Thermalization and hydrodynamics in integrable systems

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Why study macroscopics of integrable systems

- Universal laws emerge at the macroscopic level
  e.g. Entropy always increases
- These universal laws depend only on the symmetries and conservation laws and not on the microscopic details
- Necessary to study systems with different number of conservation laws to explore the richness in macroscopic behaviour

Entropy maximization

$$\rho \propto e^{-(\beta_1 I_1 + \beta_2 I_2 + ...)} \tag{1}$$

This is called Generalized Gibbs ensemble for integrable system.

Model

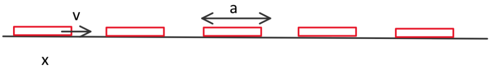
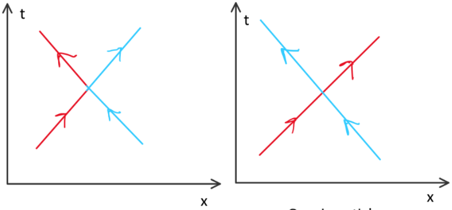
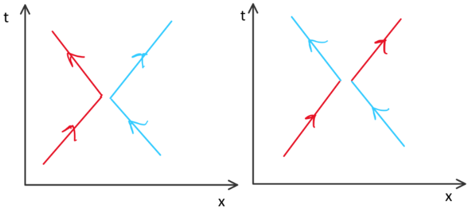


Figure 1: Hard rods, each of length a

Point particle vs hard rods: interacting vs non-interacting





Aim

- To see the sign of dissipation in a hard rod gas
- To compare predictions of hydrodynamics with those of Newtonian dynamics

Conserved quantities for hard rods

$$I_n = \sum_{i=1}^{N} p_i^n \tag{2}$$

Thus, if $f(x, p, t)$ is the single particle phase space distribution, the GGE is given by:

$$f(x, p, t) \propto e^{-(\beta_1 p + \beta_2 p^2 + \ldots)} \tag{3}$$

Thus in GGE, $f(x, p, t)$ for a given p does not depend on x

Euler equation for hard rods

$f(x, v, t)$ is the density of a conserved quantity for a given v.

$$\partial_t f(x, v, t) + \partial_x(v_{eff} f) = 0 \tag{4}$$

where,

$$v_{eff} = \frac{v - a\rho u}{1 - a\rho} \tag{5}$$

and,

$$\rho = \int f(x, v, t) dv \tag{6}$$

$$u = \frac{1}{\rho} \int v f(x, v, t) dv \tag{7}$$

Checking thermalization to GGE

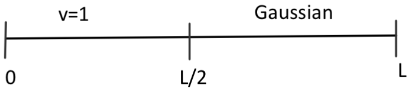
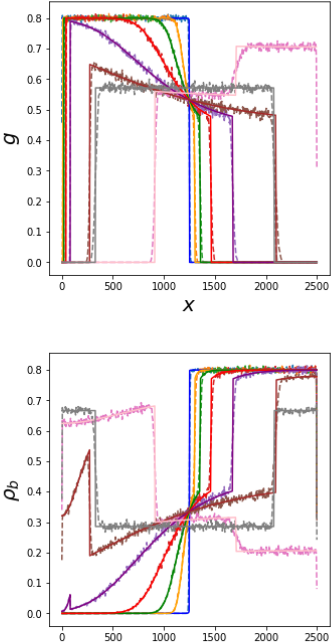
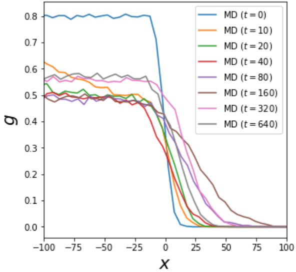


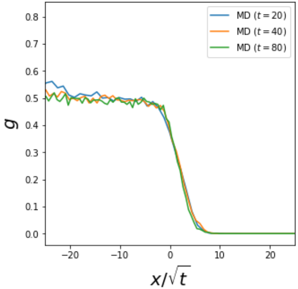
Figure 2: Initial condition.

Euler vs Molecular Dynamics

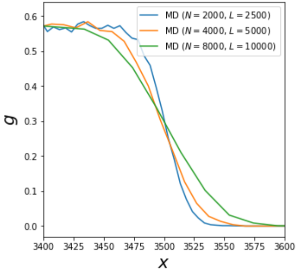


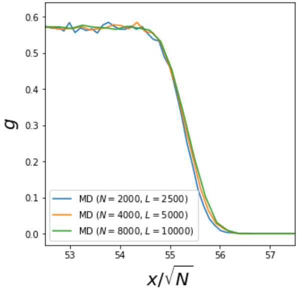
Euler vs Molecular Dynamics





Euler vs Molecular Dynamics





Diffusive scaling from central limit theorem

$$x_i = x_i' + aN_{<x_i} \tag{8}$$

$N_{<x_i} \propto t$ before saturation.
Thus according to central limit theorem, fluctuations are proportional to $\sqrt{t}$.

