Bounding fidelity in feedback control protocols for quantum state engineering

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CONTRABASS

The Problem

$$d\rho(t) = -i[H_c, \rho]dt + \sum_j \mathscr{D}[\hat{c}_j]\rho\, dt + \sum_j \sqrt{\eta_j}\mathscr{H}[\hat{c}_j]\rho dw_j$$

Collapse operators $\hat{c}_j$

$$\mathscr{D}[\hat{A}]\rho = \hat{A}\rho\hat{A}^\dagger - (\hat{A}^\dagger\hat{A}\rho + \rho\hat{A}^\dagger\hat{A})/2$$

$$\mathscr{H}[\hat{A}]\rho = \hat{A}\rho + \rho\hat{A}^\dagger - \textbf{tr}[\rho(\hat{A} + \hat{A}^\dagger)]\rho$$

The Problem

$$d\rho(t) = -i[H_c, \rho]dt + \sum_j \mathscr{D}[\hat{c}_j]\rho\, dt + \sum_j \sqrt{\eta_j}\mathscr{H}[\hat{c}_j]\rho dw_j$$

Collapse operators $\hat{c}_j$

State-based feedback

The Problem

$$d\rho(t) = -i[H_c, \rho]dt + \sum_j \mathscr{D}[\hat{c}_j]\rho\, dt + \sum_j \sqrt{\eta_j}\mathscr{H}[\hat{c}_j]\rho dw_j$$

Collapse operators $\hat{c}_j$

State-based feedback

The figure of merit

$$\mathbb{E}[\mathscr{F}] \equiv \bar{\mathscr{F}}$$

with $\mathscr{F} = \langle \psi_T | \rho(t) | \psi_T \rangle$

Bounding Fidelity

$$d\rho(t) = -i[H_c, \rho]dt + \sum_j \mathscr{D}[\hat{c}_j]\rho\, dt + \sum_j \sqrt{\eta_j}\mathscr{H}[\hat{c}_j]\rho dw_j$$

$\eta_j = 1$ $\quad$ $\rho(t) = |\varphi(t)\rangle\langle\varphi(t)|$

Assume best case

Bounding Fidelity

$$d\rho(t) = -i[H_c, \rho]dt + \sum_j \mathcal{D}[\hat{c}_j]\rho\, dt + \sum_j \sqrt{\eta_j}\mathcal{H}[\hat{c}_j]\rho dw_j$$

$\eta_j = 1$ $\qquad \rho(t) = |\varphi(t)\rangle\langle\varphi(t)|$

Assume best case

Take each part individually

Bounding Fidelity

$$d\mathcal{F} = \langle \psi_T | d\rho(t) | \psi_T \rangle = \langle \mathcal{U}\rho \rangle_{\psi_T} dt + \langle \mathcal{D}\rho \rangle_{\psi_T} dt + \langle \mathcal{H}\rho \rangle_{\psi_T, \{dw_j\}}$$

$$\langle \mathcal{U}\rho \rangle_{\psi_T} = -\, i \langle \psi_T | [H_c, \rho] | \psi_T \rangle$$

Bounding Fidelity

$$d\mathscr{F} = \langle \psi_T | d\rho(t) | \psi_T \rangle = \langle \mathscr{U}\rho \rangle_{\psi_T} dt + \langle \mathscr{D}\rho \rangle_{\psi_T} dt + \langle \mathscr{H}\rho \rangle_{\psi_T, \{dw_j\}}$$

$$\langle \mathscr{U}\rho \rangle_{\psi_T} = -i \langle \psi_T | [H_c, \rho] | \psi_T \rangle$$

$$\langle \mathscr{D}\rho \rangle_{\psi_T} = \sum_j \langle \psi_T | \mathscr{D}[\hat{c}_j]\rho | \psi_T \rangle$$

Bounding Fidelity

$$d\mathscr{F} = \langle\psi_T|d\rho(t)|\psi_T\rangle = \langle\mathscr{U}\rho\rangle_{\psi_T}dt + \langle\mathscr{D}\rho\rangle_{\psi_T}dt + \langle\mathscr{H}\rho\rangle_{\psi_T,\{dw_j\}}$$

$$\langle\mathscr{U}\rho\rangle_{\psi_T} = -i\langle\psi_T|[H_c,\rho]|\psi_T\rangle$$

$$\langle\mathscr{D}\rho\rangle_{\psi_T} = \sum_j \langle\psi_T|\mathscr{D}[\hat{c}_j]\rho|\psi_T\rangle$$

$$\langle\mathscr{H}\rho\rangle_{\psi,\{dw_j\}} = \sum_j \langle\psi|\mathscr{H}[\hat{c}_j]\rho|\psi\rangle\, dw_j$$

Bounding Fidelity

$$d\mathscr{F} = \langle \mathscr{U}\rho \rangle_{\psi_T} dt + \langle \mathscr{D}\rho \rangle_{\psi_T} dt + \langle \mathscr{H}\rho \rangle_{\psi_T,\{dw_j\}}$$

$$\mathbb{E}\left[\langle \mathscr{H}\rho \rangle_{\psi,\{dw_j\}}\right] = \sum_j \mathbb{E}\left[\langle \psi | \mathscr{H}[\hat{c}_j]\rho | \psi \rangle\right] \mathbb{E}\left[dw_j\right] = 0$$

