THE ATIYAH-SINGER INDEX THEOREM

M.S.RAGHUNATHAN

1. Introduction

This is essentially an expository account of the Atiyah-Singer Index theorem, undoubtedly one of the great theorems of the twentieth century. The theorem establishes the equality of two numbers that are associated with an elliptic linear differential operator on a smooth compact manifold, one defined with the geometric data on the operator and the other by using analysis. For the precise statement of the theorem, we refer to the next ssection. We give here a proof of the theorem broadly following the lines of the original proof given by Atiyah and Singer in their announement in [AS] offering however some new ways for dealing with the various steps that lead up to the final proof. Atiyah and Singer did not publish the details of the proof indicated in their paper. The details were however worked out in the seminar conducted by Palais at the Institute for Advanced Study, the notes of which are published in the Annals of Mathematics Studies series [P].

The proof given here follows in broad outline the ideas in the Bulletin announcement; however it deviates in many ways in the matter of details indicated there (the Palais seminar follows closely the scheme set out in [AS] even in the matter of details). Odd dimensional manifolds however are dealt with exactly as in [P] - both the analytic and topological indices are zero for all *differential operators* on odd dimensional manifolds.

The first important differece is in the proof of the all important fact that the analytic index of a linear elliptic differential operator depends only on the K-theory element defined by its symbol. The proof given here makes no use of the theory of pseudo-differential operators - there is a trade off though, in that we need to appeal to some somewhat more refined results from topology - among others, to the fact that the odd dimensional sphere is rational homotopy equivalent to the real

Date: July 2, 2009.

projective space of the same dimension under the natural map.

The second point of difference is that we make use of some qualitative information on the heat kernel to prove the "bordism-invariance" of the analytic index (which is a crucial ingredient of the proof) rather than results about boundary value problems in the theory of partial differential equations as is done in [P]. Specifically we need the fact that the terms in the asymptotic expansion of the heat kernel of a second order elliptic operator are determined by the local behaviour of the operator. This is essentally a result of Minakshisundaram and Plejel [MP]. It may be remarked that the heat equation plays a crucial role in many of the different proofs of the index theorem; however the use of the heat kernel in those proofs has a very different flavour from our use of it here.

Atiyah and Singer define an equivalence relation involving cobordism on the set consisting of elliptic operators (on all smooth manifolds and vector bundles on them); they give the set of equivalence classes a natural ring structure. Then they show that both the analytic and topological indices are constant on equivalence classes of elliptic operators (this is where the "bordism invariance" mentioned in the last paragraph is needed) and the resulting $\mathbb{Q}$-valued functions are both homomorphisms of this ring into $\mathbb{Q}$. They analyse the strucure of the ring and exhibit a set of generators; and the index theorem for these generators is a consequence on the one hand of the Gauss - Bonnet theorem and on the other hand Hirzebruch's signature theorem. We replace these arguments, by an induction argument on the dimension of the manifold. This is achieved by viewing the indices as functions on (even) cohomology with coefficients in $\mathbb{Q}$ rather than on K-theory tensored with $\mathbb{Q}$ and make use of a theorem due to Serre [S](as well as an idea employed by Thom in a different context). The theorem in question asserts that if X is a finte complex of dimension n and α is a q-cohomolgy class with coefficients in $\mathbb{Q}$, and $n < 2q - 1$ then there is a map f of of X in S^q such that α is in the image of $H^q(S^q, \mathbb{Q})$ under the map induced by f.

Finally it may be remarked that the index theorem itself is formulated in [AS] for elliptic pseudo-differential operators (for which too the topological and analytic indices are defined), but once the theorem for differential operators is established the general case follows from the following considerations: in the even dimensional case, the K-theory symbol of an elliptic pseudo- differential operator (which determines both the indices) is the K-theory symbol of a suitable elliptic *differential* operator; the odd dimensional case can be reduced to the even dimensional case by forming product with the unit circle on which there is a pseudo differential operator for which analytic and topological indices are both 1 while both these indices behave multiplicatively with respect to the formation of (the external) tensor products of the symbols and the Kunneth isomorphism in K theory.

This account confines itself to differential operators. As the remarks in the last paragraph indicate this yields the theorem in the more general case of pseudo differential operators as well.

My introduction to the index theorem took place in a seminar organised by Ramanan and M.S.Narasimhan soon after the Bulletin announcement of Atiyah and Singer appeared, in which I participated. When I was thinking about that seminar (during a spell of nostalgia a few years ago) this somewhat different way of handling the proof occured to me. This paper is the result and I am happy that it appears in this volume in honour of Ramanan.

2. The statement of the theorem

2.1. **The symbol and the analytic index.** Throughout this paper (except in the Appendix) we will be working with smooth manifolds and bundles. Unless otherwise specified explicitly all maps considered will be smooth. In particular sections of vector bundles will be smooth. Let M be a smooth *compact* closed manifold of dimension m. Let E and F be complex vector bundles on M. Recall that a linear differential operator from E to F is a $\mathbb{C}$-linear map $D : \Gamma(E) \to \Gamma(F)$ such that for any section Φ of E, the support of $D(\Phi)$ is contained in the support of Φ. *All differential operators considered in this paper will be linear; so we will drop the suffix "linear" in the sequel.* If we denote by $J_k(E)$, the bundle of k-jets of E, then for all suitably large k, D defines and is defined by a bundle homomorphism of $J_k(E)$ in F. The minimal k for which this homomorphism is defined is the *order* of D. Let T (resp. T^*) be the tangent (resp.cotangent) bundle of M and $S^k(T^*)$ the k-th symmetric power of T^*; then one has an inclusion of $S^k(T^*) \otimes E$ in $J_k(E)$. If now D is a differential operator of order k then the homomorphism it defines from $J_k(E)$ in F gives by restriction a homomorphism of $S^k(T^*) \otimes E$ in F. The diagonal inclusion of T^* in $S^k(T^*)$ enables one to view this last homomorphism as a bundle homomorphism of the pull-back $p^*(E)$ of E in the pull-back $p^*(F)$ of F under the natural projection $p : T^* \to M$. This element of $Hom(p^*(E), p^*(F))$ which on each fibre of T^* is a homogeneous polynomial of degree k is the *symbol*, $\sigma(D)$ of D. It is a basic fact from the theory of linear differential operators that the kernel (resp. cokernel) of an operator D whose symbol is an injective (resp. surjective) homomorhism outside the zero section of T^* is finite dimensional. A differential operator D (from E to F) is *elliptic*, iff the symbol of D is an isomorphism outside the zero section of T^*. This means of course that E and F have the same rank. The *analytic index* $a(D)$ of D is defined as the integer $dim.(kernel(D)) - dim.(cokernel(D))$.

