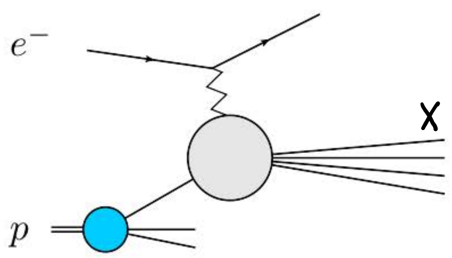
X. Applications of SCET

The formalism we have developed in this course has widespread applications in collider physics, heavy-flavor physics and other fields. Some important examples are shown below (along with some key references).

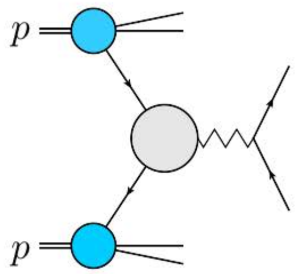
Collider physics:



DIS for $x \to 1$

$$M_X^2 = Q^2 \, \frac{1-x}{x} \ll Q^2$$

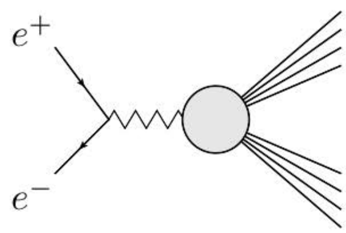
Becher, MN: hep-ph/0605050
Becher, MN, Pecjak: " /0607228



DY production

soft additional radiation

Becher, MN, Xu: hep-ph/0710.0680

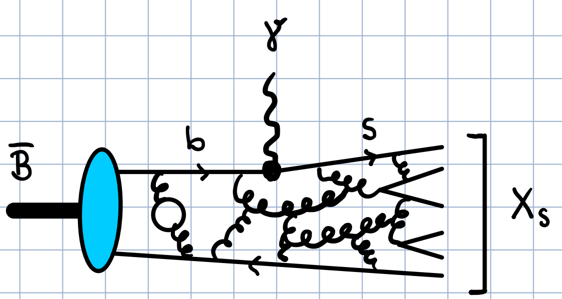


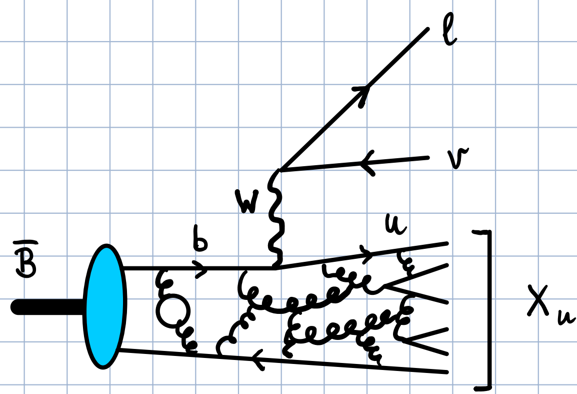
$e^+e^- \to 2$ jets

event shapes

Lee, Sterman: hep-ph/0611061
Becher, Schwartz: 0803.0342

Flavor physics:





Inclusive decays $\bar{B} \to X_s \gamma$ and $\bar{B} \to X_u \ell \bar{\nu}$ in the kinematic region where $M_X^2 \ll m_B^2$

Bauer, Pirjol, Stewart: hep-ph/0109045
Bosch, Lange, MN, Paz: hep-ph/0402094

In all of these processes the relevant modes are collinear or ultra-soft and there are at most two collinear directions (n^μ and $\bar{n}^\mu$). Jet processes at hadron colliders are more complicated, since they require introducing $(2 + n_{\text{Jets}})$ collinear directions n_i^μ, where the first two refer to the beams.

