Iwasawa Main Conjecture for Rankin-Selberg L-function at an Eisenstein prime

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Introduction

- Iwasawa main conjecture relates analytic objects (p-adic L-functions) and algebraic objects (characteristic ideal of p-Selmer groups).
- Iwasawa main conjecture for GL_1 : Relates Kubota-Leopoldt p-adic L-function and p-part of class groups in the cyclotomic tower. Proved by Mazur-Wiles, Wiles.
- Iwasawa main conjecture for modular forms (GL_2)
  1. Kato proved one divisibility characteristic polynomial divides p-adic L-function.
  2. Other divisibility in most cases proved by Skinner-Urban.
- Iwasawa main conjecture for Rankin-Selberg L-function $(\mathrm{GL}_2 \times \mathrm{GL}_2)$
  1. One divisibility proved by Lei-Loeffler-Zerbes, Kings-Loffler-Zerbes.
  2. Other divisibility in a some cases was shown by X.Wan.
- Still not proved in wide generality.
- In this talk we approach Iwasawa main conjecture for Rankin-Selberg L-function via congruences.

Rankin-Selberg L-function

- Let $f = \sum a(n,f)q^n \in S_k(\Gamma_0(N),\eta)$ and $g = \sum a(n,g)q^n \in S_l(\Gamma_0(J),\psi)$ with $2 \leq l < k$. The Rankin-Selberg L-function of $f \otimes g$ is defined as

$$D_{JN}(s,f,g) := L_{JN}(2s+2-k-l,\psi\eta)\sum_{n=1}^{\infty} a(n,f)a(n,g)n^{-s},$$

  where $L_{JN}(2s+2-k-l,\psi\eta)$ denotes the Dirichlet L-function of $\psi\eta$ with the Euler factors at the primes dividing JN omitted from its Euler product.
- By a result of Shimura, for $0 \leq j \leq k-l-1$, we have

$$\frac{D_{JN}(l+j,f,g)}{(2\pi i)^j \langle f,f\rangle_N} \in \bar{\mathbb{Q}}.$$

- For a character φ, let $\mu_g(\varphi)(z) = (g|\varphi)(z) := \sum \varphi(n)a(n,g)q^n$.

p-adic Rankin-Selberg L-function

For a p-ordinary cusp form f, let f_0 be the p-stabilization of f with $a(p, f_0)$ is a p-adic unit.

Theorem (H. Hida)

Let $f \in S_k(\Gamma_0(N), \eta; K)$ be a normalised p-ordinary eigenform with $p \nmid N$ and $g \in M_l(\Gamma_1(Jp^\alpha))$ with $k > l \geq 2$ and $p \nmid J$. Assume that the map $T(n) \mapsto a(n, f)$ induces the decomposition $h_k(\Gamma_0(N), \eta; K) = K \oplus X$. For every finite order character $\phi \in C(\mathbb{Z}_p^\times, \bar{\mathbb{Q}}_p)$ and $0 \leq j \leq k - l - 1$, we have

$$\mu_{f \times g}(x_p^j \phi) = c(f_0) t p^{\beta l/2} p^{\beta j} p^{(2-k)/2} a(p, f_0)^{1-\beta} \frac{D_{JNp}(l + j, f_0, \mu_{g^\rho}(\phi^{-1})|_l \tau_{Jp^\beta})}{(2i)^{k+l+2j} \pi^{l+2j+1} \langle f_0^\rho|_k \tau_{Np}, f_0 \rangle_{Np}},$$

where β is the smallest positive integer s.t. $\mu_{g^\rho}(\phi^{-1}) \in M_l(\Gamma_1(Jp^\beta))$ and $\tau_{Jp^\beta} = \begin{pmatrix} 0 & 1 \\ Jp^\beta & 0 \end{pmatrix}$.

Setup

- $f \in S_k(\Gamma_0(N), \eta)$ be a p-ordinary newform with $p \nmid N$. Assume
  - (i) $\bar{\rho}_f$ is irreducible i.e. T_f/π is an irreducible $G_{\mathbb{Q}}$-module.
  - (ii) f is p-distinguished $\bar{\rho}_f|G_p \cong \begin{pmatrix} \epsilon_p & * \\ 0 & \delta_p \end{pmatrix}$ with $\epsilon_p \neq \delta_p$.
- $h \in S_l(\Gamma_0(I_0 p^t), \psi)$ be a p-ordinary eigenform with $p \nmid I_0$ and $2 \leq l < k$. Assume $\bar{\rho}_h$ is reducible (i.e. p is an Eisenstein prime). More precisely

$$0 \to \frac{\mathcal{O}}{\pi}(\bar{\xi}_1) \to \frac{T_h}{\pi} \to \frac{\mathcal{O}}{\pi}(\bar{\xi}_2) \to 0 \quad \text{with } \bar{\xi}_2 \text{ unramified at } p.$$

