epsilon^{(1)}_{\mathsf{F}k}, \epsilon^{(2)}_{\mathsf{F}k}$

$$|\Psi_k(nT,0)\rangle = \textbf{exp}\left[-\frac{i}{\hbar}\epsilon^{(1)}_{\mathsf{F}k}nT\right]|1\rangle\langle 1|\Psi(0)\rangle + \textbf{exp}\left[-\frac{i}{\hbar}\epsilon^{(2)}_{\mathsf{F}k}nT\right]|2\rangle\langle 2|\Psi(0)\rangle$$

All measurements are done at these stroboscopic times $t = nT$

$\langle \hat{\tilde{c}}^{\dagger}_k \hat{\tilde{c}}_k \rangle$
Decays to a steady value

$\langle \hat{\tilde{c}}^{\dagger}_k \hat{\tilde{c}}^{\dagger}_{-k} + \ h.c.\rangle$
Keeps on oscillating

We can use FPT in the intermediate frequency regime with

$$|h_1| >> |h_0|, \Delta_k, a_3(\vec{k})$$

An example 2 qubit system

$$\mathscr{H} = \{|\downarrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\uparrow\uparrow\rangle\}, \quad \mathcal{A} = \text{qubit-1}, \; \overline{\mathcal{A}} \to \text{qubit-2}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle\right)$$

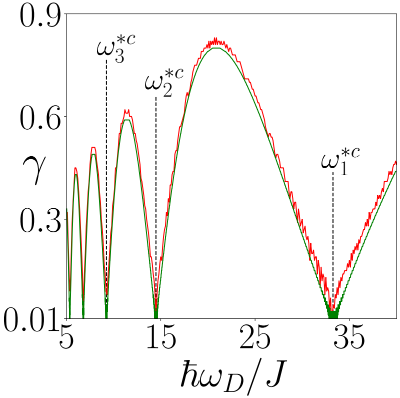
$$\begin{aligned}
\rho_{\mathcal{A}} &= \mathsf{Tr}_{\overline{\mathcal{A}}}\left(|\psi\rangle\langle\psi|\right) \\
&= \frac{1}{2} \times {}_{\overline{\mathcal{A}}}\langle\downarrow|\left(|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle\right)\left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right)|\downarrow\rangle_{\overline{\mathcal{A}}} \\
&+ \frac{1}{2} \times {}_{\overline{\mathcal{A}}}\langle\uparrow|\left(|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle\right)\left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right)|\uparrow\rangle_{\overline{\mathcal{A}}} \\
&= \frac{1}{2}\left(|\downarrow\rangle_{\mathcal{A}}\, {}_{\mathcal{A}}\langle\downarrow| + |\uparrow\rangle_{\mathcal{A}}\, {}_{\mathcal{A}}\langle\uparrow|\right)
\end{aligned}$$

$$\mathcal{S}_{\mathcal{A},\overline{\mathcal{A}}} = -\mathsf{Tr}\left(\rho_{\mathcal{A}} \ln \rho_{\mathcal{A}}\right) = \ln 2$$

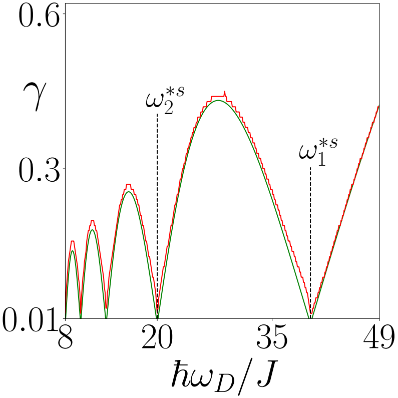
Main Result-2: Entanglement "phase diagram"