In this lecture we discuss the process

$$e^+e^- \to 2 \text{ jets}$$ ↪ see 1803.04310 by Becher

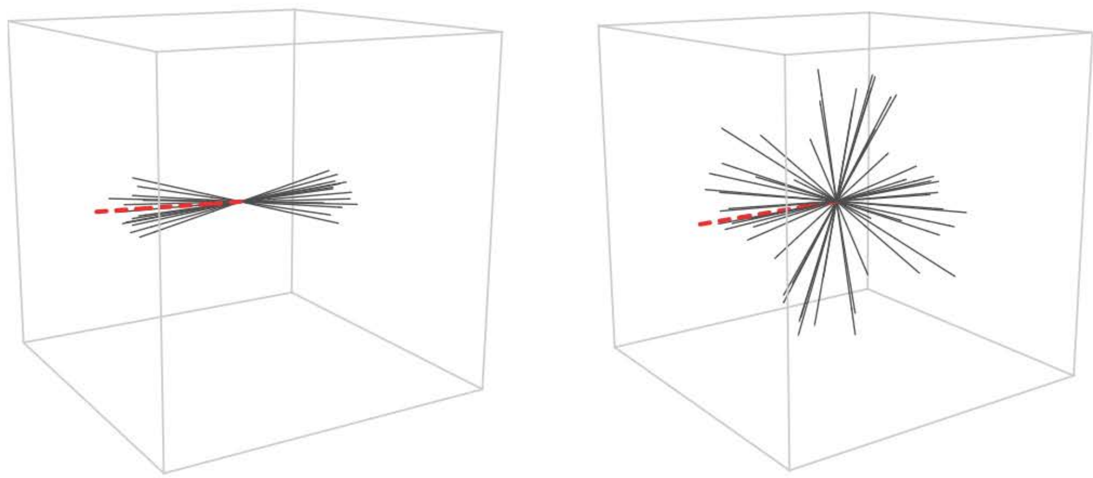
in more detail. Rather than defining the jets through some complicated jet algorithm (↪ non-global logs, see 1508.06645, 1605.02737 for a treatment in SCET) we consider an <u>event shape</u>, which characterizes the geometry of an event and measures how "pencil-like" it is. The prototypical event shape is <u>thrust</u>:

$$\boxed{T = \frac{1}{Q} \max_{\vec{n}_T} \sum_i |\vec{n}_T \cdot \vec{p}_i|}$$

thrust axis

(in CMS)

Here $Q = \sqrt{s} = \sum_i |\vec{p}_i|$ is the total CMS energy (massless particles). The event shape thrust varies between $T_{max} = 1$ (perfect alignment of two jets) and $T_{min} = 1/2$ (completely spherical event).



$T = 0.998 \quad (\tau = 0.002)$ $\qquad$ $T = 0.65 \quad (\tau = 0.35)$

One defines $\tau = 1 - T$ to measure the departure from the perfect 2-jet limit. We are interested in the region where $\tau \ll 1$.

Thrust is soft and collinear safe, meaning that its value does not change under exactly collinear splittings:

$$\vec{p}_i \longrightarrow \vec{p}_{i,a} + \vec{p}_{i,b} \quad ; \quad \vec{p}_{i,a} \parallel \vec{p}_{i,b}$$

and infinitely soft emissions:

$$\vec{p}_i \longrightarrow \vec{p}_{i,a} + \vec{p}_{i,b} \quad ; \quad |\vec{p}_{i,b}| \rightarrow 0$$

This property ensures that the cross section $d\sigma/d\tau$ is free of IR divergences. However, in the 2-jet limit $\tau \ll 1$ the cross section receives large double-logarithmic corrections $\sim (\alpha_s \ln^2 \tau)^n$, which need to be resummed to all orders of perturbation theory. A region analysis shows that the relevant modes are:

hard $\quad Q^2 \quad \leftarrow$ integrate out

(anti-) collinear $\quad \tau Q^2$

ultra-soft $\quad \tau^2 Q^2$

$\Big\}$ SCET with $\lambda = \tau^{1/2} \ll 1$

We choose the thrust axis to define the reference vectors:

$$n^\mu = (1, \vec{n}_T), \qquad \bar{n}^\mu = (1, -\vec{n}_T)$$

By definition, thrust is additive and we can separate the sum over particles into sums in the various sectors of SCET:

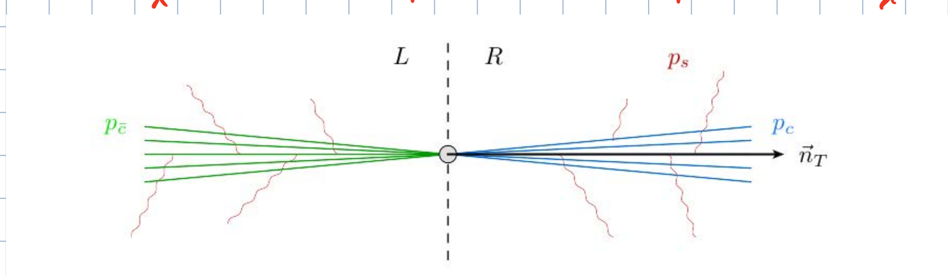
$\vec{n}_T \cdot \vec{p}_i > 0$ right movers $\quad$ $\vec{n}_T \cdot \vec{p}_i < 0$ left movers

$$\tau Q = \sum_i \left(|\vec{p}_i| - |\vec{n}_T \cdot \vec{p}_i| \right)$$

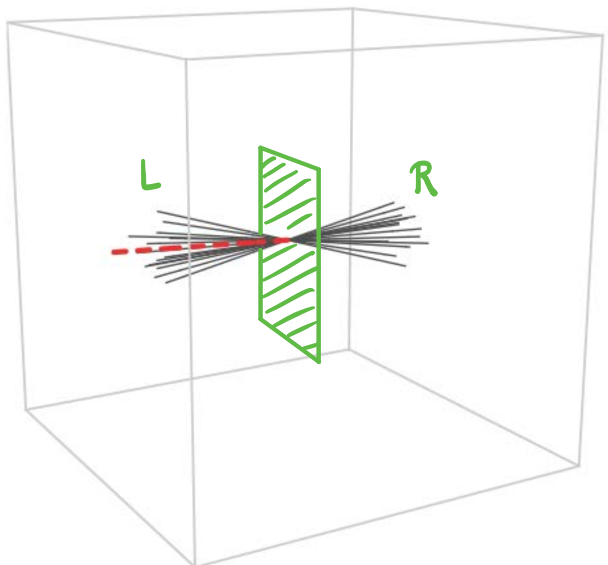
$$= \sum_i n \cdot p_{c,i} + \sum_i \bar{n} \cdot p_{\bar{c},i} + \sum_i n \cdot p^R_{s,i} + \sum_i \bar{n} \cdot p^L_{s,i}$$

collinear particles $\quad$ anti-collinear particles $\quad$ soft particles

$$\equiv \underbrace{n \cdot p_{X_c}}_{\lambda^2} + \underbrace{\bar{n} \cdot p_{X_{\bar{c}}}}_{\lambda^2} + \underbrace{n \cdot p^R_{X_s}}_{\lambda^2} + \underbrace{\bar{n} \cdot p^L_{X_s}}_{\lambda^2} \sim \lambda^2 \sim \tau$$



