Worksheet for the session on 10 Oct 2025

SUMMARY OF LAST SESSION: (We will not need these details so much for the next session but do understand the summary.)

A quick summary of our first session is as follows:

1. Given a quadratic equation in two real variables

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

there are three invariants J_1, J_2, J_3 associated with this.

$J_1 = A+C$, $J_2 = AC - B^2$, $J_3 = A(CF - E^2) - B(BF - DE) + D(BE - CD)$

By invariant we mean these quantities do not change under rotation and translation in the xy plane.

If J_3 is not zero then the describes an ellipse or a parabola or a hyperbola depending on $J_2 = AC - B^2 > 0, = 0$, or < 0 .

Thus we have a way of knowing from the equation what kind of curve it is going to be.

2. We have also discussed that the above equation after renaming variables can be brought to a **standard form**

$Ax^2 + By^2 = C$ for ellipse (A, B, C positive),
$Ax^2 - By^2 = C$ for hyperbola (A, B both positive, C nonnegative)
$y^2 = 4px$ for parabola

3. **From now on we will use the standard form in 2.) WITH RATIONAL Coefficients. Rational points as they correspond to integer solutions of quadratic equations**.

4. From here on, we will **focus more on the algebraic equation** and not worry about the geometric curve they represent.

Problem

1a. Very Simple: **Show that** every rational parabola $y^2 = 4 p x$ where p is a rational has infinite number of rational points. So these are not that interesting.

Note: The equation $Ax^2+B y^2 = C$ where A and B are rational positive or negative correspond to integer solutions of the equation $Ax^2+B y^2 =Cz^2$. We can also relax the condition that A, B C are rational we can multiple throughout by the LCM of denominators of A, B and C and get integer coefficients.

Here is a very interesting result. **Either there is no nontrivial solution or there are infinite number of such solutions.** We want to understand why.

1b) **Show that** the equation $x^2+ y^2 = z^2$ has infinite number of non trivial solutions. **Hint.** Draw a circle of radius 1. Join the line connecting (-1,0) and (o,t) and let it intersect the circle at point P (p,q). Calculate the coordinate of P and show that the point P is a rational point if t is rational**. These are Pythagorean triplets.**

1b.) **Show that** the equation $x^2- y^2 = 8z^2$ has infinite number of solutions. Hint: Take z=1 and $x^2- y^2 = 8$. (3,1) is a solution. Now draw rational lines through (3,1). All of these will give rise to nontrivial solutions of $x^2- y^2 = 8z^2$

1c.) **Generalize the argument** in 1b to show that if $Ax^2+B y^2 =Cz^2$ has one non trivial solution (p,q, r) then it has infinite number of integer solutions.

Problem2.

But there may not be any rational points.
The following example does not have integer solutions. In each case, there is a “local obstacle” to integer solutions.
$x^2 + y^2 = 3z^2$
$x^2+2y^2 = 5z^2$
The key idea: if these equations have integer solutions, then they must have integer solutions modulo n when n is any positive integer. If it fails for a single n, then the original equation does have any solution. Sufficient to test this for mod p for prime p and powers of p. p^k. (Why?)

Show that the first equation $x^2 + y^2 = 3z^2$ does not have nontrivial solutions by considering mod 3.

Problem 3.

So when do we have non trivial solutions?

Do we need to check for mod n for all n? (n positive ineger)

Do we need to check for mod p^k for all prime p and for all positive integer k? (certainly, this is sufficient to check for all n, why???)

Is it enough to check for all prime p? . Certainly it is easier to do this. The short answer is Yes. Let us understand this in the context of the equation $x^2+2y^2 = 5z^2$. This has solution mod 3.

Taking modulo 3, we get $x^2+2y^2 = 2 \bmod 3$ has a solution (0,2) .

Using this as a seed solution find solutions to the equations

$x^2+2y^2 = 2 \bmod 3^2$

$x^2+2y^2 = 2 \bmod 3^3$
and so on.

So, starting with a mod 3 solution, we have been able to **lift** it to mod 3^k where k is a positive integer. That is a lot of progress!!!

Note that we can formally write this as

$(0, 2.\ 3^0+1\cdot 3^1+2\cdot 3^2+\ldots)$ [Does the series in the y coordinate converge ?]

If you have done this far, you have essentially explored and discovered a very famous result of number theory from the year 1904. We will discuss this in our next class.

Problem 4.

Show that the equation $x^2+2y^2 = 5z^2$ DOES NOT HAVE any non trivial rational solution even though we have just solved for and found a mod 3 solution.

Hint. Try mod 5. And show that there is no mod 5 solution. Hence no rational solution of the original equation.

[**This shows that for a GLOBAL rational solution to exist we must have local solution for every mod p**.]

At this point we are at the threshold of stating two powerful results on the necessary and sufficient conditions of existence of non trivial solutions. But we will do that more fully in the next class.