Analytical vs Numerical Phase Boundary

$$\mathcal{S} = \alpha \ln \ell + \beta$$

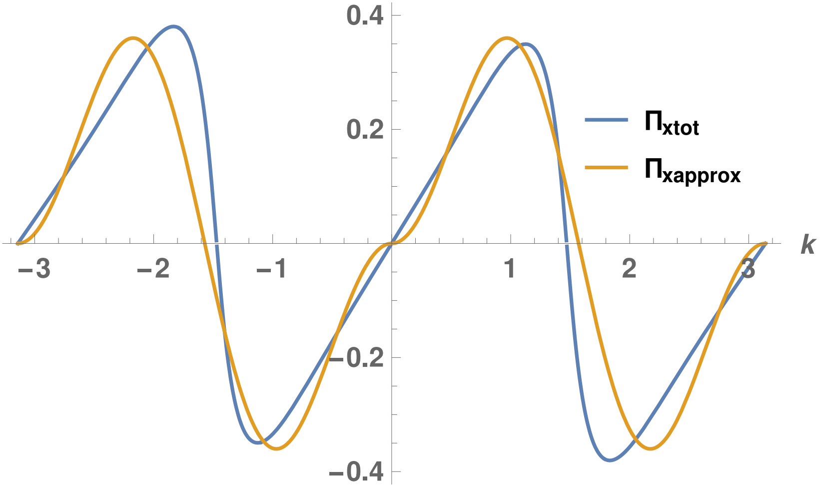


Cosine wave protocol



Square wave protocol

Smooth Generator and absence of volume law



$\mathcal{S}$ = constant

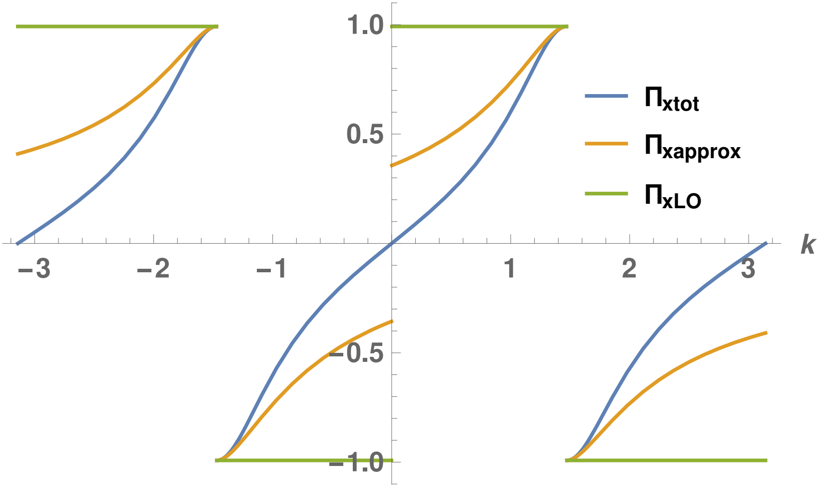
Smooth Generator $\longrightarrow$ Area Law

Szegő-Widom Strong Limit Theorem - Steady State

$$\begin{aligned}
\mathcal{S}_{\text{steady}}(\ell) &\approx \frac{\ell}{8\pi^2 i} \oint_{\mathcal{C}} d\lambda\, e(1,\lambda) \int_0^{2\pi} dk\, \frac{d}{d\lambda} \ln\left[\det\left(\mathcal{A}(k)\right)\right] + \mathcal{O}\left(\ln(\ell)\right) \\
&= \underbrace{\frac{\ell}{4\pi i} \oint_{\mathcal{C}} d\lambda \left[\frac{e(1,\lambda)}{\lambda - 1} + \frac{e(1,\lambda)}{\lambda + 1}\right]}_{\text{Volume Law Scaling}} + \underbrace{\mathcal{O}\left(\ln(\ell)\right)}_{\text{Logarithmic Scaling}} + \text{const.}
\end{aligned}$$

$$\det\left(\mathcal{A}(k)\right) = \det\left(\lambda \mathbb{I} - \tilde{\Pi}(k)\right) = \lambda^2 - 1$$