The definition of the thrust axis ensures that the total transverse momentum in each hemisphere vanishes:



$$p_X^{L,\perp} = p_{X_{\bar{c}}}^{\perp} + p_{X_s}^{L,\perp} = 0$$

$$p_X^{R,\perp} = \underbrace{p_{X_c}^{\perp}}_{\lambda} + \underbrace{p_{X_s}^{R,\perp}}_{\lambda^2} = 0$$

$$\Rightarrow \quad p_{X_c}^{\perp} = 0 = p_{X_{\bar{c}}}^{\perp}$$

up to power corrections

It follows that, at leading power:

$$M_R^2 = (p_{X_c} + p_{X_s}^R)^2 = p_{X_c}^2 + \bar{n}\cdot p_{X_c}\, n\cdot p_{X_s}^R + \dots$$

$$= \bar{n}\cdot p_{X_c} \left(n\cdot p_{X_c} + n\cdot p_{X_s}^R \right) + \dots$$

$$= Q \left(\underbrace{n\cdot p_{X_c}}_{\lambda^2 Q} + \underbrace{n\cdot p_{X_s}^R}_{\lambda^2 Q} \right) + \dots$$

$$M_L^2 = (p_{X_{\bar{c}}} + p_{X_s}^L)^2 = n\cdot p_{X_{\bar{c}}} \left(\bar{n}\cdot p_{X_{\bar{c}}} + \bar{n}\cdot p_{X_s}^L \right) + \dots$$

$$= Q \left(\underbrace{\bar{n}\cdot p_{X_{\bar{c}}}}_{\lambda^2 Q} + \underbrace{\bar{n}\cdot p_{X_s}^L}_{\lambda^2 Q} \right) + \dots$$

Up to power corrections, we thus obtain:

$$\boxed{\tau Q^2 = M_L^2 + M_R^2 = p_{X_c}^2 + p_{X_{\bar{c}}}^2 + Q \left(n\cdot p_{X_s}^R + \bar{n}\cdot p_{X_s}^L \right)}$$

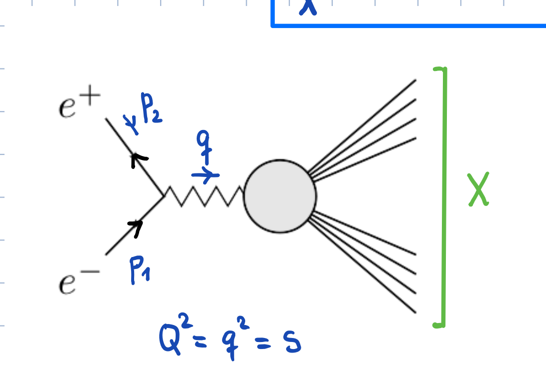
$$\lambda^2 Q^2 \qquad \lambda^2 Q^2 \qquad \lambda^2 Q \qquad \lambda^2 Q$$

The fact that thrust is additive in the collinear and ultra-soft contributions is important to establish factorization. The differential cross section is given by:

$$\frac{d\sigma}{d\tau} = \frac{1}{2Q^2} \boxed{\sum\!\!\!\!\!\!\int_X |\mathcal{M}(e^+e^- \to \gamma^* \to X)|^2 (2\pi)^4 \delta^{(4)}(p_X - q)\, \delta(\tau - \tau(X))}$$

$L^{\mu\nu} H_{\mu\nu}$

thrust of final state X



Leptonic tensor:

$$L^{\mu\nu} = \sum_q \frac{Q_q^2 e^4}{Q^4} \left(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - p_1 \cdot p_2\, g^{\mu\nu}\right)$$

(Q_q: quark electric charge)

The hadronic tensor is:

vector current

$$H_{\mu\nu} = \sum\!\!\!\!\!\!\int_X \langle 0 | J_\nu^\dagger(0) | X \rangle \langle X | J_\mu(0) | 0 \rangle (2\pi)^4 \delta^{(4)}(p_X - q)\, \delta(\tau - \tau(X))$$