Bounding Fidelity

$$d\mathscr{F} = \langle \mathscr{U}\rho \rangle_{\psi_T} dt + \langle \mathscr{D}\rho \rangle_{\psi_T} dt + \langle \mathscr{H}\rho \rangle_{\psi_T,\{dw_j\}}$$

Bounding Fidelity

$$d\mathscr{F} = \langle \mathscr{U}\rho \rangle_{\psi_T} dt + \langle \mathscr{D}\rho \rangle_{\psi_T} dt + \langle \mathscr{H}\rho \rangle_{\psi_T, \{dw_j\}}$$

$$\langle \mathscr{U}\rho \rangle_{\psi_T} = -i \langle \psi_T | [H_c, \rho(t)] | \psi_T \rangle$$

Hamiltonian contribution

Bounding Fidelity

$$d\mathcal{F} = \langle \mathcal{U}\rho \rangle_{\psi_T} dt + \langle \mathcal{D}\rho \rangle_{\psi_T} dt + \langle \mathcal{H}\rho \rangle_{\psi_T, \{dw_j\}}$$

$$\langle \mathcal{U}\rho \rangle_{\psi_T} = -i\langle \psi_T | [H_c, \rho(t)] | \psi_T \rangle \leq 2\Delta H_c \sqrt{\mathcal{F}(1-\mathcal{F})}$$

Cauchy-Schwarz

L. Mandelstam and I. Tamm, J. Phys., **9** 249 (1945)

Bounding Fidelity

$$d\mathscr{F} = \langle \mathscr{U}\rho \rangle_{\psi_T} dt + \langle \mathscr{D}\rho \rangle_{\psi_T} dt + \langle \mathscr{H}\rho \rangle_{\psi_T, \{dw_j\}}$$

$$\langle \mathscr{U}\rho \rangle_{\psi_T} = -i\langle \psi_T | [H_c, \rho(t)] | \psi_T \rangle \leq 2\Delta H_c \sqrt{\mathscr{F}(1-\mathscr{F})}$$

Constraints are necessary

$$(\Delta H_c)^2 \equiv \langle \psi_T | H_c^2 | \psi_T \rangle - \langle \psi_T | H_c | \psi_T \rangle^2 \leq (\Delta H)^2$$

L. Mandelstam and I. Tamm, J. Phys., **9** 249 (1945)

Bounding Fidelity

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$$\langle \mathscr{U}\rho \rangle_{\psi_T} = -i\langle \psi_T | [H_c, \rho(t)] | \psi_T \rangle \leq 2\Delta H_c \sqrt{\mathscr{F}(1-\mathscr{F})}$$

Now take the average

$$\mathbb{E}\left[\langle \mathscr{U}\rho \rangle_{\psi_T}\right] \leq 2\Delta H \, \mathbb{E}\left[\sqrt{\mathscr{F}(1-\mathscr{F})}\right] \leq 2\Delta H \sqrt{\bar{\mathscr{F}}(1-\bar{\mathscr{F}})}$$

L. Mandelstam and I. Tamm, J. Phys., **9** 249 (1945)

Bounding Fidelity

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Now take the average

$$\mathbb{E}\left[\langle \mathscr{U}\rho \rangle_{\psi_T} \right] \leq 2\Delta H \, \mathbb{E}\left[\sqrt{\mathscr{F}(1-\mathscr{F})} \right] \leq 2\Delta H \sqrt{\bar{\mathscr{F}}(1-\bar{\mathscr{F}})}$$

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Bounding Fidelity

$$d\mathscr{F} = \langle \mathscr{U}\rho \rangle_{\psi_T} dt + \langle \mathscr{D}\rho \rangle_{\psi_T} dt + \langle \mathscr{H}\rho \rangle_{\psi_T, \{dw_j\}}$$

Identify the optimal orthogonal state

$$|\varphi\rangle = \sqrt{\mathscr{F}}\,|\psi_T\rangle + e^{i\phi}\sqrt{1-\mathscr{F}}\,|\psi^{\perp}\rangle$$

Bounding Fidelity

$$\langle \mathscr{D}\rho \rangle_{\psi_T} \leq -\mathscr{F}(\boldsymbol{\Delta}\hat{\mathbf{c}})^2 + (1-\mathscr{F})\mathscr{A} + \mathscr{B}\sqrt{\mathscr{F}(1-\mathscr{F})}$$

Bounding Fidelity

$$\langle \mathscr{D}\rho \rangle_{\psi_T} \leq -\mathscr{F}(\boldsymbol{\Delta}\hat{\mathbf{c}})^2 + (1-\mathscr{F})\mathscr{A} + \mathscr{B}\sqrt{\mathscr{F}(1-\mathscr{F})}$$

$$(\boldsymbol{\Delta}\hat{\mathbf{c}})^2 = \sum_j \left(\langle \psi_T | \hat{c}_j^\dagger \hat{c}_j | \psi_T \rangle - |\langle \psi_T | \hat{c}_j | \psi_T \rangle|^2 \right),$$

Flow out of the target state

Bounding Fidelity

$$\langle \mathcal{D}\rho \rangle_{\psi_T} \leq -\mathcal{F}(\boldsymbol{\Delta}\hat{\mathbf{c}})^2 + (1-\mathcal{F})\mathcal{A} + \mathcal{B}\sqrt{\mathcal{F}(1-\mathcal{F})}$$

$$(\boldsymbol{\Delta}\hat{\mathbf{c}})^2 = \sum_j \left(\langle \psi_T | \hat{c}_j^\dagger \hat{c}_j | \psi_T \rangle - |\langle \psi_T | \hat{c}_j | \psi_T \rangle|^2 \right),$$