Singular generator



$$\mathcal{S} = \alpha \ln \ell + \beta$$

Singular Generator $\longrightarrow$ Logarithmic Law

Analytical estimate of coefficient of **ln** ℓ

$$\tilde{\Pi}^{(1)}_{\text{steady}}(k) = \frac{k-k^*}{|k-k^*|}\frac{\sqrt{(g_0^p)^2Y^2-\gamma^2}}{g_0^pY}\sigma^x + \frac{\gamma}{g_0^pY}\sigma^z = \Pi^{\text{steady}}_{xk}\sigma^x + \Pi^{\text{steady}}_{zk}\sigma^z$$

$$\tilde{\Pi}^{(2)}_{\text{steady}}(k) = \frac{k+k^*}{|k+k^*|}\frac{\sqrt{(g_0^p)^2Y^2-\gamma^2}}{g_0^pY}\sigma^x - \frac{\gamma}{g_0^pY}\sigma^z = \Pi^{\text{steady}}_{xk}\sigma^x + \Pi^{\text{steady}}_{zk}\sigma^z$$

$$\begin{aligned}\tilde{\Pi}^{k}_{\text{steady}}\Big|_{|g_0^pY|\gg\gamma} &= \frac{k-k^*}{|k-k^*|}\,\text{sgn}(g_0^pY)\,\sigma^x \qquad 0<k<\pi \\ &= \frac{k+k^*}{|k+k^*|}\,\text{sgn}(g_0^pY)\,\sigma^x \qquad -\pi<k<0\end{aligned}$$

$$\Gamma_{2\ell\times 2\ell} = B_{\ell\times\ell}\otimes\sigma^x,\ g_0^p = \sqrt{4-h_0^2},\ Y = \mathcal{J}_0\left(\frac{2g_1}{\hbar\omega_D}\right)$$

Fisher-Hartwig Conjecture

The conjecture states that the generator/symbol of a Toeplitz matrix having m jump and m root singularities can be cast in the following form

$$f(z) = e^{V(z)} \prod_{j=0}^{m} |z - z_j|^{2\alpha_j} \left(\frac{z}{z_j}\right)^{\beta_j} g_{z_j,\beta_j}(z),\, z = e^{i\theta}, 0 \le \theta \le 2\pi$$

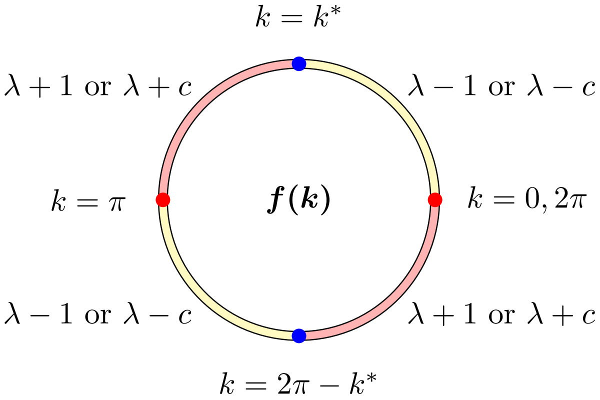
With the *singularities* located at $\left\{z_j = e^{i\theta_j},\;\; j = 0, 1, 2, ..., m\right\}$ and $0 \le \theta_0 < \theta_1 < \theta_2 < ... < \theta_m < 2\pi$. Here the $|z - z_j|^{2\alpha_j}$ specifies the root type singularities and

$$g_{z_j\beta_j} := g_{\beta_j}(z) = \begin{cases} e^{+i\pi\beta_j} & 0 \le \theta < \theta_j \\ e^{-i\pi\beta_j} & \theta_j \le \theta < 2\pi \end{cases}$$