We wish to compute the differential cross section in the angle θ of the thrust axis $\vec{n}_T$ with respect to the momentum $\vec{p}_1$ of the electron. To do so, we insert:

orthogonal to thrust axis

$$1 = \int d^3\vec{n}\, \delta^{(3)}(\vec{n} - \vec{n}_T) = \int_0^{2\pi} d\varphi \int d\cos\theta\, \delta^{(2)}(\vec{n}^\perp)$$

contains $\delta(|\vec{n}| - 1)$

$$= 2\pi \left(\frac{Q}{2}\right)^2 \int d\cos\theta\, \underbrace{\delta^{(2)}(p_{X_c}^\perp)}_{\text{required by definition of thrust axis}} ; \qquad \vec{p}_{X_c} = \frac{Q}{2}\vec{n} \parallel \vec{n}_T \quad \text{up to power cors.}$$

Combining this δ-function with the δ-function from momentum conservation gives:

$$\delta^{(4)}(p_X - q)\,\delta^{(2)}(p^\perp_{X_c}) = \delta^{(4)}(p_{X_c} + p_{X_{\bar c}} + p_{X_s} - q)\,\delta^{(2)}(p^\perp_{X_c})$$

$$= 2\,\delta(\bar n\cdot p_{X_c} - Q)\,\delta(n\cdot p_{X_{\bar c}} - Q)\,\delta^{(2)}(p^\perp_{X_c})\,\delta^{(2)}(p^\perp_{X_{\bar c}}) + \dots$$

We finally introduce new variables

$$M_c^2 = p_{X_c}^2\,, \quad M_{\bar c}^2 = p_{X_{\bar c}}^2\,, \quad \omega = n\cdot p^R_{X_s} + \bar n\cdot p^L_{X_s}$$

$$\Rightarrow \quad \tau Q^2 = M_c^2 + M_{\bar c}^2 + Q\omega$$

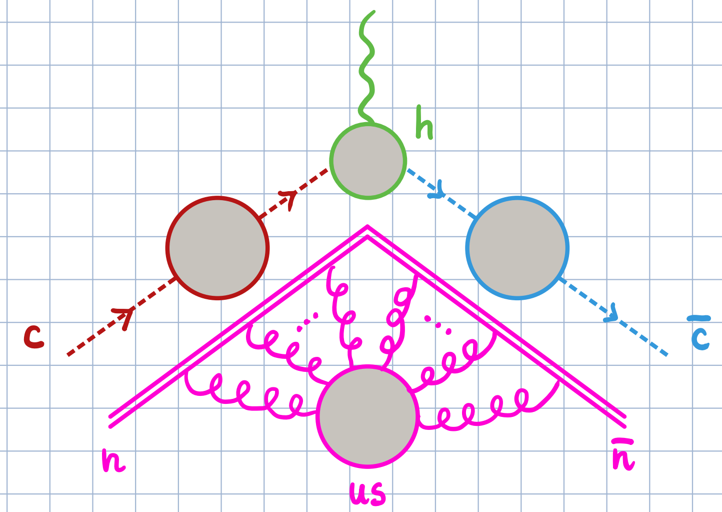
by multiplying the hadronic tensor with the dummy integral:

$$1 = \int dM_c^2\,\delta(M_c^2 - p_{X_c}^2)\int dM_{\bar c}^2\,\delta(M_{\bar c}^2 - p_{X_{\bar c}}^2)\int d\omega\,\delta(\omega - n\cdot p^R_{X_s} - \bar n\cdot p^L_{X_s})$$

This leads to:

$$\boxed{\begin{aligned}\frac{d\sigma}{d\tau\, d\cos\theta} =& \frac{\pi}{2}\, L_{\mu\nu}\,|C_V(-Q^2 - i0,\mu)|^2 \int dM_c^2 \int dM_{\bar c}^2 \int d\omega\,\delta\Big(\tau - \overbrace{\frac{M_c^2 + M_{\bar c}^2 + Q\omega}{Q^2}}^{\tau(x)}\Big)\\
&\times \sum\!\!\!\!\!\!\int_{X_c} \underline{\langle 0|\,\chi^a_{c,\delta}(0)|X_c\rangle\langle X_c|\,\bar\chi^b_{c,\alpha}\,|0\rangle}\,\delta(M_c^2 - p^2_{X_c})\,\delta^{(2)}(p^\perp_{X_c})\,\delta(\bar n\cdot p_{X_c} - Q)\\
&\times \sum\!\!\!\!\!\!\int_{X_{\bar c}} \underline{\langle 0|\,\bar\chi^d_{\bar c,\gamma}(0)|X_{\bar c}\rangle\langle X_{\bar c}|\,\chi^e_{\bar c,\beta}\,|0\rangle}\,\delta(M_{\bar c}^2 - p^2_{X_{\bar c}})\,\delta^{(2)}(p^\perp_{X_{\bar c}})\,\delta(n\cdot p_{X_{\bar c}} - Q)\\
&\times \sum\!\!\!\!\!\!\int_{X_s} \underline{\langle 0|\,[S_n^\dagger S_{\bar n}]_{da}\,|X_s\rangle\langle X_s|\,[S_{\bar n}^\dagger S_n]_{be}\,|0\rangle}\,\delta(\omega - n\cdot p^R_{X_s} - \bar n\cdot p^L_{X_s})\\
&\times (2\pi)^4\,(\gamma^\mu_\perp)_{\alpha\beta}\,(\gamma^\nu_\perp)_{\gamma\delta}\end{aligned}}$$

Here $\chi_c \equiv W_c^{(0)\dagger} \xi_n^{(0)}$ and $\chi_{\bar{c}} = W_{\bar{c}}^{(0)\dagger} \xi_{\bar{n}}^{(0)}$ are the gauge-invariant collinear building blocks after decoupling of the ultra-soft gluons, and $S_n, S_{\bar{n}}$ are the soft Wilson lines.



At this stage, one defines jet and soft functions via the matrix elements in the different sectors of SCET.

Jet functions:

$$\sum\!\!\!\!\!\!\int_{X_c} \langle 0|\, \chi^a_{c,\delta}(0) |X_c\rangle \langle X_c|\, \bar{\chi}^b_{c,\alpha}\, |0\rangle\, \delta(M_c^2 - p^2_{X_c})\, \delta^{(2)}(p^\perp_{X_c})\, \delta(\bar{n}\cdot p_{X_c} - Q)$$

$$= \frac{\delta^{ab}}{2(2\pi)^3} \left[\frac{\not{n}}{2}\right]_{\delta\alpha} J(M_c^2)$$

color conservation

since $\not{n}\, \xi_n^{(0)} = 0$

Note that this is the spectral representation of a collinear quark propagator (c.f. Section 7.1 in Peskin & Schroeder).

At lowest order, X_c are single quark states and we find:

$$\int \frac{d^3\vec{p}}{(2\pi)^3\, 2p^0} \underbrace{\sum_s u_{n,\delta}(p,s)\, \bar{u}_{n,\alpha}(p,s)}_{\left(\frac{\not{n}\bar{\not{n}}}{4}\not{p}\frac{\bar{\not{n}}\not{n}}{4}\right)_{\delta\alpha} = \left(\frac{\not{n}}{2}\right)_{\delta\alpha} \bar{n}\cdot p} \delta^{ab}\, \delta(M_c^2)\, \delta^{(2)}(p_\perp)\, \delta(2p^3 - Q)$$

$$= \frac{\delta^{ab}}{2(2\pi)^3} \left(\frac{\not{n}}{2}\right)_{\delta\alpha} \delta(M_c^2)$$

It follows that:

$$J(M^2) = \delta(M^2) + \mathcal{O}(\alpha_s)$$

This jet function is known to 3-loop order in QCD.