- $M = \mathrm{cond}(\bar{\xi}_1)\mathrm{cond}(\bar{\xi}_2)$ and M_0 is prime to p-part of M.
- $\mathcal{M} := \{r \text{ is prime} : r \mid I_0/M_0, r^2 \mid M_0\}$ and $m := \prod_{r \in \mathcal{M}} r$.
- Let L be a number field containing $a(n,f), a(n,h)$ and values of η, ψ. Let π be a uniformizer of the ring integers of $L_{\mathfrak{p}}$.
- $\Sigma^K = \{\text{primes of } K : \mathfrak{q} \mid pNI_0\infty\}$ and $\Sigma_0^K = \{\text{primes dividing of } K : \mathfrak{q} \mid m\}$.

Congruence with an Eisenstein Series

- Let ξ_1 and ξ_2 be the Teichmüller lift of $\bar{\xi}_1$ and $\bar{\xi}_2$ respectively.
- Let $g(z) := E_l(\xi_1\omega_p^{1-l}, \xi_2) \in M_l(\Gamma_0(M), \xi_1\xi_2\omega_p^{1-l})$. Then
  - (i) $a(n,g) = \sum_{d|n} \xi_1\omega_p^{1-l}(n/d)\xi_2(d)d^{l-1}$ for $n \geq 2$.
  - (ii) g is p-ordinary, p-minimal and $\bar{\rho}_g \cong \bar{\xi}_1 \oplus \bar{\xi}_2$.
  - (iii) $L(s,g) = L(s-l+1, \xi_1\omega_p^{1-l})L(s,\xi_2)$.

Lemma

We have $h|\iota_{pm} \equiv g|\iota_{pm} \mod \pi$. As a consequence, we following congruence of p-adic L-functions $\mu_{f|\iota_m \times h|\iota_m} \equiv \mu_{f|\iota_m \times g|\iota_m} \mod \pi$.

Relating Special values

Proposition

Let $\tilde{f}_0 = f_0|\iota_m$, $g = E_l(\xi_1\omega_p^{1-l}, \xi_2)$, $W(g^\rho|\bar{\phi})$ be the root number of $g^\rho|\bar{\phi}$ and $0 \le j \le k-l-1$. There exists a p-adic unit $c(m)$ such that

(i) For trivial character ι_p, we have $\beta = 2$ and

$$p^{\beta(l+2j)/2} D_{M_0 mNp}(l+j, \tilde{f}_0, \mu_{g^\rho|\iota_m}(\iota_p)|\tau_{M_0 m^2 p^\beta}) = p^{s(l+2j)/2} W(g^\rho) c(m) u_f^{\beta-s} P_p(g^\rho, p^j u_f^{-1}) L(j+1, \tilde{f}_0, \xi_2) L(l+j, \tilde{f}_0, \xi_1\omega_p^{1-l}).$$

(ii) For a non-trivial finite order character ϕ on $\mathbb{Z}_p^\times$ with $(\xi_1\omega_p^{1-l}\phi)_0(p) = 0$, we have $p^\beta = \mathrm{cond}_p(\xi_1\omega_p^{1-l}\phi)\mathrm{cond}_p(\xi_2\phi)$ and

$$D_{M_0 mNp}(l+j, \tilde{f}_0, \mu_{g^\rho|\iota_m}(\bar{\phi})|\tau_{M_0 m^2 p^\beta}) = W(g^\rho|\bar{\phi}) c(m) L(j+1, \tilde{f}_0, \xi_2\phi) L(l+j, \tilde{f}_0, \xi_1\omega_p^{1-l}\phi).$$

Congruence of p-adic L-functions

- Recall $\Sigma_0 = \{\text{primes dividing } m\}$.

Theorem A

Let $f \in S_k(\Gamma_0(N), \eta)$ be a p-ordinary newform with $p \nmid N$. Let h be a p-ordinary eigenform and $(T_h/\pi)^{ss} \cong \bar{\xi}_1 \oplus \bar{\xi}_2$. Assume that

(i) $\bar{\rho}_f$ is irreducible i.e. T_f/π is an irreducible $G_{\mathbb{Q}}$-module.

