ICTS-RRI Math Circle
Discrete Dynamical Systems

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1 Introduction

Informally, a *discrete dynamical system* can be thought of as a system which evolves by composing a fixed function f with itself again and again. Given a function f, we write $f^{(1)}(x) = f(x)$, $f^{(2)}(x) = f(f(x))$, $f^{(3)}(x) = f(f(f(x)))$, and in general, $f^{(n)}(x) = f(f^{(n-1)}(x))$. We can ask the following question: given a real number x, how does the sequence $x, f(x), f^{(2)}(x), \ldots$, behave as n increases?

Example 1.1. *An ecologist is studying the population growth of certain insects in a pond and finds out that each day the insect population becomes twice as that of the previous day. Suppose X_i is the population of the insects on the i-th day, then we can represent the ecologist's findings mathematically as*

$$X_{i+1} = 2X_i.$$

If the initial population of the insects was x_0, what is the population on the n-th day? The answer is $X_n = 2^n(x_0)$.

The purpose of this exploration is to demonstrate that discrete dynamical systems governed by very simple functions can be extremely complicated and can exhibit rich behavior.
Suppose, for a given function f, there exists a point x_0 such that after applying n-iterations of f to x_0, we get the same point x_0 back, such a point is called a periodic point (we will see a formal definition in the next section). One famous result related to periodic points says that if a continuous function has a periodic point of period 3 then it has periodic points of all periods. We will explore this in Section 3.

2 Fixed points and periodic points

The set of real numbers is denoted by $\mathbb{R}$. Let f be a function from $\mathbb{R}$ to $\mathbb{R}$.

Definition. *A point $x_0 \in \mathbb{R}$ is a called a* **fixed point** *of f if $f(x_0) = x_0$.*

Example 2.1.

1. *The function $f(x) = x^2$ has two fixed points,* 0 *and* 1.
2. *The function $g(x) = 4x(1 - x)$ has fixed points at* 0 *and* 3/4.

Problem 2.2. *In each of the following, find a function $f : \mathbb{R} \to \mathbb{R}$*

(i) such that every real number is a fixed point

(ii) such that only $x = 0$ is a fixed point.

(iii) which doesn't have any fixed points.

(iv) which has exactly 3 fixed points.

(v) which has infinitely many fixed points (but not all points are fixed points).

Sometimes, a function contains a certain parameter and the occurrence of fixed points can change depending on the range of the parameter.

Problem 2.3. *Let $q_c(x) = x^2 + c$, where c is a (real) parameter. For what range of c will q_c have fixed points? How many fixed points does q_c have in each range considered?*

As you will observe, the behavior of the function changes as the value of c decreases through a certain value. This is an example of what is called a **bifurcation**.
A natural generalisation of the notion of a fixed point is that of a periodic point.

Definition. *Given a function f and a point x_0 in its domain, the* **orbit of x_0 under f** *is defined to be the sequence $x_0, x_1, x_2, x_3, \ldots, x_n$ where $x_n = f^n(x_0)$.*

Example 2.4. *Consider the function $f(x) = x^2$.*

- *The orbit of the point $x_0 = 3$ is the set of points $3, 9, 81, \ldots, 3^{2n}, \ldots$. Thus, the orbit of 3 goes to infinity.*
- *The orbit of $x_0 = 0$ is $0, 0, 0, \ldots, 0$. Thus, the orbit of 0 stays at 0. Likewise, the orbit of 1 stays at 1.*
- *The orbit of $x_0 = \frac{1}{3}$ is the collection of points $\frac{1}{3}, \frac{1}{9}, \frac{1}{81}, \ldots, \frac{1}{3^{2n}}, \ldots$. Thus, this orbit tends to 0.*

If x_0 is a point such that the orbit of x_0 under f comes back to x_0 after n iterations, then we say that the orbit is an *n-cycle.*

Definition. *A point x_0 in the domain of a function f is called a* **periodic point** *of f of* **period** *n (or simply an n-periodic point) if $f^{(n)}(x_0) = x_0$ and n is the smallest positive integer satisfying this property. In other words, there's no $0 < k < n$ such that $f^{(k)}(x_0) = x_0$. In such a case, we say that* f has period n at x_0. *In general, we say that* f has period n *if there exists a periodic point x_0 of f of period n.*

Example 2.5.