2.2. **The K-theoretic symbol.** We fix a Riemanniaan metric on M and dneote by B (resp. S) the unit disc (resp. sphere) sub-bundle of T^*. We denote by p the projection of T^* on M as well as its restriction to B. If now D is an elliptic operator from E to F its symbol $\sigma(D)$ defines an isomorphism of the restrictions to S of the bundle $p^*(E)$ on the bundle $p^*(F)$ (here for a bundle V on M, $p^*(V)$ is the pull-back of V to B under p). The "difference construction" in K-theory (see [P], p.15) now yields an element an $\sigma_0(D)$ in the relative K-group $K^0(B, S)$ which we will refer to as the *K-theoretic symbol* (or *K-Theory symbol*) of D in the sequel. The following properties of the K-theoretic symbol follows from its definition (via the difference construction).

Lemma 1. *If $D : \Gamma(E) \to \Gamma(F)$ and $D' : \Gamma(F) \to \Gamma(G)$ are elliptic differential operators from E to F and from F to G respectively, then $D'D$ is an elliptic operator from E to G and and $\sigma_0(D'D) = \sigma_0(D) + \sigma_0(D')$. Also if the operaotor $D(resp. D')$ is an elliptic operator from E (resp. E') to F (resp. F') and $D \oplus D'$ is the operator from $E \oplus E'$ to $F \oplus F'$defined by setting*

$$(D \oplus D')(\sigma \oplus \sigma') = D(\sigma) \oplus D(\sigma')$$

for sections σ, σ' respectively of E and E', then $D \oplus D'$ is elliptic and

$$\sigma_0(D \oplus D') = \sigma_0(D) + \sigma_0(D')$$

2.3. **The topological index.** Now, we have a natural ring homomorphism, the Chern character (see [P], p.14), Ch from $K^0(B, S)$ to the sum of even dimensional singular cohomology groups, $H^{even}(B, S; \mathbb{Q})$ of the pair (B, S) with coefficients in $\mathbb{Q}$ (which in fact gives an *isomorphism* of $K^0(B, S) \otimes \mathbb{Q}$ on $H^{even}(B, S; \mathbb{Q})$).We get thus a cohomology class $Ch(\sigma_0(D))$ in the last group. Let $Td(M)$ denote the Todd class of M as well as its pull-back in $H^*(B;\mathbb{Q})$. The cup-product gives a bilinear pairing $H^*(B,S;\mathbb{Q}) \times H^*(B;\mathbb{Q}) \to H^*(B,S;\mathbb{Q})$ and we thus obtain a cohomology class $Ch(\sigma_0(D).Td(M)$ in $H^*(B,S;\mathbb{Q})$.Now the pair (B,S) has a canonical orientation and hence $H_{2n}(B,S;\mathbb{Z})$ which is isomorphic to $\mathbb{Z}$ has a canonical generator. The *topological index* $t(D)$ of the elliptic operator D is defined as the (*rational*) number obtained by evaluating $Ch(\sigma_0(D).Td(M)$ on this generator. With these definitions and notation we can state the Atiyah-Singer index theorem.

Theorem 2. *Let D be an elliptic operator on a compact manifold from a vector bundle E to a vector bundle F. Then $a(D) = t(D)$.*

2.4. **Remarks.** Note that the theorem implies that the analytic index of D depends only on the K-theoretic symbol of D (as this is true for $t(D)$ by its very definition). This fact is proved as the first step in the proof of the theorem. Also note that the topological index which we know only to be a rational number, turns out to be an integer as a consequence of the theorem.

3. Construction of some differential operators

3.1. For a vector bundle W on M and a point x in M, W_x will denote the fibre of W at X. Let E ad F be vector bundles on M and D a differential operator from E to F. We introduce a Riemannian metric on M and denote by μ the Borel measure defined by it on M. We also intoduce hermitian inner products along the fibres of E and F. In the sequel these inner products on E and F as well as the inner products on T and T^* defined by the Riemannian metric will be denoted $<,>$. With this notation, we have the notion of the adjoint D^* of the operator: this is the unique differential operator from F to E which satifies the following condition: for sections α of E and β of F,

$$\int_M < D(\alpha), \beta > d\mu = \int_M < \alpha, D^*(\beta >) d\mu$$

That such a D^* exists and has the same order as D is a standard fact and is proved easily by integration by parts on local charts over which the bundles E and F are trivial. Also the adjoint D^{**} of D^* is D. When D is elliptic, so is D^* and one has natural isomorphisms of kernel D (resp. cokernel D) on cokernel D^* (resp. Kernel D^*). It follows that $a(D) = -a(D^*)$. In particular if F is the same as E and D is self-adjoint, i.e, $D = D^*$, then $a(D) = 0$. The symbol of the operator $\Delta = D^*D$ from E to itself assigns to each v in T^*_x, x in M, the automorphism $\sigma^*(v)\sigma(v)$ of E_x, where for an endomorphism U of E_x, x in M, U^* denotes its conjugate transpose with respect to the inner product on E_x. It follows that $\sigma(D^*D)$ is at every point of S is a Hermitian symmetric positive definite automorphism and we can therefore raise it to the power t for every t in the closed interval [0,1]. This yields a homotopy between the symbol of D^*D and the Identity automorphism of $p^*(E)$ over S. As this last automorphism (evidently) extends over all of B, we conclude (from the definition of the difference construction) that the K-theory symbol of D^*D is zero. We have the following proposition.

Proposition 3. *Given any vector bundle E on a manifold M there is a self adjoint elliptic operator Δ_E from E to E of order 2 such that $\sigma_0(\Delta_E)$ is zero (and $a(\Delta_E) = 0$).*

3.2. In the light of the disscussion in the last paragraph, to prove the proposition, we need only exhibhit operator D of order 1 from E to a suitable bundle F such that the symbol $\sigma(D)$ is an injective bundle homomorphism outside the zero section of T^*- one can then take for Δ the operator D^*D for some hermitian inner products along the fibres of E and F and a Riemannian metric on M. To construct a D of order 1, we take F to be $Hom(T, E)$ $(= T^* \otimes E)$, the bundle of E-valued 1-forms on M. We fix a hermitian inner product on E and a Riemannian metric on E; these give rise to a natural inner product on F as well. We then take D to be the exterior differentiation with respect to a unitary connection ω on E. One sees then that for this operator Δ, the symbol associates to each v in T_x^*, x in M, the automorphism $|| \, v \, ||^2$.Identity.

3.3. For a vector bundle V and a non-negative integer r, we denote by $r.V$, the direct sum of r copies of V. A differentail operaator D from E to another vector bundle F evidently defines a differential operator $r.D$ from $r.E$ to $r.F$ which on each component of $r.E$ is the operator D from E to F, the latter considered as the corresponding component of F. It follows from the definition of the K-theoretic symbol that $\sigma_0(r.D) = r.\sigma_0(D)$. With this notation we will now establish the following.

