Group Theory using Rubik's cube

Here are some explorations that arose during the first session.

1. We saw that a square has 8 symmetries and a cube has 24 symmetries. Find objects (2D or 3D) with the following number of symmetries. There could be multiple correct answers
   a) 7 b) 8 c) 9 d) 15 e) 16

2. Find the number of symmetries of the following objects. Also describe how each symmetry was obtained (as we did for square)
   a) Rectangle
   b) Tetrahedron
   c) Cuboid of dimensions $l \times b \times h$
   d) Cuboid of dimensions $l \times b \times b$ (i.e. breadth = height)
   e) Hexagon

3. Find the most 'scrambled' configuration of the cube.
   a) How would you describe the 'scrambledness' of a configuration? Find at least three different ways to measure scrambledness.
   b) How do you know if a configuration is the most scrambled configuration?

Here is some reading material to go through before the session. It is regarding the terms and notations we will use during the sessions. This will help us describe the configuration of the cube and the operations we perform on the cube. The following content is excerpted from lecture notes by Janet Chen. While we will be using her notes extensively during the sessions, it will be better if you read just this sheet instead of her entire notes online. It will help you keep an open mind during the sessions.

Cube notation

The Rubik's cube is composed of 27 small cubes, which are typically called **cubies**. 26 of these cubies are visible (if you take your cube apart, you'll find that the 27th cubie doesn't actually exist). When working with the Rubik's cube, it's helpful to have a systematic way of referring to the individual cubies. Although it seems natural to use the colors of a cubie, it is actually more useful to have names which describe the locations of the cubies. The cubies in the corners are called, appropriately enough, **corner cubies**. Each corner cubie has 3 visible faces, and there are 8 corner cubies. The cubies with two visible faces are called **edge cubies**; there are 12 edge cubies. Finally, the cubies with a single visible face are called **center cubies**, and there are 6 center cubies.

Now, let's name the 6 faces of the Rubik's cube. Following the notation developed by David Singmaster, we will call them right (r), left (l), up (u), down (d), front (f), and back (b). The advantage of this naming scheme is that each face can be referred to by a single letter. To name a corner cubie, we simply list its visible faces in clockwise order. For instance, the cubie in the upper, right, front corner is written urf. Of course, we could also call this cubie rfu or fur. Sometimes, we will care which face is listed first; in these times, we will talk about **oriented cubies**. That is, the oriented cubies urf, rfu, and fur are different. In other situations, we won't care which face is listed first; in these cases, we will talk about **unoriented cubies**. That is, the unoriented cubies urf, rfu, and fur are the same.

Similarly, to name edge and center cubies, we will just list the visible faces of the cubies. For instance, the cubie in the center of the front face is just called f, because its only visible face lies on the front of the cube.

We will also frequently talk about **cubicles.** These are labelled the same way as cubies, but they describe the space in which the cubie lives. Thus, if the Rubik's cube is in the start configuration (that is, the Rubik's cube is solved), then each cubie lives in the cubicle of the same name (the urf cubie lives in the urf cubicle, the f cubie lives in the f cubicle, and so on). If you rotate a face of the Rubik's cube, the cubicles don't move, but the cubies do. Notice, however, that when you rotate a face of the Rubik's cube, all center cubies stay in their cubicles.

Finally, we want to give names to some moves of the Rubik's cube. The most basic move one can do is to rotate a single face. We will let R denote a clockwise rotation of the right face (looking at the right face, turn it 90° clockwise). Similarly, we will use the capital letters L, U, D, F, and B to denote clockwise twists of the corresponding faces. More generally, we will call any sequence of these 6 face-twists a **move** of the Rubik's cube. For instance, rotating the right face counterclockwise is a move which is the same as doing R three times. Later, we will describe a notation for these more complicated moves.

A couple of things are immediately clear. First, we already observed that the 6 basic moves keep the center cubies in their cubicles. Since any move is a sequence of these 6 basic moves, that means that every move of the Rubik's cube keeps the center cubies in their cubicles. Also, any move of the Rubik's cube puts corner cubies in corner cubicles and edge cubies in edge cubicles; it is impossible for a corner cubie to ever live in an edge cubicle or for an edge cubie to live in a corner cubicle.

Cycle notation

In this section, we will lay down the notation needed to understand permutations.

We can put the objects 1, 2, 3 in the order [3 1 2]. Here, 3 is in the first slot, 1 is in the second slot, and 2 is in the third slot. So, this ordering corresponds to the function

$F : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$ defined by $f(1) = 3$, $f(2) = 1$, and $f(3) = 2$.

This can also be viewed as 1→3 →2 →1, a kind of loop. This loop or cycle is denoted by (1 3 2). This is the **cycle notation** of the permutation. It is the same as (3 2 1) or (2 1 3). But (1 3 2) is different from (1 2 3). Can you see why?

Consider a permutation P on {1, 2, …, 12} defined by

P (1) = 12 , P (2) = 4, P (3) = 5, P (4) = 2, P (5) = 6, P (6) = 9

P (7) = 7, P (8) = 3, P (9) = 10, P (10) = 1, P (11) = 11, P (12) = 8

We will write "i → j" (i maps to j) to mean P (i) = j. Then,

1 → 12, 12 → 8, 8 → 3, 3 → 5, 5 → 6, 6 → 9, 9 → 10, 10 → 1.

2 → 4, 4 → 2.

7 → 7.

11 → 11.

This data tells us what P does to each number, so it defines P. As shorthand, we write

P = (1 12 8 3 5 6 9 10) (2 4) (7) (11).

Here, (1 12 8 3 5 6 9 10), (2 4), (7), and (11) are called **cycles**. When writing the disjoint cycle decomposition, we leave out the cycles with just one number, so the disjoint cycle decomposition of P is P = (1 12 8 3 5 6 9 10) (2 4).

Cycle notation for the cube

We can write each move of the Rubik's cube using a slightly modified cycle notation. We want to describe what happens to each oriented cubie; that is, we want to describe where each cubie moves and where each face of the cubie moves. For example, if we unfold the cube and draw the down face, it looks like

	f	f	f	
l	d	d	d	r
l	d	d	d	r
l	d	d	d	r
	b	b	b	

If we rotate this face clockwise by 90° (that is, we apply the move D), then the down face looks like

	l	l	l	
b	d	d	d	f
b	d	d	d	f
b	d	d	d	f
	r	r	r	

So, D(dlf) = dfr because the dlf cubie now lives in the dfr cubicle (with the d face of the cubie lying in the d face of the cubicle, the l face of the cubie lying in the f face of the cubicle, and the f face of the cubie lying in the r face of the cubicle). Similarly, D(dfr) = drb, D(drb) = dbl, and D(dbl) = dlf.

If we do the same thing for the edge cubies, we find D = (dlf dfr drb dbl)(df dr db dl).

Remember that D, U, L, R, F, and B are defined to be clockwise twists of the down, up, left, right, front, and back faces, respectively.

Exercise:

1. Check that the disjoint cycle decomposition of R is (rfu rub rbd rdf)(ru rb rd rf).
2. Write U, L, F, and B as products of disjoint cycles.