Euler vs Newtonian dynamics for the free expansion problem

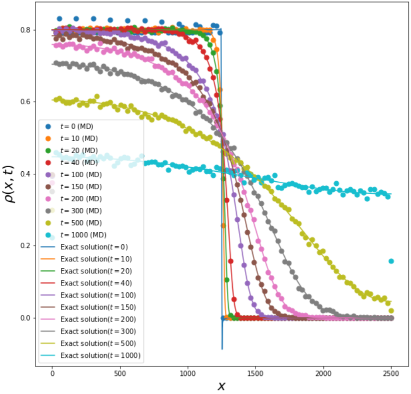


Figure 3: Plot comparing Euler vs MD for $\rho(x, t)$.

Euler vs Newtonian dynamics

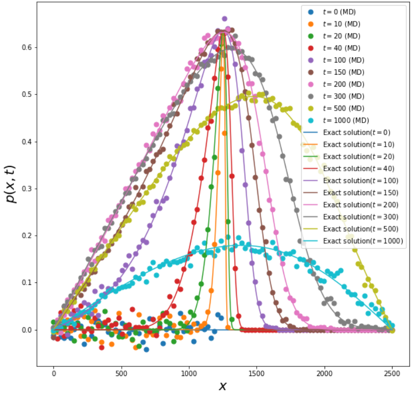


Figure 4: Plot comparing Euler vs MD for $p(x, t)$.

Lebowitz, Percus, Syke (LPS) initial condition

LPS like initial condition: $f(x, v, t = 0) = \delta(x)\delta(v - 1) + \rho_0 h(v)$,
where $h(v) = \frac{e^{-v^2/2}}{\sqrt{2\pi}}$.
Solution at other times is of the form
$f(x, v, t) = \delta\rho(x, t)\delta(v - 1) + \rho_0 h(v)$.

LPS initial condition

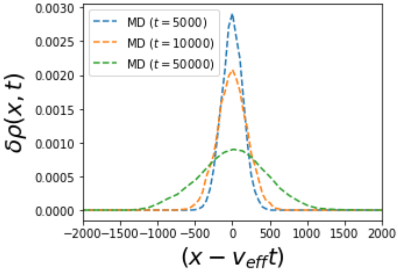


Figure 5: We have taken $N = 2 \times 10^6$, $L = 2.5 \times 10^6$, $a = 1.0$. For MD, ensemble averaging has been done over 10000 realizations. Note that the times are much before the pulse hits the walls of the box, and thus the system is effectively infinite for the times considered.

Navier-Stokes vs Newtonian dynamics

$$\partial_t f + \partial_x(v_{eff} f) = \partial_x \mathcal{N} \tag{9}$$

where

$$\mathcal{N} = \frac{a^2}{2(1 - a\rho)} \int dw |v - w| (f(w)\partial_x f(v) - f(v)\partial_x f(w)) \tag{10}$$

(Spohn and Doyon)
We again consider LPS-like initial condition:

$$f(x, v, t) = \delta\rho(x, t)\delta(v - 1) + \rho_0 h(v) \tag{11}$$

Navier-Stokes

In the long time limit, $\delta\rho(x,t) << \rho_0$. $\rho \approx \rho_0$ and $u \approx 0$. We use this in the Navier-Stokes equation to get:

$$\partial_t(\delta\rho) + v_e \partial_x(\delta\rho) = \frac{n\mu(v')a^2}{2} \partial_x^2(\delta\rho), \tag{12}$$

where $v_e = \frac{v'}{1-a\rho_0}$, $n = \frac{\rho_0}{1-a\rho_0}$, $\mu(v') = \int dv|v - v'|h(v)$. Can be solved exactly:

$$\delta\rho(x,t) = \frac{1}{\sqrt{2\pi n a^2 \mu(v')t}} e^{-\frac{(x-v_e t)^2}{2a^2 n\mu(v')t}} \tag{13}$$

Central limiting arguments can be used to explain the diffusive scaling.

Navier-Stokes vs Newtonian dynamics

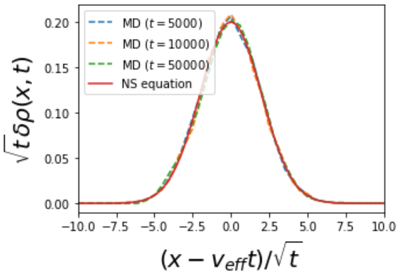


Figure 6: $N = 2 \times 10^6$, $L = 2.5 \times 10^6$.

Conclusion

- Good agreement between GHD and simulation.
- GGE not observed for some initial conditions.
- Central limiting arguments can be used to explain diffusive scaling.

The end