1. *A fixed point of a function is a periodic point of period 1.*
2. *Consider $f(x) = 5x^2 - 16x + 3$. It is easy to see that f has period 2 since $f(0) = 3$ and $f^{(2)}(0) = f(f(0)) = f(3) = 5(9) - 16(3) + 3 = 0$.*

Problem 2.6.

1. *Consider the function $g(x) = 3x^2 - \frac{7}{2}x + 1$. Find the orbits of the points $x_0 = 1$ and $x_0 = 0$ respectively. Comment on the periodicity of these points.*
2. *Consider function $h(x) = \sqrt{2} - \frac{1}{x}$. Find the orbits of the points $x_0 = 1$ and $x_0 = -1$ respectively. Comment on the periodicity of these points.*

Finding all periodic points of f of order n requires us to solve the equation $f^{(n)}(x) = x$, and in general this is quite difficult. However, if we know that $k|n$, then each solution of $f^{(k)}(x) = x$ is also a solution of $f^{(n)}(x) = x$. This trick is illustrated in the next example.

Example 2.7. *Suppose f is a quadratic polynomial and we want to find periodic points of period 2. Now, every point x_0 which satisfies $f(x_0) = x_0$ also satisfies $f^{(2)}(x_0) = x_0$. That means we would like to find solutions of the equation $f^{(2)}(x) = x$ which are not the solutions of $f(x) = x$. Since $f^{(2)}(x) - x$ is a degree 4 polynomial which is divisible by $f(x) - x$, we can find $g(x) := \frac{f^{(2)}(x) - x}{f(x) - x}$, which is a polynomial of degree 2 which can then be solved to get points of period 2.*

Remark. *Recall the* division algorithm *for integers: if a and b are integers, with $b \neq 0$, then there exist unique integers q (called the quotient) and r (called the remainder) such that*

$$a = qb + r, \quad 0 \leq r < |b|.$$

The division algorithm also holds for polynomials: if $a(x)$ and $b(x)$ are two polynomials with $b(x)$ non-zero. Then there are unique polynomials $q(x)$ and $r(x)$ such that

$$a(x) = q(x)b(x) + r(x), \quad \text{with } r(x) = 0 \text{ or } degree(r(x)) < degree(b(x)).$$

This allows us to divide one polynomial with another non-zero polynomial.

Example 2.8. *Let us divide $x^3 - x^2 + x - 1$ by $x^2 + 1$. In this case we can factorize $x^3 - x^2 + x - 1 = (x-1)(x^2+1)$ and hence we get*

$$\frac{(x^3 - x^2 + x - 1)}{(x^2 + 1)} = (x - 1).$$

If we are not able to factorize and cancel common factors, we may also use polynomial long division, which works exactly like long division for integers.

Problem 2.9. *Determine the values of λ, with $0 < \lambda \leq 4$, for which the map $f_\lambda(x) = \lambda x(1-x)$ has a real-valued 2-cycle.*

Problem 2.10 (The Tent Map)**.** *Let the function $T : [0,1] \to [0,1]$ be defined by*

$$T(x) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2} \\ 2 - 2x, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

1. *Draw the graphs of the function $T(x)$ and find all of its fixed points.*
2. *Write down the function $T^2(x)$ explicitly and draw its graph. Find all 2-periodic points of T.*
3. *What does the graph of T^n look like? What can you say about the n-periodic points of T?*

We end this exploration with a remarkable theorem due to Li and Yorke.

Theorem 2.11. *Suppose f is a continuous function from an interval to itself. If f has a 3-periodic point, then it also has a k-periodic point for all $k \geq 1$.*

The theorem says that period-3 implies all periods. We may explore this result further in a session in the future. You can use this theorem to answer the following:

Problem 2.12. *Which functions considered in this exploration have periodic points of all periods?*