Flow out of the target state

$$\mathcal{A} = \sum_j |\langle \psi_T | \hat{c}_j | \psi^\perp \rangle|,$$

Flow into of the target state

Bounding Fidelity

$$\langle \mathscr{D}\rho \rangle_{\psi_T} \leq -\mathscr{F}(\boldsymbol{\Delta}\hat{\mathbf{c}})^2 + (1-\mathscr{F})\mathscr{A} + \mathscr{B}\sqrt{\mathscr{F}(1-\mathscr{F})}$$

$$(\boldsymbol{\Delta}\hat{\mathbf{c}})^2 = \sum_j \left(\langle \psi_T | \hat{c}_j^\dagger \hat{c}_j | \psi_T \rangle - |\langle \psi_T | \hat{c}_j | \psi_T \rangle|^2 \right),$$

Flow out of the target state

$$\mathscr{A} = \sum_j |\langle \psi_T | \hat{c}_j | \psi^\perp \rangle|,$$

Flow into of the target state

$$\mathscr{B} = \left| \langle \psi_T | \sum_j \left(2\hat{c}_j^\dagger | \psi_T \rangle \langle \psi_T | \hat{c}_j - \hat{c}_j^\dagger \hat{c}_j \right) | \psi^\perp \rangle \right| .$$

Bounding Fidelity

$$\frac{d\bar{\mathscr{F}}}{dt} \leq -\bar{\mathscr{F}}(\boldsymbol{\Delta}\hat{\mathbf{c}})^2 + (1-\bar{\mathscr{F}})\mathscr{A}^* + (\mathscr{B}^* + 2\Delta H)\sqrt{\bar{\mathscr{F}}(1-\bar{\mathscr{F}})}$$

Bounding Fidelity

$$\frac{d\bar{\mathscr{F}}}{dt} \leq -\bar{\mathscr{F}}(\Delta\hat{\mathbf{c}})^2 + (1-\bar{\mathscr{F}})\mathscr{A}^* + (\mathscr{B}^* + 2\Delta H)\sqrt{\bar{\mathscr{F}}(1-\bar{\mathscr{F}})}$$

Solving for $\frac{d\bar{\mathscr{F}}}{dt} = 0$

$$\bar{\mathscr{F}}_{ss} \leq B_Q \equiv \frac{2\mathscr{A}^*(\mathscr{A}^* + (\Delta\hat{\mathbf{c}})^2) + (\mathscr{B}^* + 2\Delta H)\left((\mathscr{B}^* + 2\Delta H) + \sqrt{4\mathscr{A}^*(\Delta\hat{\mathbf{c}})^2 + (\mathscr{B}^* + 2\Delta H)^2}\right)}{2\left((\mathscr{A}^* + (\Delta\hat{\mathbf{c}})^2)^2 + (\mathscr{B}^* + 2\Delta H)^2\right)}$$

K. Kobayashi and N. Yamamoto, Phys. Rev. A, **99** 052347 (2019)

Bounding Fidelity

$$\frac{d\bar{\mathscr{F}}}{dt} \leq -\bar{\mathscr{F}}(\boldsymbol{\Delta}\hat{\mathbf{c}})^2 + (1-\bar{\mathscr{F}})\mathscr{A}^* + (\mathscr{B}^* + 2\Delta H)\sqrt{\bar{\mathscr{F}}(1-\bar{\mathscr{F}})}$$

$$\bar{\mathscr{F}}_{ss} \leq B_Q \equiv \frac{2\mathscr{A}^*(\mathscr{A}^* + (\Delta\hat{\mathbf{c}})^2) + (\mathscr{B}^* + 2\Delta H)\left((\mathscr{B}^* + 2\Delta H) + \sqrt{4\mathscr{A}^*(\Delta\hat{\mathbf{c}})^2 + (\mathscr{B}^* + 2\Delta H)^2}\right)}{2\left((\mathscr{A}^* + (\Delta\hat{\mathbf{c}})^2)^2 + (\mathscr{B}^* + 2\Delta H)^2\right)}$$

Kobayashi and Yamamoto [1] derive a bound, B_{KY}, under the same assumptions, we prove that $B_Q \leq B_{KY}$

[1] K. Kobayashi and N. Yamamoto, Phys. Rev. A, **99** 052347 (2019)

Bounding Fidelity

$$\frac{d\bar{\mathcal{F}}}{dt} \leq -\bar{\mathcal{F}}(\boldsymbol{\Delta}\hat{\mathbf{c}})^2 + (1-\bar{\mathcal{F}})\mathscr{A}^* + (\mathscr{B}^* + 2\Delta H)\sqrt{\bar{\mathcal{F}}(1-\bar{\mathcal{F}})}$$

If $\mathscr{A} = \mathscr{B} = 0$

$$B_Q \leq \frac{4(\Delta H)^2}{4(\Delta H)^2 + (\boldsymbol{\Delta}\hat{\mathbf{c}})^2}$$

This is also independent of the orthogonal state

Dicke states

$$d\rho(t) = -i[H_c, \rho]dt + \kappa\mathscr{D}[J_z]\rho\, dt + \gamma\mathscr{D}[J_-]\rho + \ldots$$