Proposition 4. *For a positive integer n, let $\lambda(n) = 2^{2n+2}$. Then given any vector bundle E on a manifold M of dimension m there is an elliptic operator D_E of order 1 from $\lambda(m).E$ to itself such that $a(D_E) = 0$ and $\sigma_0(D_E) = 0$.*

3.4. Let $I' : M \to \mathbb{R}^{2m+1}$ be an imbedding and I the imbedding in $\mathbb{R}^{2m+2}$ obtained by composing the standard inclusion of $\mathbb{R}^{2m+1}$ in $\mathbb{R}^{2m+2}$ with I'. We have then an inclusion of T in the trivial rank $(2m+2)$ real vector bundle on M and hence (using the standard inner product on $\mathbb{R}^{2m+2}$) an inclusion of T^* in the trivial real vector bundle of rank 2m+2. Now let V denote the trivial complex vector bundle of rank 2m+2. As I factors through the inclusion of R^{2m+1} in R^{2m+2}, we note that T^* is contained in a trivial (complex) vector sub-bundle V' such that V is the direct sum of V' and a trivial line bundle equipped with an everywhere non-zero section s. Now let x in M and v an element of T^*_x. Define a homomrphism $e(v)(resp.i(v)) : T^*_x \otimes \Lambda^p(V)_x :\to \Lambda^{p+1}(V)_x$ (resp. $i(v) : T^*_x \otimes \Lambda^p(V)_x :\to \Lambda^{p-1}(V)_x$) (where for a non-negative integer l, $\Lambda^l(V)$ is the lth exterior power of V) is the exterior (resp. interior) multiplication by v (note that T^*_x is a subspace of V_x and V being the trivial bundle carries a natural inner product). Let $\Lambda^e(V)$ (resp. $\Lambda^o(V)$) be the direct sum of the even (resp. odd) exterior powers of V. Then $e(v) + i(v)$ as v varies in T^* defines homomorphisms of $T^* \otimes V^e$ in V^o and $T^* \otimes V^o$ in V^e which we denote σ^e and σ^o respectively in the sequel. If now E is any vector bundle on M, $\sigma^e \otimes (Identity)$ is a homomorphism of $p^*(V^e \otimes E)$ in $p^*(V^o \otimes E)$ which is an isomorphism outside the zero-section: this is because $\sigma^o \otimes (Identity) \cdot \sigma^e \otimes (Identity)$ is the endomorphism $|| \, v \, ||^2 \, .(Identity)$ of $p^*(V^e \otimes E)$ (here v is an element of T^*_x, x in M). Now if we introduce a hermitian inner product on E and a connection on E compatible with it, one sees that $\sigma^e \otimes (Identity)$ and $\sigma^o \otimes (Identity)$ can be lifted to (elliptic) differential operators D_E and D^*_E which are adjoints of each other. It follows that the direct sum $\widehat{D}$ of D_E and D^*_E gives an elliptic self adjoint operator of order 1 from $\Lambda(V) \otimes E$ ($\Lambda(V)$ is the exterior algebra bundle of V) to itself. Moreover $t(e(v) + i(v)) + (1-t)(e(s(x)) + i(s(x)))$, v in T^*_x, x in M, gives a homotopy over S of the symbol of $\widehat{D}$ with an isomorphism which extends over all of T^*. This proves the proposition: since V is a trivial bundle of rank $2m+2$, $\Lambda(V)$ is trivial of rank $\lambda(m)$ and $\Lambda(V) \otimes E$ is isomorphic to a direct sum of $\lambda(m)$ copies of E.

4. K-theoretic symbol and the analytic index

Our aim in this section is to establish the following:

Theorem 5. *Let D and D' be elliptic operators from vector bundles E and E' to F and F' respectively. If the K-theoretic symbols of D and D' are equal, then $a(D) = a(D')$.*

4.1. We will first prove the following much weaker statement:

Proposition 6. *Suppose given two elliptic differential operators D and D', both from E to F of the same order k such that there is a smooth homotopy σ_t, t in [0,1], of isomorphisms of $p^*(E)$ on $p^*(F)$ such that each σ_t is homgeneous polynomial of degree k along the fibres of T^* then $a(D) = a(D')$. In particular if D and D' have the same symbol, $a(D) = a(D')$.*

4.2. For any vector bundle W on M with a Hermitian inner product and a non-negative integer k, one defines a pre-Hilbert struture on $\Gamma(W)$ with the inner product$<,>_k$ defined as follows. Fix once and for all a finite covering $\{U_i \mid i \in I\}$ of M by cooordinate open sets and isomorphisms of the restrictions of E and F to the U_i, with trivial bundles. Fix a shrinking $\{V_i \mid i \in I\}$ of $\{U_i \mid i \in I\}$. Then each section of W defines a collection $\mathbf{F} = \{F_i \mid i \in I\}$, each F_i being a $\mathbb{R}^n$ valued smooth function on V_i. For a pair $\mathbf{F}$, **F'** of sections of W, let $< \mathbf{F}, \mathbf{F}' >_k = \sum_{i\in I,\ |\alpha|\leq k} < \partial^\alpha F_i/\partial x^\alpha, \partial^\alpha/\partial x^\alpha >$. The completion of this pre-Hilbert strucure on $\Gamma(W)$ is denoted $H_k(W)$. The topological vector space structure on $H_k(W)$ is independent of the choice of the covering, its shrinking and the trivialisations of W over the open sets of the covering.

We now fix vector bundles E and F on M. With the notation introduced above, one then sees that a differentaial operator $D : \Gamma(E) \to \Gamma(F)$ of order k extends to a continuous linear map of $H_k(E)$ in $H_0(F)$ and when D is elliptic, there is a constant $C = C(D) > 0$ such that for all σ in the orthogonal complement of kernel$D$, one has $|| D(\sigma) ||_0 \geq C(D).\ || \sigma ||_k$. Suppose now that D_t, t in [0,1], is a smooth family of elliptic operators. Let t_0 in [0,1] and let H' be the orthogonal complement of kernel D_{t_0} in $H_k(E)$ Then there is a neighbourhood U of t_0 in [0,1] and a constant $C > 0$ such that $C(D_t) \geq C$ for all t in U. From this it follows easily that $a(D_t)$ is independent of t. Now if D and D' have the same symbol, $t.D+(1-t)D'$, t in [0,1] provides a smooth family of elliptic operators leading to the conclusion that $a(D) = a(D')$. Now since the inclusion of the pull-back of $S^k(T^*) \otimes E$ to $M \times [0,1]$ in the pull-back of $J^k(E)$ admits a right inverse vector bundle morphism, we see that given σ_t as in the statement of the proposition, we can find a smooth family of (elliptic) operators D_t, t in [0,1] such that $\sigma(D_t) = \sigma_t$. It follows now that $a(D) = a(D')$.