(ii) f is p-distinguished $\bar{\rho}_f|G_p \cong \begin{pmatrix} \epsilon_p & * \\ 0 & \delta_p \end{pmatrix}$ with $\epsilon_p \neq \delta_p$.

(iii) $T(n) \mapsto a(n, \tilde{f}_0)$ induces the decomposition $h_k(\Gamma_0(N_{\tilde{f}_0}), \eta\iota_m; K) \cong K \oplus X$.

Then for $l-1 \leq j \leq k-1$, we have the following congruence in $\mathbb{Z}_p[[T]]$

$$(L^{\Sigma_0}_{p,f\otimes h,j}) \equiv (L^{\Sigma_0}_{p,f,\xi_1,j})(L^{\Sigma_0}_{p,f,\xi_2,j}) \mod \pi\mathbb{Z}_p[[T]].$$

Some notations

- $p \geq 5$ be a prime.
- For a field K, $G_K = \mathrm{Gal}(\bar{K}/K)$.
- $G_p = \mathrm{Gal}(\bar{\mathbb{Q}}_p|\mathbb{Q}_p) =$ decomposition group and $I_p =$ inertia group at p.
- $\chi_p : G_{\mathbb{Q}} \to \mathbb{Z}_p^{\times}$ be the p-adic cyclotomic character.
- $\omega_p := G_{\mathbb{Q}} \xrightarrow{\chi_p} \mathbb{Z}_p^{\times} \to \mu_{p-1}$ be the Techimüller character.
- $\mathbb{Q}_{\mathrm{cyc}}$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, i.e., $\mathbb{Q}_{\mathrm{cyc}} \subset \cup_{n=1}^{\infty} \mathbb{Q}(\zeta_{p^n})$ and $\Gamma := \mathrm{Gal}(\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}) \cong \mathbb{Z}_p$. We have $\mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[T]]$.
- For a discrete module $\mathbb{Z}_p[[\Gamma]]$-module M, let $M^{\vee} := \mathrm{Hom}_{\mathrm{cont}}(M, \mathbb{Q}_p/\mathbb{Z}_p)$.
- Let (ρ_f, V_f) be the p-adic Galois representation attached to f. Let $T_f \subset V_f$ be a $G_{\mathbb{Q}}$-stable lattice.

Selmer groups

- $V_f(\varphi) = V_f \otimes \varphi$ ($\cong L_{\mathfrak{p}}^{\oplus 2}$), $A_f(\varphi) = \frac{V_f(\varphi)}{T_f(\varphi)}$ and $A_f^-(\varphi) = \frac{V_f^-(\varphi)}{T_f^-(\varphi)}$ ($\cong \frac{L_{\mathfrak{p}}}{\mathcal{O}}$).
- $V_j = V_f \otimes V_h \otimes \chi_p^{-j}, A_j = \frac{V_j}{T_j} = T_h(\chi_p^{-j}) \otimes A_f$ and $A_j^- = T_h(\chi_p^{-j}) \otimes A_f^-$.
- For $B_j \in \{A_f(\varphi\chi_p^{-j}), A_j\}$ and $r \geq 0$. Define

$$S_{\mathrm{Gr}}(B_j[\pi^r]/K) := \mathrm{Ker}\Big(H^1(G_\Sigma(K), B_j[\pi^r]) \to \prod_{v \in \Sigma} \frac{H^1(K_v, B_j[\pi^r])}{H^1_{\mathrm{Gr}}(K_v, B_j[\pi^r])}\Big),$$

  where $\Sigma = \{$primes of $K : \mathfrak{q} \mid pNI_0, \mathfrak{q} \mid \infty\}$, $G_\Sigma(K) = \mathrm{Gal}(K_\Sigma/K)$.
- $S_{\mathrm{Gr}}^{\Sigma_0}(B_j[\pi^r]/K) := \mathrm{Ker}\Big(H^1(G_\Sigma(K), B_j[\pi^r]) \to \prod_{v \in \Sigma\setminus\Sigma_0} \frac{H^1(K_v, B_j[\pi^r])}{H^1_{\mathrm{Gr}}(K_v, B_j[\pi^r])}\Big)$.
- $S_{\mathrm{Gr}}(B_j[\pi^r]/\mathbb{Q}_{\mathrm{cyc}}) := \varinjlim_n S_{\mathrm{Gr}}(B_j[\pi^r]/\mathbb{Q}_n)$, where $\mathbb{Q}_n = \mathbb{Q}_{\mathrm{cyc}}^{\Gamma^{p^n}}$.

