Maths Circle: Random Walks (Part I)

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Random Walks

Think of a particle (or an organism) moving on the set of all integers ($\mathbb{Z}$) as follows.

1. It starts at 0.
2. It tosses a coin.
3. If head appears, then it takes a step $+1$ and if tail appears, then it takes a step -1.
4. It tosses the coin once again.
5. If head appears, then it takes a step $+1$ and if tail appears, then it takes a step -1.
6. Steps 4 and 5 are repeated again and again.

In the above situation, the particle (or the organism) is said to perform a random walk. Random walks have applications to many scientific fields including physics, biology, ecology, computer science, economics, engineering. etc. The term random walk was first introduced by Karl Pearson in 1905.

For each nonnegative integer n, let S_n denote the position of the particle at time n, i.e., after n steps. This means $S_0 = 0$ always, and the values of $S_1, S_2, \ldots$ would depend on the outcomes of the tosses. For example, if the outcomes of the first five tosses are

Head, Tail, Tail, Tail, , Head

(we shall denote the above outcome by $HTTTH$), then $S_0 = 0$, $S_1 = +1$, $S_2 = 0$, $S_3 = -1$, $S_4 = -2$ and $S_5 = -1$.

Problem 1 *Write down the values of $S_0, S_1, \ldots, S_5$ when the outcomes of the first five tosses are*

(a) $HHTTT$

(b) $HTHTH$

(c) $THHTT$

(d) $HTTTT$

(e) $THTTH$

Representation of a Random Walk

Fix a positive integer n_0. We can represent the first n_0 steps of a random walk in three possible ways:

1. We can write down the sequence of outcomes (each being a head (H) or a tail (T)) of the first n_0 tosses. For example, if $n_0 = 5$, the sequence of outcomes can be $HTTHH$. This will ensure $S_0 = 0$, $S_1 = +1$, $S_2 = 0$, $S_3 = -1$, $S_4 = 0$ and $S_5 = +1$ (please check!).

2. We can give the entire path with the help of arrows:

$$0\ (= S_0) \to S_1 \to S_2 \to \cdots \to S_{n_0}.$$

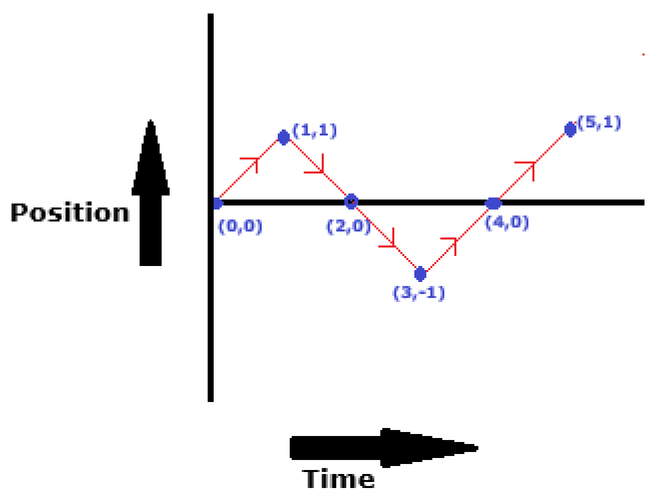
   For example, when $n_0 = 5$ and the sequence of outcomes is $HTTHH$, the corresponding random walk path can be represented as follows:

$$0 \to +1 \to 0 \to -1 \to 0 \to +1$$

3. **Graphical Representation:** We can also represent the random walk path graphically by joining the points

$$\big(0, 0\big),\ \big(1, S_1\big),\ \big(2, S_2\big),\ \ldots\ \big(n_0, S_{n_0}\big)$$

   on the Cartesian plane. Obviously, the horizontal axis represents time and the vertical axis represents the position of the particle at that time. For instance, if $n_0 = 5$ and the sequence of outcomes is $HTTHH$, then the corresponding random walk path can be represented graphically as follows:



Remark 1 Since the random walk starts at $S_0 = 0$. the above graph starts at the origin $(0, 0)$.

Remark 2 Each time a head appears, it contributes a northeast path $\nearrow$ in the graphical representation. On the other hand, a tail contributes a southeast path $\searrow$ in the graphical representation.

Remark 3 During this maths circle session, you will be asked to simulate the first $n_0 = 10$ steps of a random walk and then represent it using all of the above methods.

Problem 2 *Show the following.*

(a) If n is odd, then so is S_n.

(b) If n is even, then so is S_n.

A Basic Counting Problem

For each positive integer n and each integer k, let $N_{n.k}$ denote the total number of random walk paths from $\big(0,0\big)$ to $\big(n,k\big)$. This means that there are $N_{n.k}$ ways for the random walk starting at 0 to be at k in n steps. Clearly $N_{1,1} = N_{1,-1} = N_{2,2} = N_{2,-2} = 1$ and $N_{2,0} = 2$ (why?).

Problem 3 *Compute $N_{n,k}$ for each positive integer n and each integer k.*

Step 1: If $|k| > n$, then show that $N_{n,k} = 0$.

Step 2: If $n \not\equiv k \ (mod\ 2)$ (i.e., if $n - k$ is odd), then by Problem 2, $N_{n,k} = 0$.

Step 3: Finally, take k such that $|k| \leq n$ and $n \equiv k \ (mod\ 2)$ and compute $N_{n,k}$