4.3. Suppose now that D is an elliptic differential operator from a vector bundle E to a vector bundle F. It is obvious that $t(r.D) = r.t(D)$ and $a(r.D) = r.a(D)$ for any non-negative integer r. Thus equality of $a(D)$ and $t(D)$ holds if it holds for $a(r.D)$ and $t(r.D)$ for some integer r greater than zero. Observe next that if D' is an elliptic operator from F to a vector bundle G of the same rank then $a(D'D) = a(D') + a(D)$ and $t(D'D) = t(D') + T(D)$. Now suppose that D (resp. D')is an elliptic operaor from E to F (resp. from E' to F') of order k (resp. k') are such that their k-theoretic symbols are the same. We want to show that they have the same analytic index. Observe first that we may replace the operators D and D' by $r.D$ and $r.D'$ for any positive integer r and by choosing $r = \lambda(n)$ as in the proposition above and composing with a suitable power of D_F (resp. D'_F), we may assume that $k = k' = 2l$ is an even integer. Let W (resp. W') be a vector bundle such that the direct sum of E (resp. E') and W (resp. W') is trivial of rank q. Let Δ_E and Δ'_E) be as in Proposition 3 and let L and L' denote their respective lth powers. Let $\widehat{D}$ (resp. $\widehat{D}'$) be the direct sum of D (resp. D') and L (resp. L'). Clearly $\widehat{D}$ and $\widehat{D}'$ have the same K-theoretic symbol and we need to prove that their analytic indices are the same.

4.4. This means that we can assume that we are in the following situation. Let E denote the trivial bundle of rank q and D (resp. D') be an elliptic operator from E to a rank-q vector bundle F (resp. F') of even order $2l$ such that $\sigma_0(D) = \sigma_0(D')$. We have to show that $a(D) = a(D')$. Since E is trivial, so is F (resp F') over S. This means that the bunndles F and F' are stably isomorphic: this follows from the fact that the K-theoretic symbols of D and D' obtained by the difference construction are the same. Using Proposition 4 for the trivial bundle, one finds that by forming direct sum with suitable number of copies of the trivial line bundle **1** equipped with a suitable power of $\Delta_{\mathbf{1}}$, we may assume that both D and D' are elliptic operators from the (trivial) bundle E to the *same bundle* F.

4.5. That the two operators have the same K-theoretic symbol implies the following: there exists a positive integer N and an automorphism $u : E \oplus 1_N \to E \oplus 1_N$ such that $\Sigma_1 = \sigma(D) \oplus Id_N).u$ and $\Sigma_2 = \sigma(D') \oplus Id_N$ are homtopic through a 1-paramaeter family Σ_t, t in $[0, 1]$ of sections of $Hom(p^*(E) \oplus 1_N, p^*(F) \oplus 1_N)$ over S, each Σ_t being an isomorphism. Σ_1 and Σ_2 are both symbols of differential operators

D_1 and D_1' respectively of order $k = 2l$ (see Proposition 3 and the discussion in 3.2) with the same K-theoretic symbols as D and D' respectively; further $a(D_1) = a(D_1')$ as u, a 0^{th} order operator has index 0. Now let P denote the real projective space bundle associated to T^*: this is a quotient of S by the involutive automorphism which maps each unit vecor in S into its negative. Let q denote the natural projection of P on M. Then since the order k of D and D' is even, $\sigma(D)$ and $\sigma(D')$ are pull backs of sections σ and σ' of $Hom(q^*(E), q^*(F))$. It is shown in the Appendix that the existence of the homotopy σ_t implies that replacing D_1 and D_1' by $r.D_1$ and $r.D_1'$ we can assume that Σ_1 and Σ_2 are homotopic through sections of $Hom(q^*(E), q^*(F))$ which are all isomorphisms of $q^*(E)$ on $q^*(F)$. Equivalently, we can assume that the homotopy Σ_t above is such that for v in T_x^*, x in M, $\Sigma_t(v) = \Sigma_t(-v)$.

4.6. Fix some extension of the homotopy Σ_t to a homotopy of sections over all of T^* of $Hom(p^*(E), p^*(F)$ and denote this extension by $\tilde{\Sigma}_t$. We assume that the $\tilde{\Sigma}_t$ vanish outside a compact neighbourhood of S in T^*. Now convolving $\tilde{\Sigma}_t$ with the function $(c/\pi)^{1/2} exp(-c \mid\mid (.) \mid\mid^2$ on T^* along the fibres of T^* we obtain a homotopy $\tilde{\Sigma}_{(c,t)}$ such that on every fibre of T^*, it admits a Taylor expansion which converges uniformly on compact sets. As c tends to ∞, $\tilde{\Sigma}_{(c,t)}$ converges uniformy on compact sets to $\tilde{\Sigma}_t$. It follows that for all large c, $\tilde{\Sigma}_{(c,t)}$ is invertible over S as also suitable a truncation $\theta_{(c,t)}$ of its Taylor expansion (along the fibres). Let $\tau_{(c,t)} = (\theta_{(c,t)} + \theta_{(c,-t)})/2$. Since Σ_t is an even function on the fibres of T^*, we can choose c and the truncation so that $\tau_{(c,t)}$ is invertible over all of S and $\tau_{(c,0)}$ (resp. $\tau_{(c,1)}$) equals $\tilde{\Sigma}_{(c,0)}$ (resp.$\tilde{\Sigma}_{(c,1)}$. We replace $\tilde{\Sigma}_t$ now by such a $\tau_{(c,t)}$ and denote the latter by Σ_t in the sequel.

4.7. It is clear that Σ_t is now a polynomial along the fibres of T^* with every homogeneous component of even degree. It can therefore be made homgeneous with the function $\mid\mid . \mid\mid^2$ on T^* (which is homogeneous of degree 2 along the fibres of T^* and is identically 1 on S. In other words we can aassume that the homotopy σ_t is a homogeneous polynomial function along the fibres of T^* of fixed degree. This may mean however that the degrees of Σ_0 and Σ_1 have been raised. However from Proposition 4 we know that the composites of $\Delta_{F \oplus 1_N}$ raised to any power with D_1 and D_1' have the same analytic index as well as K-theoretic symbol as D and D' respectively. We are now in a situation where the homotopy between the symbols of D and D' is through isomorphisms of $p^*(E)$ on $p^*(F)$ all of which are homogeneous polynomials of the same even degree. Theorem 5 now follows from Proposition 6.

5. The signature operators

In this section we will describe the so called twisted signature operators.