$N = 2l$ non-interacting spins

Dicke states:
common eigenstates of

$$\mathbf{J}^2 | l, m\rangle = l(l+1) | l, m\rangle,$$
$$J_z | l, m\rangle = m | l, m\rangle$$

- γ: Collective damping rate
- κ: Collective dephasing rate

Dicke states

$$d\rho(t) = -i[H_c, \rho]dt + \kappa\mathscr{D}[J_z]\rho\, dt + \gamma\mathscr{D}[J_-]\rho + \ldots$$

$$H_c = u(t)J_y$$

Fixed-phase (FP)

$$H_c = u(t)\left(\cos\phi(t)J_x + \sin\phi(t)J_y\right)$$

Optimized-phase (OP)

u: Control strength, $|u(t)| \leq \tilde{u}$

Dicke states

$$d\rho(t) = -i[H_c, \rho]dt + \kappa\mathcal{D}[J_z]\rho\, dt + \gamma\mathcal{D}[J_-]\rho + \dots$$

$$H_c = u(t)J_y$$

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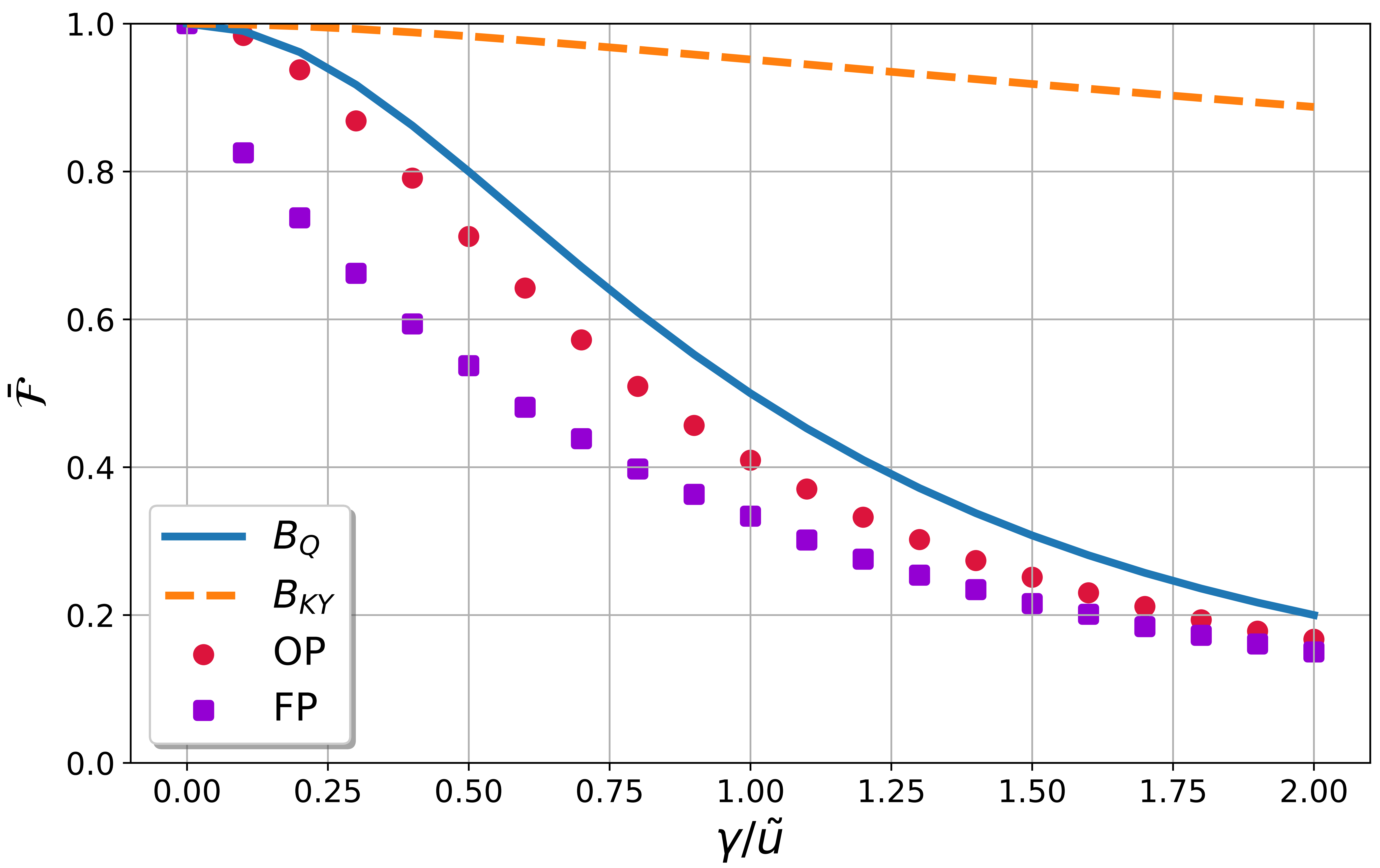
Optimized-phase (OP)

u: Control strength, $|u(t)| \leq \tilde{u}$

$$N = 1$$

$$\bar{\mathcal{F}} \leq \frac{1}{1 + \left(\frac{\gamma}{\tilde{u}}\right)^2}$$

Dicke states



$$\frac{\kappa}{\tilde{u}} = 0.4$$

Only Jensen gap

Phase optimization is important

Dicke states

Target state

$$|\,l, l\rangle$$

Maximally excited

Bound

$$\bar{\mathcal{F}} \leq \frac{1}{1 + N\left(\frac{\gamma}{\tilde{u}}\right)^2}$$