Lemma

If $H^0(G_{\mathbb{Q}_n}, B_j) = 0$, then the natural map $S_{\mathrm{Gr}}^{\Sigma_0}(B_j[\pi]/\mathbb{Q}_n) \to S_{\mathrm{Gr}}^{\Sigma_0}(B_j/\mathbb{Q}_n)[\pi]$ is an isomorphism. Hence $S_{\mathrm{Gr}}^{\Sigma_0}(B_j[\pi]/\mathbb{Q}_{\mathrm{cyc}}) \cong S_{\mathrm{Gr}}^{\Sigma_0}(B_j/\mathbb{Q}_{\mathrm{cyc}})[\pi]$.

Exact sequence of Selmer groups

Theorem

Let $(N, M_0) = 1$ and $\psi|_{I_{\mathrm{cyc},v}}$ has order co-prime to p for all $v \mid pI_0$. Assume that $\bar{\rho}_f$ is an irreducible $G_{\mathbb{Q}}$-module, $(T_h/\pi)^{ss} \cong \bar{\xi}_1 \oplus \bar{\xi}_2$, $S^{\Sigma_0}_{\mathrm{Gr}}(A_j/\mathbb{Q}_{\mathrm{cyc}})^\vee$ is a f.g. torsion $\mathcal{O}[[\Gamma]]$-module and $H^2(\mathbb{Q}_\Sigma/\mathbb{Q}_{\mathrm{cyc}}, A_f(\xi_1\chi_p^{-j})[\pi]) = 0$. If $(p-1) \mid j$, then assume that $\bar{\rho}_h|I_{\mathrm{cyc},v} \cong \bar{\xi}_1 \oplus \bar{\xi}_2$ whenever $H^0(G_{\mathbb{Q}_{\mathrm{cyc},v}}, A_f^-(\xi_2)[\pi]) \neq 0$ for all $v \mid p$. Then we have the following exact sequence

$$0 \to S^{\Sigma_0}_{\mathrm{Gr}}(A_f(\xi_1\chi_p^{-j})[\pi]/\mathbb{Q}_{\mathrm{cyc}}) \to S^{\Sigma_0}_{\mathrm{Gr}}(A_j[\pi]/\mathbb{Q}_{\mathrm{cyc}}) \to S^{\Sigma_0}_{\mathrm{Gr}}(A_f(\xi_2\chi_p^{-j})[\pi]/\mathbb{Q}_{\mathrm{cyc}}) \to 0.$$

As a consequence, we have

$$0 \to \frac{S^{\Sigma_0}_{\mathrm{Gr}}(A_f(\xi_2\chi_p^{-j})/\mathbb{Q}_{\mathrm{cyc}})^\vee}{\pi} \to \frac{S^{\Sigma_0}_{\mathrm{Gr}}(A_j/\mathbb{Q}_{\mathrm{cyc}})^\vee}{\pi} \to \frac{S^{\Sigma_0}_{\mathrm{Gr}}(A_f(\xi_1\chi_p^{-j})/\mathbb{Q}_{\mathrm{cyc}})^\vee}{\pi} \to 0.$$

Furthermore, $S^{\Sigma_0}_{\mathrm{Gr}}(A_f(\xi_1\chi_p^{-j})/\mathbb{Q}_{\mathrm{cyc}})^\vee$, $S^{\Sigma_0}_{\mathrm{Gr}}(A_j/\mathbb{Q}_{\mathrm{cyc}})^\vee$ and $S^{\Sigma_0}_{\mathrm{Gr}}(A_f(\xi_2\chi_p^{-j})/\mathbb{Q}_{\mathrm{cyc}})^\vee$ have no non-zero pseudo-null submodules.

Congruence of the characteristic ideals

- For a f.g. torsion $\mathcal{O}[[\Gamma]]$-module M, let $C_{\mathcal{O}[[\Gamma]]}(M)$ denote the characteristic ideal.