5.1. Let M be a manifold of dimension m. We fix a Riemannian metric on M which gives rise to a Borel measure μ on M. Let Ω_M denote the line bundle of exterior m-forms on M. When there is no ambiguity about the underlying manifold M, we denote Ω_M simply Ω. The metric gives a canonical reduction of the strcure group $\mathbb{R}^*$ of the line bundle Ω to the subgroup (1,-1) so that a 2-sheeted cover of M imbeds in Ω, the two sheeted covering being trivial or non-trivial according as M is orientable or not; in the former case an orienttion gives a natural trivialisation of the covering. This reduction of structure group leads also to a canonical isomorphism of $\Omega \otimes \Omega$ with the trivial bundle $M \times \mathbb{R}$. If the manifold is oriented, the metric gives a natural trivialisation of Ω. For an integer p with $0 \leq p \leq m$, let Ω^p denote the p-th exterior power of T^*;$\Omega^m = \Omega$. The exerior multiplication then defines a pairing $\Omega^p \otimes \Omega^{m-p} \to \Omega$ which is non-degenerate and hence gives a non-degenerate pairing of Ω^p with $\Omega^{m-p} \otimes \Omega$. Using the isomorphism of Ω^{m-p} with its dual given by the Riemannian metric, we obtain a bundle isomorphism of Ω^p on $\Omega^{m-p} \otimes \Omega$ which is denoted $*$ in the sequel. The natural isomorphism of $\Omega \otimes \Omega$ on the trivial bundle yields (by taking tensor product of with the Identity morphism of Ω) an isomorphism of $\Omega^p \otimes \Omega$ on Ω^{m-p} which will also be denoted $*$. One then finds that $*^2$ on Ω^p equals $(-1)^{p(m-p)}$. Suppose now that α and β are two p-forms on M; then $\alpha \wedge * \beta$ is a section of $\Omega \otimes \Omega$ and - as this last line bundle is naturally isomorphic to the trivial bundle - is a function on M. On the other hand the Riemannian metric gives an inner product denoted $<,>$ on the Ω_p. One checks easily that $<\alpha, \beta> = \alpha \wedge * \beta$. For α, β as above we set

$$(\alpha, \beta) = \int_M <\alpha, \beta> d\mu$$

5.2. Suppose now that E is a complex vecor bundle with a hemitian inner product along its fibre. Let ω be a unitary connection on E. We then have the differential operator of order 1, viz., exterior differentiation with respect to ω of E-valued p-forms on M - these are nothing but sections of the bundle $\Omega^p \otimes E$.

$$d_\omega : \Gamma(\Omega^p \otimes E) \to \Gamma(\Omega^{p+1} \otimes E)$$

The hermitian inner product on E gives a conjugate-linear isomorphism of E on the dual E^*. Forming the tensor product of this isomorphism with $*$ we obtain a conjugate linear isomorphism of $\Omega_p \otimes E$ on $\Omega_{m-p} \otimes \Omega \otimes E^*$ which also we denote $*$.

$$* : \Omega^p \otimes E \to \Omega^{m-p} \otimes \Omega \otimes E^*$$

Now, if α (resp. β) is a p-form (resp. $(m-p)$-form) on M with values in E (resp. E^*), then using the pairing between E and E^*, we get a m-form $\alpha \wedge \beta$ with values in $\Omega \otimes \Omega$. One checks easily that for $p-forms$ α, β with values in E, $< \alpha, \beta >$ (the inner product on $\Omega^p \otimes E$ is the one deduced from that given by the Riemannian metric on M and the Hemitian inner product on E) equals $\alpha \wedge * \beta$ (treated as a function via the canonical trivialisation of $\Omega \otimes \Omega$). It follows moreover from Stoke's theorem that for a p-form α and a $p+1$-form β, we have, setting $\delta_\omega = (-1)^{p+1} * d_\omega *$,

$$\int_M < d_\omega \alpha, \beta > d\mu = \int_M < \alpha, \delta_\omega \beta > d\mu$$

Thus we see that δ_ω is the adjoint of d_ω. Let Ω^e (resp. Ω^o the direct sum of the even (resp. odd) exterior powers of T^*. Then $d_\omega + \delta_\omega$ is an elliptic differential operator (of order 1) from $\Omega^e \otimes E$ to $\Omega^o \otimes \Omega \otimes E$. In the sequel we denote this operator $D_{E,deRh}$ and refer to it as the de Rham - Hodge operator for E (or the E-twisted de Rham - Hodge operator). When E is the trivial line bundle, we drop the E in the suffix and refer to it simmply as the de Rham - Hodge operaator.

5.3. When the dimension m of M is even, equal to $2n$, say, there is another way of decomposing the bundle ΛT^* of (all) exterior differential forms into a direct sum of two vector bundles Ω^+ and Ω^-: if we define $'* : \Omega^p \to \Omega^{m-p} \otimes \Omega$ by setting $'*(\xi) = i^{p(p-1)} * (\xi)$, one checks easily that $('*)^2 = (-1)^n$ (here i is a square root of -1 fixed once and for all). It follows that when n is even (resp. odd), that ΛT^* decomposes as a direct sum of two sub-bundles Ω^+ and Ω^- where Ω^+ (resp. Ω^-)is the eigen-sub-bundle of ΛT^* corresponding to the eigen-value θ (resp. $-\theta$) for $'*$, θ being 1 or i according as n is even or odd. The operator $d_\omega + \delta_\omega$ is then seen to define an elliptic operator from $\Omega_+ \otimes E$ to $\Omega_- \otimes E$ (which will be denoted $D_{E,\omega}$ in the sequel) is the *E-twisted signature operator* on M. We have then the following crucial fact from K-theory. For a proof see [P], pp. 225-26.

Theorem 7. *Let $Th : K^0(M) \to K^0(B, S)$ be the homorphism defined by setting $Th(E) = \sigma_0(D_{E,\omega})$ for a vector bundle E on M. Then Th gives an isomorphism of $K^0(M) \otimes \mathbb{Q}$ on $K^0(B, S) \otimes \mathbb{Q}$.*

Note that for a vector bundle E, $Th(E)$ depends only on the K-theory class of E and is independent of the choice of the hermitian metric and the connection on E and defines a homorphissm of $K^0(M)$ into $K^0(B, S)$. Theorems 5 and 7 together show that to prove Theorem 2, it suffices to prove that $a(D_{E,\omega}) = t(D_{E,\omega})$ and this is what will be done in the rest of this paper.