Dicke states

Target state | **Bound**

$$|l, l\rangle$$

Maximally excited

$$\bar{\mathcal{F}} \leq \frac{1}{1 + N\left(\frac{\gamma}{\tilde{u}}\right)^2}$$

$$|l, 0\rangle$$

Highly entangled

$$\bar{\mathcal{F}} \leq \frac{1}{2} + \frac{1}{\sqrt{4 + 2N(N+2)\left(\frac{\gamma}{\tilde{u}}\right)^2}}$$

Dicke states

Target state	**Bound**
$\lvert l, l\rangle$ *Maximally excited*	$\bar{\mathcal{F}} \leq \dfrac{1}{1 + N\left(\frac{\gamma}{\tilde{u}}\right)^2}$
$\lvert l, 0\rangle$ *Highly entangled*	$\bar{\mathcal{F}} \leq \dfrac{1}{2} + \dfrac{1}{\sqrt{4 + 2N(N+2)\left(\frac{\gamma}{\tilde{u}}\right)^2}}$

Uncontrollable in large N limit

Dicke states

Target state | **Bound**

$|l,l\rangle$

Maximally excited

$$\bar{\mathscr{F}} \leq \frac{1}{1 + N\left(\frac{\gamma}{\tilde{u}}\right)^2}$$

Uncontrollable in large N limit

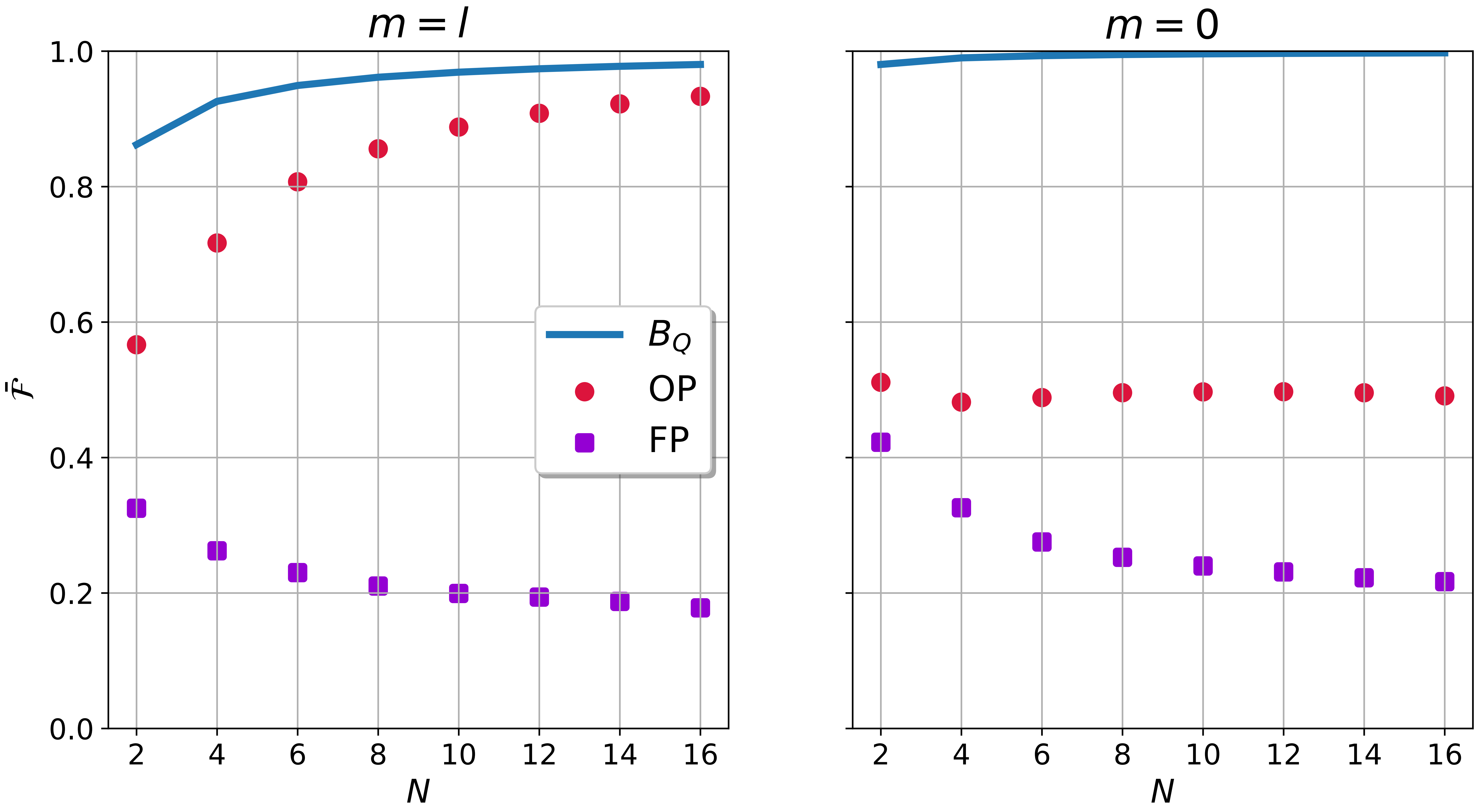
$|l,0\rangle$

Highly entangled

$$\bar{\mathscr{F}} \leq \frac{1}{2} + \frac{1}{\sqrt{4 + 2N(N+2)\left(\frac{\gamma}{\tilde{u}}\right)^2}}$$

