Maths Circle: Random Walks (Part II)

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In the last session, we learnt about random walks and how to represent them graphically. We also observed (in Problem 2) that the position S_n of the random walk has the same parity as n. This happens because the random walk starts at 0, i.e., $S_0 = 0$. Following the same method, solve the problem below.

Problem 4 *Suppose a random walk starts at $k_0 \in \mathbb{N}$ and proceeds as before. Then show the following.*

(a) *If $n + k_0$ is odd, then so is S_n.*

(b) *If $n + k_0$ is even, then so is S_n.*

That is, $S_n \equiv n + k_0$ (mod 2).

Easy Counting Problems

Before solving Problem 3 (from the first session), let us try to solve a few easy counting problems.

Problem 5 *Suppose you have m distinguishable (i.e., non-identical) balls. Show that the number of ways of arranging these balls in a line is $m! = 1 \times 2 \times \cdots \times m$.*

Problem 6 *Now assume that you have m_1 indistinguishable (i.e., identical) red balls and m_2 indistinguishable green balls. Show that the number of ways of arranging these balls in a line is*

$$\frac{(m_1 + m_2)!}{m_1!\, m_2!}.$$

Counting Random Walk Paths

Recall that for each positive integer n and each integer k, $N_{n.k}$ denotes the total number of random walk paths from $(0, 0)$ to (n, k). This means that there are $N_{n.k}$ ways for the random walk starting at 0 to be at k in n steps. We would like to compute $N_{n.k}$ (in order to solve Problem 3 from the first session). Recall that we discussed the following steps:

Step 1: If $|k| > n$, then $N_{n,k} = 0$.

Step 2: If $n \not\equiv k$ (mod 2) (i.e., if $n - k$ is odd), then by Problem 2, $N_{n,k} = 0$.

Step 3: Finally, take k such that $|k| \le n$ and $n \equiv k$ (mod 2) and compute $N_{n,k}$.

Problem 7 *Use Problem 6 to carry out Step 3 above and solve Problem 3. More specifically, show that*

$$N_{n,k} = \begin{cases} \frac{n!}{\left(\frac{n+k}{2}\right)!\left(\frac{n-k}{2}\right)!} & \text{if } |k| \le n \text{ and } k \equiv n \ (mod\ 2), \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Note that from (1), we get the total number of random walk paths from $\big(0,0\big)$ to $\big(n,k\big)$. In other words, if it is known that $S_0 = 0$, then (1) gives the number of ways for the random walker be on the integer k at time n (i.e., $S_n = k$). Now suppose it is known that the random walker is on k_0 ($\in \mathbb{Z}$) at time n_0 ($\in \mathbb{N} \cup \{0\}$) (i.e., $S_{n_0} = k_0$). In how many ways can it reach k_1 ($\in \mathbb{Z}$) at time n_1 ($> n_0$)? This is answered in the following problem.

Problem 8 *Fix $k_0, k_1 \in \mathbb{Z}$ and $n_0, n_1 \in \mathbb{N}$ such that $n_1 > n_0$. Show that the total number of random walk paths from $\big(n_0, k_0\big)$ to $\big(n_1, k_1\big)$ is equal to*

$$N_{n_1-n_0,\, k_1-k_0} = \begin{cases} \frac{(n_1-n_0)!}{\left(\frac{n_1-n_0+k_1-k_0}{2}\right)!\left(\frac{n_1-n_0-k_1+k_0}{2}\right)!} & \text{if } |k_1-k_0| \le n_1-n_0 \text{ and } k_1-k_0 \equiv n_1-n_0 \ (mod\ 2), \\ \\ 0 & \text{otherwise.} \end{cases}$$

The importance of Problem 8 will be understood later. At this point, we shall turn our attention towards the classical definition of probability.

Classical Definition of Probability

Suppose you have an experiment whose outcome depends on chance. Such an experiment is called a random experiment. A prototypical example of a random experiment is tossing a coin - the outcome (i.e, H or T) depends on chance. Similarly, throwing a die is another example of a random experiment. Here the possible outcomes are $1, 2, \ldots, 6$.

Definition 1 *Fix any random experiment. The set Ω of all outcomes is called the sample space (for the random experiment).*

When the random experiment is tossing a coin, the sample space $\Omega = \{H, T\}$. When the random experiment is throwing a die, the sample space $\Omega = \{1, 2, \ldots, 6\}$, and so on.

Definition 2 *Any subset A of the sample space Ω is called an event.*

We typically denote events using capital letters A, B, C, etc. When the random experiment is tossing a coin, then "head appears" is an event. This event is mathematically denoted by $\{H\}$. On the other hand, when the random experiment is throwing a die, then "an even number appears" is an event. This event is mathematically denoted by $\{2, 4, 6\}$. More generally, an event A is the set of all *favourable outcomes* (i.e., the outcomes that ensure A occurs).

Problem 9 *Suppose that a particle starts at* 0 *and takes n ($\in \mathbb{N}$) random walk steps from there. Write down the sample space Ω for this random experiment and show that $|\Omega| = 2^n$. Describe an event in this case.*

For the classical definition of probability, we need to assume that the sample space Ω is finite and all the outcomes of the random experiment is equally likely. In case of tossing a coin (or thorwing a die), this is same as assumning that the coin (the die, respectively) is fair.

Definition 3 (Classical Definition of Probability) *If the sample space Ω is finite and all the outcomes are equally likely, then the probability of any event A is*

$$P(A) = \frac{|A|}{|\Omega|} = \frac{\textit{Number of Favourable Outcomes}}{\textit{Total Number of Outcomes}}\,.$$

For example, if the random experiment is throwing a fair die and E denotes the event that an even number appears, then

$$P(E) = \frac{|E|}{|\Omega|} = \frac{\text{Number of Favourable Outcomes}}{\text{Total Number of Outcomes}} = \frac{3}{6} = \frac{1}{2}\,.$$

Problem 10 *Go back to the the random experiment considered in Problem 9. Assume that all possible outcomes are equally likely. Let $E_{n,k}$ denote the event that the random walker is on the integer k at time n. Compute $P\big(E_{n,k}\big)$.*