6. "Bordism Invariance" of the index

We adopt the following notation in the sequel. For a vector bundle E on a mnaifold M, $a(E)$ (resp.$t(E)$) will be the analytic (resp. topological) index of the E-twisted signature operator $D_{E,\omega}$. This notation is justified as the indices in question are independent of the connection ω as the K-theoretic symbol of $D_{E,\omega}$ itself is independent of ω. We then have:

Theorem 8. *Suppose now that M and M' are two smooth manifolds and E and E' are vector bundles on them. Suppose further that there is a manifold with boundary W such that the boundary of W is the disjoint union of M and M' and there is a vector bundle $\mathbf{E}$ on W such that $\mathbf{E}$ restricts to E (resp. E') on M (resp. M'). Then $a(E) = a(E')$ and $t(E) = t(E')$.*

6.1. For the result for the topological index which is easy to prove, see [P], p.228. For the analytic index we will make use of some results about the heat kernel essentially due to Minakshisundaram and Plejel [MP]. We recall these now. The operator $D_{E,\omega}$ has for its adjoint the operator $D_{E\otimes\Omega,'\omega}$ with $'\omega$ denoting the connection on $E \otimes \Omega$ obtained from ω and the flat connection on Ω. We set $'E = E \otimes \omega$ in the sequel.Then $D_{'E,'\omega}D_{E,\omega}$ (resp. $D_{E,\omega}D_{'E,'\omega}$) is an elliptic second order operator of $\Omega_+ \otimes E$ (resp.$\Omega_- \otimes' E$) to itself which we denote simply Δ^+ (resp. Δ^-) in the rest of this section. For $t \geq 0$, the operator $exp(-t\Delta^+)$ (resp. $exp(-t\Delta^-)$ has a smooth kernel $K^+(t, x, y)$ (resp.$K^-(t, x, y)$) a smooth section of $Hom(p_1^*(\Omega^+ \otimes E), p_2^*(\Omega^+ \otimes E))$ (resp. $Hom(p_1^*(\Omega^- \otimes' E), p_2^*(\Omega^- \otimes' E))$ which depends smoothly on the parameter t as well. Further $K^+(t, x, x)$ (resp. $K^-(t, x, x)$) has an asymptotic expnasion $\sum_{0\leq r\leq\infty} t^{r-m/2}K^+_{r-m/2}(x, x)$ (resp. $\sum_{0\leq r\leq\infty} t^{r-m/2}K^-_{r-m/2}(x, x)$) as t

goes to zero. The analytic index of $D^+_{E,\omega}$ is then equal to

$$\int_M Trace(K_0^+(x,x))d\mu(x) - \int_M Trace(K_0^-(x,x))d\mu(x)$$

The crucial fact we need about the asymptotic expansion is the following. If E and E' are two vector bundles with hermitian inner products $<,>$ and $<,>'$ and unitary connections ω and ω' respectively such that on an open set U of M, we have a Hermitian isomorphism of the restriction of E on the restriction of E', which carries ω into ω', then on $U \times U$, all the terms in the asymptotic expansions of the two heat kernels coincide. In particular, the function $Trace(K_0^+(x,x)) - Trace(K_0^-(x,x))$ on M (which we will denoe T_E in the sequel) is determined in the neighbourhood of any point by the local data on the Riemannian metric, the Hermitian inner product on E and the unitary connection in that neighbourhood. T_E is naturally a section of $\Omega \otimes \Omega$ and so is to be regarded as a closed m-form on M with values in Ω.

6.2. Since any cobordism between manifolds is a composition of successive elementary cobordisms, we may assume for proving Theorem 8 that W is an elementary cobordism. In other words, there is smooth function $f : W \to [-1,1]$ such that $M = f^{-1}(-1)$, $M' = f^{-1}(1)$ and f has exactly one non-degenerate critical point w (of index r, say) with $f(w) = 0$. One then has an open neighbourhood U of w in W and a diffeomorhism F of of an open disc $D(4\epsilon), \epsilon > 0$, of radius 4ϵ around the origin 0 in $\mathbb{R}^m$, on U with $F(0) = w$ and such that f composed with F is the fuction Q defined on $D(4\epsilon)$ as follows: for a suitable decomposition of $\mathbb{R}^m$ as an orthogonal direct sum of $\mathbb{R}^r$ and $\mathbb{R}^s$ $(r+s=m)$, denoting by $x_1, x_2, ...x_r$ (resp. $y_1, y_2, ...y_s$) the coordinates in $\mathbb{R}^r$ (resp. $\mathbb{R}^s$) one has

$$Q(x_1, x_2....x_r, y_1, y_2, ...y_s) = \sum_{1\le i\le s} y_i^2 - \sum_{1\le j\le r} x_j^2.$$

.

For any $c > 0$, let $D(c)$ denote the disc of radius c around the origin in $\mathbb{R}^m$. Introduce a Riemannian metric $\mathbf{g}'$ on W such that it induces the standard metric on $D(3\epsilon)$ under the map F. Let X be the vector field on $W' = W \setminus \{w\}$ such that $< X, Y >= 0$ for all vector fields Y on W' with $Yf = 0$ and $< X, X > = 1$. The local 1-parameter group ϕ_t of local diffeomorphisms of W determined by X then defines for $0 < c < 3\epsilon$ a smooth map $\Phi : (M \setminus F(D(c))) \times [0,1] \to W$ given by $\Phi(x,t) = \phi_{2t}(x)$ which is a diffeomorphism onto a closed subset $W(c)$ in

W; the interior of the manifold with boundary $(M \setminus F(D(3\epsilon))) \times [0,1]$ maps onto the interior of $W(c)$ as a subset of W . This last open set is also diffeomorphic to $(M' \setminus F(D(c))) \times [0,1]$ under the map that takes (x', t) to $\phi_{-2t}(x')$. We now replace the metric $\mathbf{g}'$ by a metric $\mathbf{g}$ which coincides with $\mathbf{g}'$ on $F(D(\epsilon))$ and induces on $(M \setminus F(D(2\epsilon))) \times [0,1]$ a product Riemannian metric under the map Φ - note that $F(D(\epsilon))$ and $W(2\epsilon)$ have disjoint closures. The vector bundle $\mathbf{E}$ on W when pulled back to $(M \setminus F(D(\epsilon))) \times [0,1]$ is necessarily isomorphic to the pull back of a bundle on $(M \setminus F(D(\epsilon)))$ under the cartesian projection on $(M \setminus F(D(\epsilon)))$. We assume the Hermitian inner product to be the pull back of one on the bundle on $(M \setminus F(D(\epsilon)))$. We note that the complement of $W(3\epsilon)$ in W is contaied in $F(D(4\epsilon))$ and hence $\mathbf{E}$ is trvial over this copmlement. Observe that if ν denotes the unit inward normal field along the boundary, the interior product with ν yields isomorphism of) the bundle Ω_W of $m+1$-forms on W restricted to M (resp.M') on Ω_M (resp. Ω'_M), the bundle of m-forms on M (resp. M'), which in addition is compatible with the flat connections on thsee bundles. Now consder the closed Ω_M-valued m-form T_E on M and let α denote the pull-back of its restriction to $M \setminus F(D(3\epsilon))$ to $W(3\epsilon)$ (identified with $(M \setminus F(D(3\epsilon))) \times [0,1]$ via Φ). Then α restricted to $\Phi((M \setminus F(D(3\epsilon))) \times 1 = M' \setminus F(D(3\epsilon))$ is the same as the restriction of T'_E (on M'). Let α be a m-form on W which equals α on $W(3\epsilon)$, equals T_E on M and T'_E on M'.
Applying Stoke's theorem to the Ω_W-valued m form α, we get:

$$\int_{M'} \alpha - \int_M \alpha = \int_W d\alpha.$$

Now since the form α is closed, so is its pull back to $W(3\epsilon)$.) It follows that the integral on the right hand side equals the integral over $F(D(4\epsilon))$. Now there is an elementary cobordism V between the sphere S^m and $S^r \times S^s$ admitting a Morse function $g : V \to [-1,1]$ with exactly one non-degenerate critical point of index r admitting a neighbourhood with properties entirely analogous to the neighbourhood U of w in W. Let $\mathbf{F}$ (resp. F, F') be the trivial bundle of rank equal to rank-E. Then we see that the same arguments as above shows that the integral above over $F(D(4\epsilon))$ is equal to

$$\int_{S^r \times S^s} T'_F - \int_{S^m} T_F$$

and this is zero as the analytic index of the signature operator for the trivial bundle on the product of two spheres is zero as can be checked easily (using the Hodge- De Rham theorem). This proves Theorem 8.

7. Proof of the Atiyah - Singer theorem

In this section we complete the proof of the theorem. We first dispose off the case of odd dimensional M in the same way as is done in [P]: we show that both the analytic and topological indices vanish.

7.1. The odd dimensional case. When M is odd dimensional the asymptotic expansion of the heat kernel is in odd powers of $t^{1/2}$ so that for any vector bundle E on M (with a unitary connection ω), T_E is zero. Hence $a(D_E) = 0$. On the other hand $t(D_E)$ is zero as the K-theoretic symbol of any elliptic differential operator itself is zero. This is seen as follows. As we have seen we may assume that the operator has even degree. But this means that the symbol and hence also the K-theoretic symbol is invariant under the bundle automorphism $-(Identity)$ while this automorphism induces the map $-(Identity)$ on $K^0(B,S)$ as the antipodal map on *even* dimensional spheres is orientation reversing.

7.2. The case of spheres. When M is an *even dimensional* sphere, the group $K^0(B,S)\otimes\mathbb{Q}$ is ismorphic to $\mathbb{Q}^2$ and is generated as a vector space over $\mathbb{Q}$ by $D_{\mathbf{1}}$ and the de Rham-Hodge operator. Here $\mathbf{1}$ denotes the trivial line bundle (see [P]). For the first of these two operators, it is easily checked (as was observed earlier) that both indices vanish; for the second the equality of the two operators follows from the Gauss-Bonnet theorem (see [P]). Thus the theorem holds for all spheres.

7.3. Products. If M and M' are two manifolds for which the analytic and topological indices are equal the same holds for $M \times M'$. This is easily seen from the following facts: (i) $K^0(M \times M') \otimes \mathbb{Q}$ is isomorphic to $(K^0(M) \otimes \mathbb{Q}) \otimes (K^0(M') \otimes \mathbb{Q})$ under the natural map induced by the formation of "external" tensor products and this is compatible with the Kunneth isomorphism in cohomology and the Chern character; note also that the isomorphism Th for $M \times M'$ is the tensor product of the Th for the two factors. This implies that the topological index is multiplicative: If E (resp. E') is a vector bundle on M (resp. M') and p (resp. p') is the Cartesian projection of $M \times M'$ on M (resp. M'), the $t(p^*(E) \otimes p'^*(E')) = t(E).t(E')$. (ii) That $a(p^*(E) \otimes p'^*(E'))$ $= a(E).a(E')$ follows from the fact that the twisted signature operator on $(p^*(E) \otimes p'^*(E'))$ is obtained from the E and E' twisted operators on M and M' respectively by taking tensor product of these last operators. Since we have proved the theorem for spheres and on the even dimensional sphere there is bundle E with $a(E)$ non-zero - the one which corresponds to the de Rham-Hodge operator - it follows that the theorem holds for M if (and only if) it holds for $M \times S^{2q}$.

7.4. Spherical cohomology classes. We need the following result due to Serre (see [S]):

Theorem 9. *Let X be a finite C-W complex of dimension n and α an element of $H^q(X, \mathbb{Q})$. Assume that $n < 2q - 1$. Then there is a continous map $f : X \to S^q$ and an element α^0 in $H^q(S^q, \mathbb{Q})$ such that $f^*(\alpha^0) = \alpha$.*

.

7.5. Conclusion of the proof. Since the Chern character is an isomorphism of $K^0(M) \otimes \mathbb{Q}$ on $H^{even}(M; \mathbb{Q})$, we may view the analytic and topological indices as homomorphisms of $H^{even}(M; \mathbb{Q})$ in $\mathbb{Q}$. Suppose now that the theorem holds for all manifolds of (even) dimension less than m, the dimension of M. Let α be an element of $H^q(M; \mathbb{Q})$ with q an even integer greater than or equal to zero. We need to prove that $a(\alpha) = t(\alpha)$. If $q = 0$, this is the Hirzebruch signature theorem (as is shown by an explicit computation of the topological index - the analytic index, it is easy to show, is the signature of the manifold). Assume then that $q \geq 2$. Let l be an even integer such that $m + q < 2(q + l)$. Let β be a generator of $H^l(S^l; \mathbb{Q})$ and let γ be the class in $H^{q+l}(M \times S^l; \mathbb{Q})$ which corresponds to $\alpha \otimes \beta$ under the Kunneth isomorphism. It suffices to show that $a(\gamma) = t(\gamma)$ since the theorem holds for spheres, $a(\beta) = t(\beta)$ is non-zero and one has $a(\gamma) = a(\alpha).a(\beta)$ and $t(\gamma) = t(\alpha).t(\beta)$.

Now, in view of Serre's thoerem there is a smooth map $f : M \times S^l \to S^{q+l}$ such that γ is the pull back of a $q + l$ cohomology class η of S^{q+l} under f. Now $M \times S^l$ is the boundary of $W = M \times D^{l+1}$ and we can extend f to a smooth map F of W into the disc D^{q+l+1}. Let Z be an interior point of the $q+l+1$ disc which is not a critical value of F. Let U be an open disc containing Z such that the closure of U is disjoint with the set of critical values. Then the inverse image of $D^{q+l+1} \setminus U$ under F gives a cobordism W' between $M \times S^l$ and the inverse image N of the boundary of U in D^{q+l+1}. This last manifold is diffeomrphic to the product of a manifold M' of dimension $m + l + 1 - (q + l + 1) = m - q < m$ and a sphere of dimension $q + l$. If $\widehat{\gamma}$ (resp. γ') denotes the inverse image of η in the cohomology of W' (resp. N), by the bordism invariance theorem $a(\gamma) = a(\gamma')$ and $t(\gamma) = t(\gamma')$. On the other hand since $dim.M' < dim.M'$, the theorem holds for M' and hence also for its product N with S^l. Hence $a(\gamma) = t(\gamma)$. Hence the theorem.