Assumption: $\frac{\gamma}{\tilde{u}} \propto \frac{1}{N}$

Dicke states



$$\frac{\kappa}{\tilde{u}} = 0.4, \frac{N\gamma}{\tilde{u}} = 0.8 \qquad \frac{\kappa}{\tilde{u}} = 2, \frac{N\gamma}{\tilde{u}} = 0.8$$

Tighter bounds for Dicke state preparation

A **tighter** bound requires **stricter** constraints

$$d\rho(t) = -i[H_c, \rho]dt + \kappa\mathscr{D}[J_z]\rho\, dt + \gamma\mathscr{D}[J_-]\rho + \dots$$

Collapse operators

$$\sqrt{\kappa}J_z, \sqrt{\gamma}J_-$$

Homodyne *Heterodyne*

Tighter bounds for Dicke state preparation

A **tighter** bound requires **stricter** constraints

$$d\rho(t) = -i[H_c, \rho]dt + \kappa\mathscr{D}[J_z]\rho\, dt + \gamma\mathscr{D}[J_-]\rho + \dots$$

Collapse operators

$$\sqrt{\kappa}J_z, \sqrt{\gamma}J_-$$

$$H_c = u(t)\Big(\cos\phi(t)J_x + \sin\phi(t)J_y\Big)$$

Collective control

Tighter bounds for Dicke state preparation

Transfer occurs only between Dicke states with neighbouring angular momentum

$$|\varphi^{(c)}(t)\rangle = \sum_{m=-l}^{l} e^{i\phi_m(t)}\sqrt{a_m(t)}\,|l,m\rangle$$

$$da_m = \left(T_m - T_{m-1}\right) dt + dS$$

Tighter bounds for Dicke state preparation

$$da_m = \left(T_m - T_{m-1}\right) dt + dS$$

$$T_m = u\sqrt{a_m a_{m+1}}\cos(\alpha_m)h_m + \gamma a_{m+1}h_m^2$$

Transfer rate

Where

$$\alpha_m = \theta(t) - (\phi_m - \phi_{m+1}) - \frac{\pi}{2}$$

$$h_m = \sqrt{(l-m)(l+m+1)}$$

Tighter bounds for Dicke state preparation

$$da_m = \left(T_m - T_{m-1}\right) dt + dS$$

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Transfer rate

Where

$$\alpha_m = \theta(t) - (\phi_m - \phi_{m+1}) - \frac{\pi}{2}$$

$$h_m = \sqrt{(l-m)(l+m+1)}$$

Best case:

All transitions drive towards the target state

Tighter bounds for Dicke state preparation

$$\mathbb{E}\left[\frac{da_m}{dt}\right] = 0 \quad \forall m \implies \mathbb{E}\left[T_m\right] = 0 \quad \forall m$$

Tighter bounds for Dicke state preparation

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$$\bar{a}_{m_T} = \bar{\mathcal{F}}_{ss} \leq B_D = \frac{1}{1 + \sum_{m=-l}^{m_T-1} \prod_{k=m}^{m_T-1} \left(\frac{h_k \gamma}{\tilde{u}}\right)^2}$$

$$h_m = \sqrt{(l-m)(l+m+1)}$$

Tighter bounds for Dicke state preparation

$$\mathbb{E}\left[\frac{da_m}{dt}\right] = 0 \quad \forall m \implies \mathbb{E}\left[T_m\right] = 0 \quad \forall m$$

