Two-Player Games

Punishing the Opponent

This exploration sheet consists of a few well-known two-player games. We will figure out a winning strategy for these games. In addition, we will figure out a way to win when the opponent makes a mistake. The challenges in this sheet are inspired from previous math circle explorations.

There are two levels of challenges to work. Choose the problem set which you like. You can also work on the Number Sandwich puzzles if you wish.

General Rules for the session:

1. You can play the games with partners, or work alone.
2. It's okay to be competitive but you must maintain decorum at all times.
3. You must not use any coding/programming during the session. You are encouraged to take up coding after the session.
4. You are NOT allowed to seek help from the internet during the session.

Number Sandwich (Source: Alex Bellos column in The Guardian)

A **number sandwich** is a line of digits such that there is one digit *sandwiched* between the 1s, two digits *sandwiched* between the 2's, three digits *sandwiched* between the 3's, and so on. For example, **[3 1 2 1 3 2]** is a number sandwich with the digits 1,2 and 3.

a) Find a number sandwich with digits 1, 2, 3, 4.
b) Find a number sandwich with digits 1 to 5.
c) Considering consecutive numbers 1 to *n*, the number sandwich is possible only for certain *n*. Find all such *n* for which a number sandwich is possible.
d) Find an algorithm to generate the number sandwich for 1 to *n*. (if it exists)
e) Is a number sandwich possible for any consecutive numbers *m* to *n*?

A **number club sandwich** is a number sandwich in which each digit appears exactly three times. The same rules as above apply: one digit is sandwiched between any two consecutive 1s, two digits are sandwiched between any two consecutive 2s, and so on.

f) Construct a number club sandwich with the digits 1 to 9. To help you out, I've placed five digits in their correct positions.

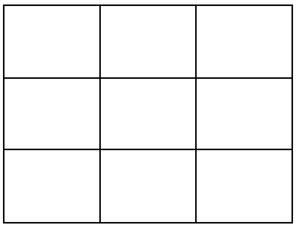
   [_ _ _ _ _ _ 4 _ 3 _ _ _ _ _ _ 5 _ 2 _ _ _ _ 1 _ _ _ _]

g) For which numbers *n*, is it possible to construct a club sandwich using numbers 1 to *n*?

I am a beginner

A player is said to have a **winning strategy** if the player can win no matter what the opponent does. We assume that both players play the best move available to them, unless specified otherwise.

1. **Tic-tac-toe**



a. Show that the first player cannot lose, if they play their best moves.
b. Show that the second player cannot lose, if they play their best moves.
c. Show that if the second player chooses any other move except their best first move, they will lose the game.
d. If the first player does not make their best move, can the second player win for sure?

2. Nim

a) There is a pile of 100 stones. Players can take 1, 2 or 3 stones from the pile in each move. The player who cannot make a move loses the game. Who has a winning strategy? Player 1 or 2?
b) Suppose there are *n* stones. For what values of *n*, will the first player have a winning strategy? For what values of *n*, will the second player have a winning strategy?
c) Suppose the players can take any number of stones from 1 to *k* in their move. For what values of *n*, will the first player have a winning strategy? For what values of *n*, will the second player have a winning strategy?
d) Suppose during the game, the player with winning strategy makes a mistake, that is, plays a move other than the best move. Can the opponent win the game for sure?

I’m familiar with Nim

There are several piles of stones. Player’s move is to take any nonzero number of stones from any one pile. The player who cannot make a move loses the game. Who has a winning strategy?

1. two piles: 20 and 20 stones

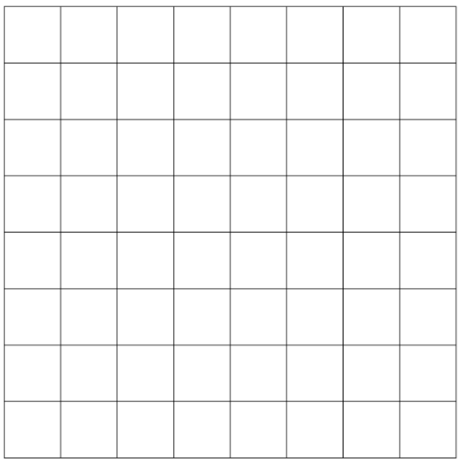
2. two piles: 30 and 20 stones

3. three piles, 20 stones each

4. four piles, 20 stones each

In each case above, if the player with the winning strategy makes a mistake, that is, plays any move other than the best move, can the opponent win for sure?

2D Nim



On a chessboard, players take turns placing a 2x1 tile. Whoever cannot place a tile loses.

a) Which player has a winning strategy?
b) For any n x n board, with 2 x 1 tiles, who has the winning strategy?
c) For any n x n board, with k x 1 tiles, who has the winning strategy?
d) If the player with winning strategy makes a mistake, can the other player win for sure?