Theorem B

We keep the hypotheses and setting as in the previous theorem. For $l-1 \leq j \leq k-2$, we have

$$C_{\mathcal{O}[[\Gamma]]}\left(S_{\mathrm{Gr}}^{\Sigma_0}(A_j/\mathbb{Q}_{\mathrm{cyc}})^{\vee}\right) = C_{\mathcal{O}[[\Gamma]]}\left(S_{\mathrm{Gr}}^{\Sigma_0}(A_f(\xi_1\chi_p^{-j})/\mathbb{Q}_{\mathrm{cyc}})^{\vee}\right)C_{\mathcal{O}[[\Gamma]]}\left(S_{\mathrm{Gr}}^{\Sigma_0}(A_f(\xi_2\chi_p^{-j})/\mathbb{Q}_{\mathrm{cyc}})^{\vee}\right) \bmod \pi.$$

Iwasawa main conjecture modulo π

Theorem (Jha, Shekhar, V)

Under the hypotheses of Theorems A,B and the Iwasawa main conjecture holds for $f \otimes \xi_i$ we have $C_{\mathcal{O}[[\Gamma]]}\left(S_{\mathrm{Gr}}(A_j/\mathbb{Q}_{\mathrm{cyc}})^\vee\right) \equiv (L_{p,f\otimes h,j}) \mod \pi$.

Sketch of the proof

- Theorem $A \implies (L^{\Sigma_0}_{p,f\otimes h,j}) \equiv (L^{\Sigma_0}_{p,f,\xi_1,j})(L^{\Sigma_0}_{p,f,\xi_2,j}) \mod \pi$.
- Theorem $B \implies C_{\mathcal{O}[[\Gamma]]}\left(S^{\Sigma_0}_{\mathrm{Gr}}(A_j/\mathbb{Q}_{\mathrm{cyc}})^\vee\right) =$ $C_{\mathcal{O}[[\Gamma]]}\left(S^{\Sigma_0}_{\mathrm{Gr}}(A_f(\xi_1\chi_p^{-j})/\mathbb{Q}_{\mathrm{cyc}})^\vee\right)C_{\mathcal{O}[[\Gamma]]}\left(S^{\Sigma_0}_{\mathrm{Gr}}(A_f(\xi_2\chi_p^{-j})/\mathbb{Q}_{\mathrm{cyc}})^\vee\right)$ mod π.
- If Iwasawa main conjecture holds for a modular form, i.e. $(L^{\Sigma_0}_{p,f,\xi_i,j}) \equiv C_{\mathcal{O}[[\Gamma]]}\left(S^{\Sigma_0}_{\mathrm{Gr}}(A_f(\xi_i\chi_p^{-j})/\mathbb{Q}_{\mathrm{cyc}})^\vee\right)$ then

$$C_{\mathcal{O}[[\Gamma]]}\left(S^{\Sigma_0}_{\mathrm{Gr}}(A_j/\mathbb{Q}_{\mathrm{cyc}})^\vee\right) \equiv (L^{\Sigma_0}_{p,f\otimes h,j}) \mod \pi.$$

Example

- $p = 11$ and $f = \Delta_{12} \otimes \chi_K$ where $K = \mathbb{Q}(\sqrt{-23})$.
- $h \in S_2(\Gamma_0(23))$ (LMFDB label $23.2.a$).
- $a(n,h) \equiv a(n,g') \mod \mathfrak{p}$ where $g'(z) = E(1_p, 1)(z) = E_2(z) - 23E_2(23z)$ and $\mathfrak{p} = (11, 4 - \sqrt{5})$.
- $m = 1$ and $\bar{\rho}_h \equiv 1 \oplus \omega_p$.
- Vanishing of μ-invariant and Iwasawa main conjecture can be checked for the twist of f.
- Hence Iwasawa main conjecture modulo π holds for $f \otimes h$.

References

[JSV] S. Jha, S. Shekhar, R. Vangala, *Iwasawa main conjecture for Rankin-Selberg L-function at an Eisenstein prime*, `https://arxiv.org/pdf/.pdf`

[H1] H. Hida, *A p-adic measure attached to the zeta functions associated with two elliptic modular forms. II*, Ann. Inst. Fourier (Grenoble) **38**(3): 1–83, 1988.

[H2] H. Hida, *Modular forms and Galois cohomology*, Cambridge studies in advanced Mathematics, **69**, 2010.

[Gr] R. Greenberg, *Iwasawa Theory and p-adic deformations of motives*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., Amer. Math. Soc. **55** (1994), 193–223.

[GV] R. Greenberg, V. Vatsal, *On the Iwasawa invariants of elliptic curves*, Invent.Math. **142** (2000), 17–63.

[SU] C. Skinner and E. Urban, *The Iwasawa main conjectures for* GL_2, Invent. Math. **195**(1) (2014), 1–277.

Thank You!