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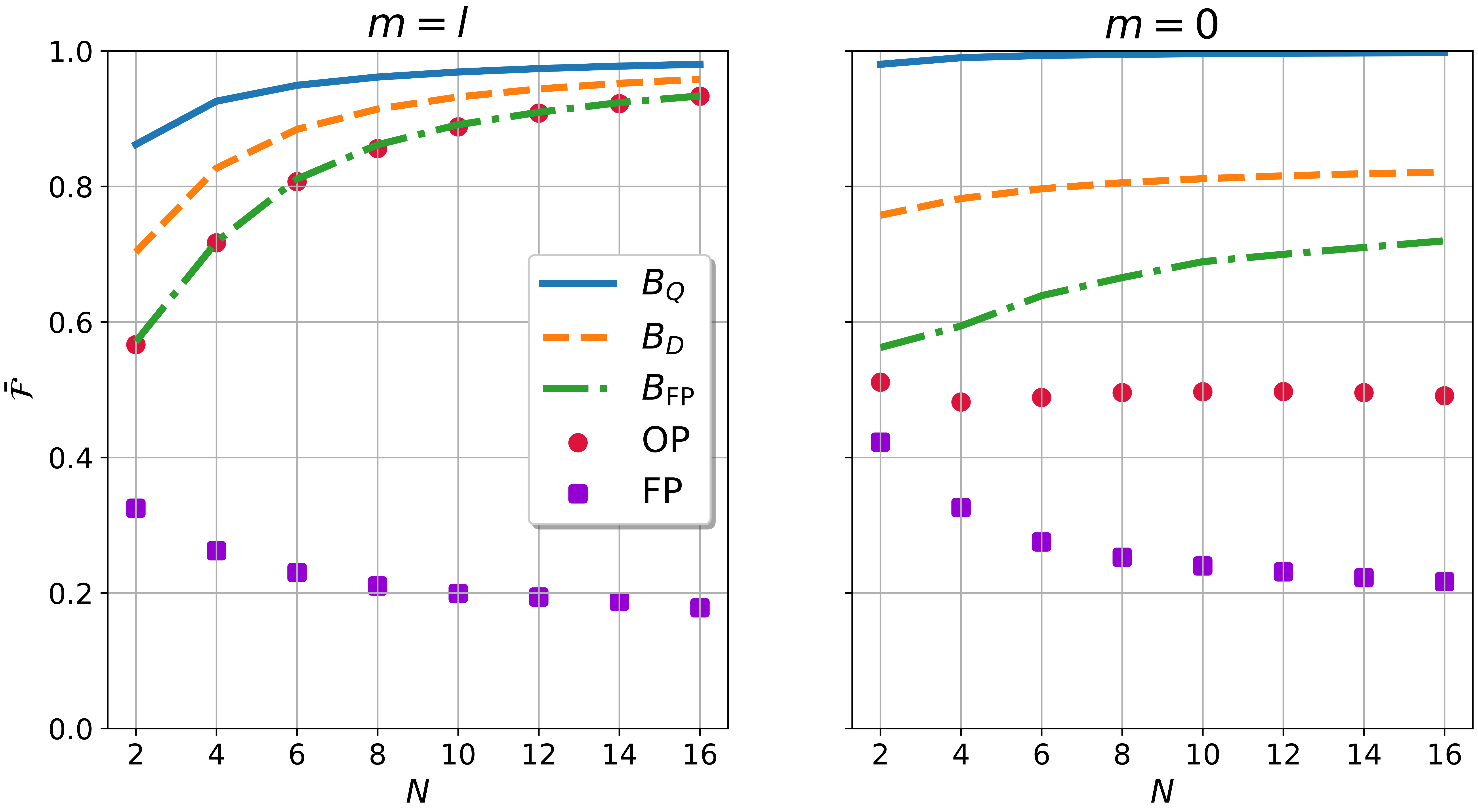
$$h_m = \sqrt{(l-m)(l+m+1)}$$