$\beta_j \in \mathbb{C}$, $\theta = \arg(z)$, $\mathrm{Re}(\alpha_j) > -1/2$ (ensures integrability) and $e^{V(\theta)}$ is a sufficiently smooth function on S^1. For a non-singular scalar symbol $j = 0, z_0 = 1, \theta_0 = 0$

$$g_{z_0,\beta_0}(z) = e^{-i\pi\beta_0} \qquad 0 \le \theta < 2\pi$$

Application of Fisher-Hartwig Conjecture



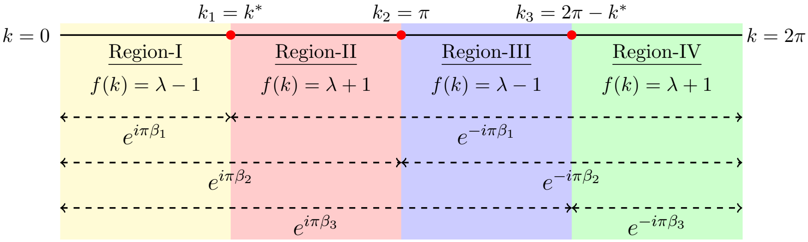
$$f(k) = e^{V(e^{ik})} \left(\prod_{i=0}^{3} \left(\frac{z}{z_i} \right)^{\beta_i} \right) e^{-i\pi\beta_0} \left(\prod_{i=1}^{3} g_{z_i \beta_i}(z) \right)$$

Casting symbol in Fisher-Hartwig form

$$\beta_0 + \beta_1 + \beta_2 + \beta_3 = 0$$

$$e^{V(e^{ik})} e^{-ik_1\beta_1} e^{-ik_2\beta_2} e^{-ik_3\beta_3} e^{-i\pi\beta_0} e^{i\pi(\pm\beta_1+\beta_2+\beta_3)} = \lambda \mp 1$$

$$e^{V(e^{ik})} e^{-ik_1\beta_1} e^{-ik_2\beta_2} e^{-ik_3\beta_3} e^{-i\pi\beta_0} e^{-i\pi(\beta_1+\beta_2\pm\beta_3)} = \lambda \pm 1$$

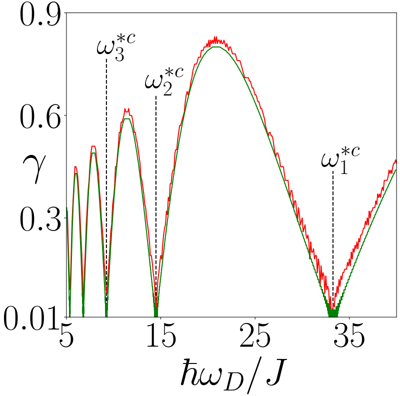


$$\beta_0 = \beta_2 = \frac{1}{2\pi i} \ln\left(\frac{\lambda+1}{\lambda-1}\right), \beta_1 = \beta_3 = \frac{1}{2\pi i} \ln\left(\frac{\lambda-1}{\lambda+1}\right), e^{V(e^{ik})} = \sqrt{\lambda^2 - 1}$$

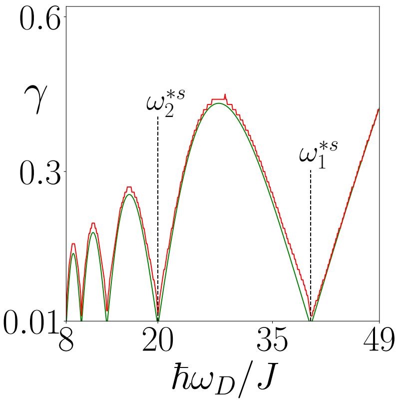
$$\boxed{\ln\left(\det\left(\mathcal{D}_\ell\left(\lambda\right)\right)\right) = \ell \ \ln\left(F\left[e^{V(e^{ik})}\right]\right) - \ln(\ell) \sum_{i=1}^{4} \beta_i^2(\lambda)} + \text{sub-leading terms}$$

Main Result-2: Entanglement "phase diagram"

Analytical vs Numerical Phase Boundary



Cosine wave protocol



Square wave protocol