8. Appendix

LeX be a finte C-W complex of dimension N and W a real vector bundle on X of rank m equipped with an inner product. Let B (resp. S) be the unit disc (resp. sphere) bundle in W. Let ϵ denote the antipodal map along the fibres of S. Let P denote the real projective space bundle associated to W: P is the same as the quotient of S obtained by identifying each point s of S with $\epsilon(s)$. Let p (resp. q) be the projection of S (resp. P) on M and u the natural map of S on P so that $q \cdot u = p$. Let E and F be complex vector bundles on X of rank n and E_S and F_S (resp. E_P and F_P), their pull-backs to S (resp. P). Let σ_t, t in [0,1], be a homtopy of sections of the bundle $Iso(E_S, F_S)$ of isomorphisms betwee fibres of E_S and F_S over S. With this notation we have the folowing.

Proposition 10. *Assume that σ_0 and σ_1 are lifts of sections τ_0 and τ^1 of $Iso(E_P, F_P)$. In other words σ_0 and σ_1 are invariant under the antipodal map ϵ of S. Let $\nu(n) = 2^{(m-1).n}$. Then there is a family $\theta_{(t,s)}$, (t, s) in $[0, 1] \times [0, 1]$ of sections of $Iso(\nu(n).E_S, \nu(n).F_S)$ depending continuously on (t, s) such that $\theta_{(t,0)} = \nu.\sigma_t$, $\theta_{(0,s)}$ and $\theta_{(1,s)}$ are independent of s, and $\theta_{(t,1)}$ is ϵ invariant.*

We argue by induction on dimension of X: when dim. $X = 0$, the assertion is obvious. Asume then that it is proved for all finite complexes of dimension less than $n = \dim.(X)$. Let Y denote the $n-1$ skeleton of X; then X is obtained by attaching the disjoint union $A = \coprod_{1\leq i\leq r} D_i$ of n-discs through a map ϕ of its boundary $\partial A = \coprod_{1\leq i\leq r} \partial D_i$ in Y. Fix trivialisations of the pull-backs of W, E and F to A. We may then regard the pull-backs of σ_t (resp. τ_0, τ_1) to A as $GL(N, \mathbb{C})$-valued functions on $A \times S^{m-1}$ (resp. $\partial A \times P^{m-1}$) which we denote by f_t (resp. g_0, g_1)(here P^{m-1} is the $(m-1)$-dimensional real projective space). By induction hypothesis, there is a family $'\sigma_(t, s)$ of sections of $Iso(E_S, F_S)$ over the inverse image S_Y of Y in S with $'\sigma_{(t,0)} = \nu(n-1).\sigma_t$, $'\sigma_{(0,s)} = \nu(n-1).\sigma_0$, $'\sigma_{(1,s)} = \nu(n-1).\sigma_1$. and $'\sigma_{(t,1)}$ invariant under ϵ. This means that there is a family $'f_{(t,s)}$ of maps of $\partial A \times S$ in $GL(\nu(n-1)N, \mathbb{C})$ such that $'f_{(t,0)} = \nu(n-1).f_t$, $'f_{(t,1)}$ is

ϵ-invariant, ${}'f_{(0,s)} = g_0$ and ${}'f_{(1,s)} = g_1$. Now using the homotopy extension property, one sees that there is a family $F_{(t,s)}$ of maps of $A \times S$ in $GL(\nu(n-1)N, \mathbb{C})$ such that $F_{(t,s)} =' f_{(t,s)}$ on $\partial A \times S$, $F_{(0,s)} = f_0$ and $F_{(1,s)} = f_1$. Let $h_t = F_{(t,1)}$. Then for each i with $1 \leq i \leq r$, h_t, t in [0,1], defines a map of the boundary of $D_i \times P^{m-1} \times [0,1]$ (which can be identified with $S^{n+1} \times P^{m-1}$) in $GL(\nu(n-1)N, \mathbb{C})$. Equivalently we have a map H of P^{m-1} in $\Omega^p(GL(\nu(n-1)N, \mathbb{C})$, the p-th free loop space of $GL(\nu(n-1)N, \mathbb{C})$ which when composed with the projection u of S^{m-1} on P^{m-1} factors through the space of all maps of D^{n+1} in $GL(\nu(n-1)N, \mathbb{C})$. Now homotopy classes of maps of P^{m-1} in $\Omega^p(GL(\nu(n-1)N, \mathbb{C}))$ can be identified with the group $K^{-p-1}(P^{m-1})$ so that H defines an element $[H]$ in that group. Moreover the image of $[H]$ in $K^{-p-1}(S^{m-1})$ under the map u^* induced by u is trivial. Since the kernel of the map $u^* : K^{-p-1}(P^{m-1}) \to K^{-p-1}(S^{m-1})$ is annihilated by 2^{m-1} (this is seen easily as a consequence of the Atiyah-Hirzebruch spectral sequence for K-theory [AH] and the fact that the kernel of the cohomolgy map from P^{m-1} to S^{m-1} is a direct sum of $m-2$ copies of $\mathbb{Z}/2$), the proposition follows.

References

[1] [AH] M.F.Atiyah and F.Hirzebruch, Vector bundles and homogeneous spaces, 1961 Proc. Symp. Pure Math. **3**, 7-38, A.M.S, Providence, RI.

[2] [AS] M.F.Atiyah and I.M.Singer, The index of elliptic operators on compact manifolds, Bull. A.M.S **69** (1963), 422-433.

[3] [M] J.Milnor and D.Stasheff, Characterestic Classes, Ann. Math. Studies **76**, Princeton University Press, 1974.

[4] [MP] S.Minakshisundaram and A.Pleijel, Some properties of the eigen functions of the Laplace operator on Riemannian manifolds, Canadian j. Math. **1** (1949), 242-256.

[5] [P] Richard S.Palais, Seminar on the Atiyah-Singer index theorem, Ann. Math. Studies **57**, Princeton University Press, 1965.

[6] [S] J.-P.Serre, Groupes d'homotopie et classes des groupes abeliens, Ann. Math **58** (1953), 258-294.

School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road

Colaba, Mumbai 400 005
India
e-mail:msr@math.tifr.res.in