Independent of κ

Dicke states

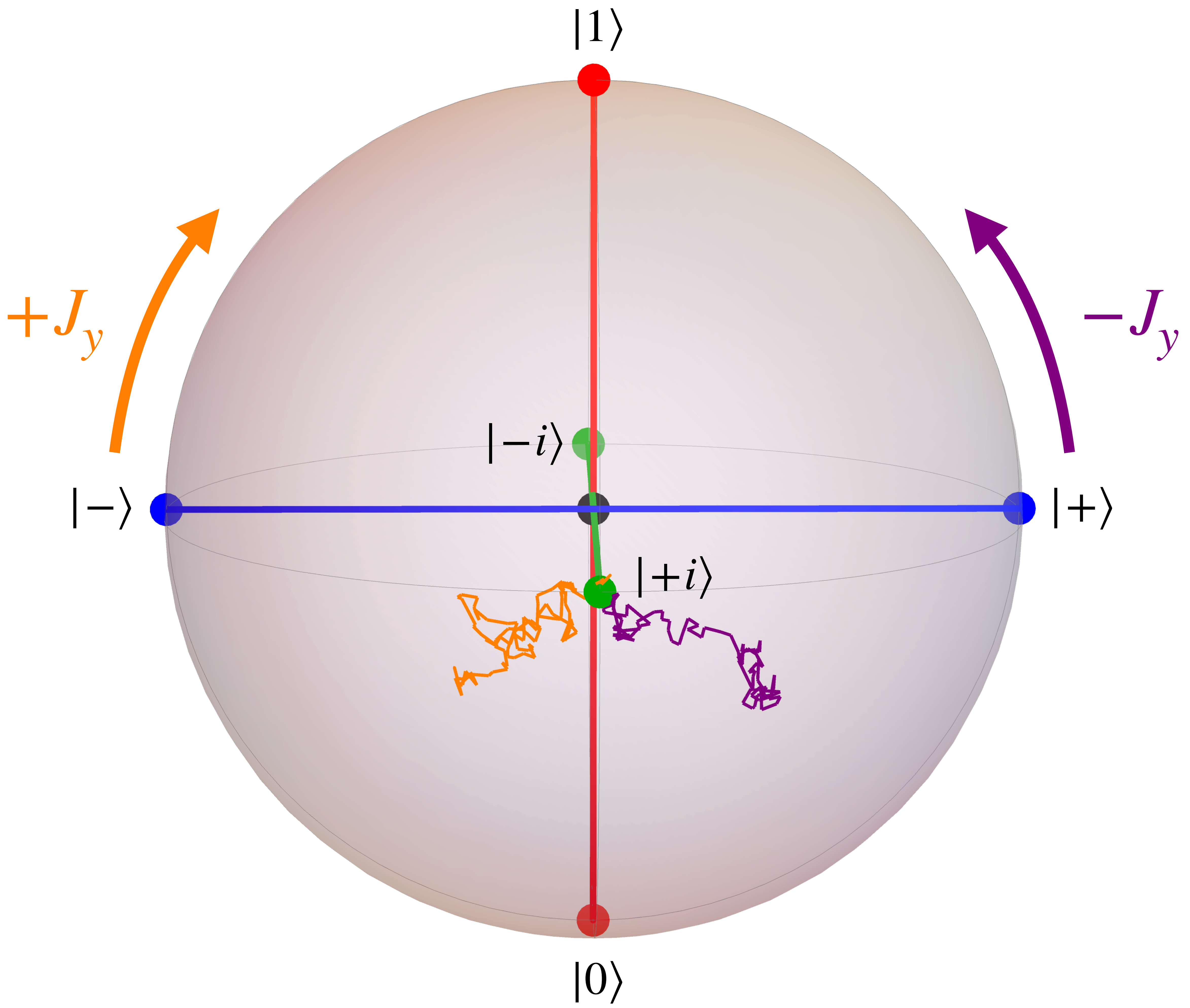


$$\frac{\kappa}{\tilde{u}} = 0.4, \frac{N\gamma}{\tilde{u}} = 0.8 \qquad \frac{\kappa}{\tilde{u}} = 2, \frac{N\gamma}{\tilde{u}} = 0.8$$

Conclusions

- Our bounds can be informative without the need for intensive numerical simulation
- They highlight scaling behaviour
- Tighter bounds require stricter constraints

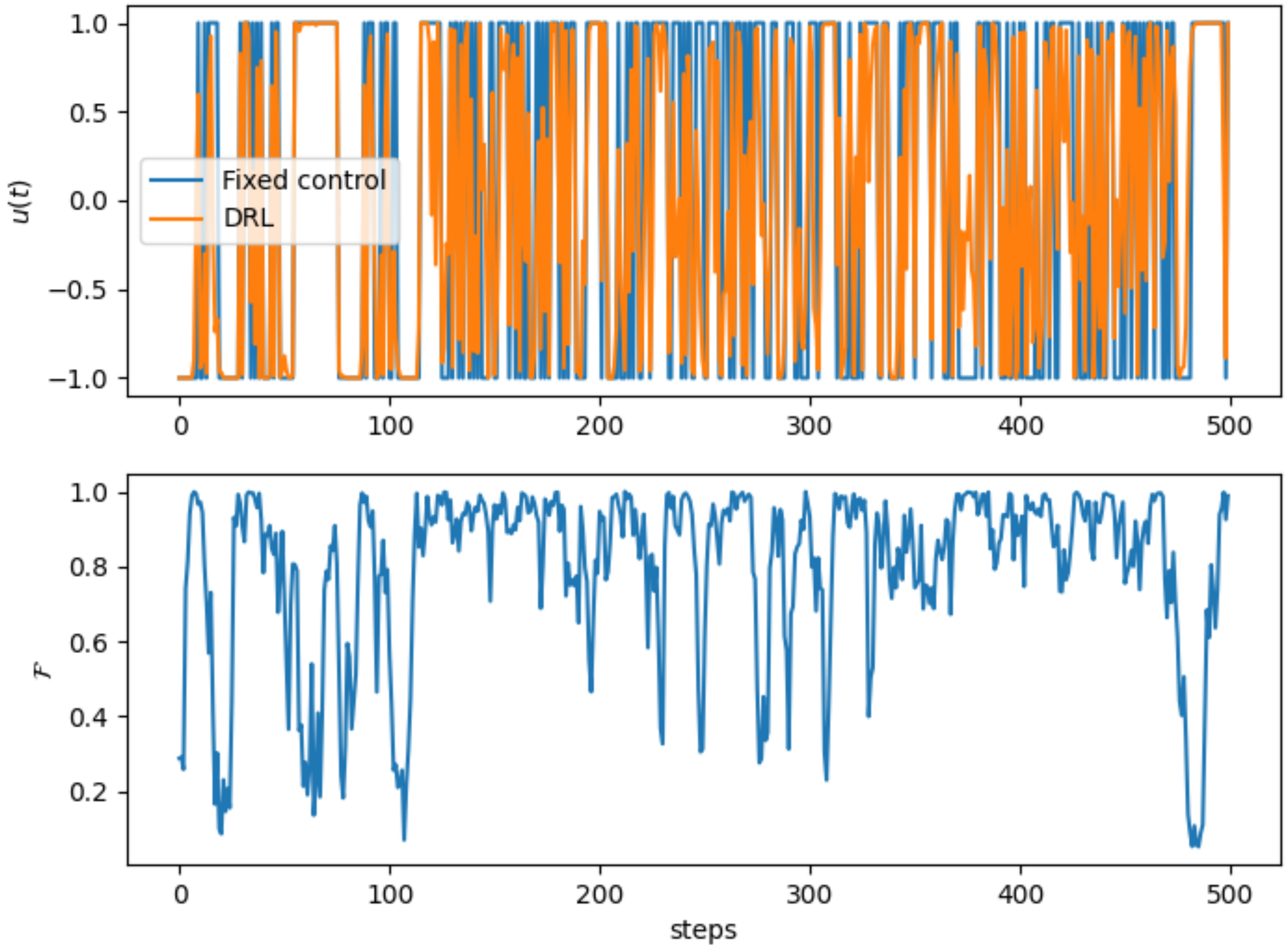
Control



The bound highlights the optimal control

$$u(t) = \mathrm{sgn}\left(\mathrm{Tr}\left[-\sigma_x \rho^{(c)}\right]\right)\tilde{u}\,.$$

Control



DRL converges to FP control