↪ Becher, MN: hep-ph/0603140 (2-loop)
Brüser, Liu, Stahlhofen: 1804.09722 (3-loop)

At 1-loop order, one finds: (hep-ph/0402094)

$$J(M^2) = \delta(M^2) + \frac{C_F\alpha_s}{4\pi}\left[(7-\pi^2)\,\delta(M^2) + 4\left(\frac{\ln M^2/\mu^2}{M^2}\right)_*^{[\mu^2]} - 3\left(\frac{1}{M^2}\right)_*^{[\mu^2]}\right]$$
$$+ \mathcal{O}(\alpha_s^2)$$

generalized plus distributions

The jet function in the anti-collinear sector is defined as:

$$\sum\!\!\!\!\!\!\int_X \langle 0|\, \bar{\chi}^d_{\bar{c},\gamma}(0)\, |X_{\bar{c}}\rangle\langle X_{\bar{c}}|\, \chi^e_{\bar{c},\beta}\, |0\rangle\, \delta(M_{\bar{c}}^2 - p_{X_{\bar{c}}}^2)\, \delta^{(2)}(p^\perp_{X_{\bar{c}}})\, \delta(n\cdot p_{X_{\bar{c}}} - Q)$$

$$= \frac{\delta^{de}}{2(2\pi)^3}\left[\frac{\not{\bar{n}}}{2}\right]_{\beta\gamma} J(M_{\bar{c}}^2)$$

At this point, we obtain the following trace over Dirac matrices:

$$\mathrm{tr}\left(\gamma_\perp^\mu \frac{\bar{\not{n}}}{2} \gamma_\perp^\nu \frac{\not{n}}{2}\right) = -g_\perp^{\mu\nu}\, n\cdot\bar{n} = -2\, g_\perp^{\mu\nu}$$

Also, the four color indices in the ultra-soft matrix element get contracted in pairs.

Soft function:

$$S(\omega) = \frac{1}{N_c} \sum_{X_s} \langle 0| \left[S_n^\dagger S_{\bar{n}}\right]_{ab} |X_s\rangle\langle X_s| \left[S_{\bar{n}}^\dagger S_n\right]_{ba} |0\rangle\, \delta(\omega - n\cdot p_{X_s}^R - \bar{n}\cdot p_{X_s}^L)$$

The prefactor $1/N_c$ has been introduced such that at leading order:

$$S(\omega) = \delta(\omega) + \underbrace{\mathcal{O}(\alpha_s)}_{\text{calculable only if } \omega \gg \Lambda_{QCD}}$$

"shape function"

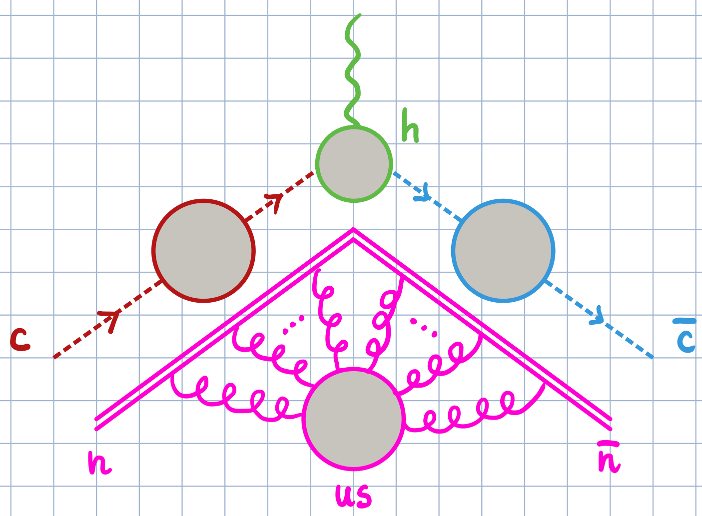
hep-ph/9311325
9902341

This function is known at 2-loop order in perturbation theory. (Becher, Schwartz: 0803.0342) Note that for $\omega \sim \Lambda_{QCD}$, i.e. $\tau \sim \frac{\Lambda_{QCD}}{Q}$, the shape function is a genuinely nonperturbative object, which must be extracted from data.

Cross section:

Combining all pieces, we obtain the cross section:

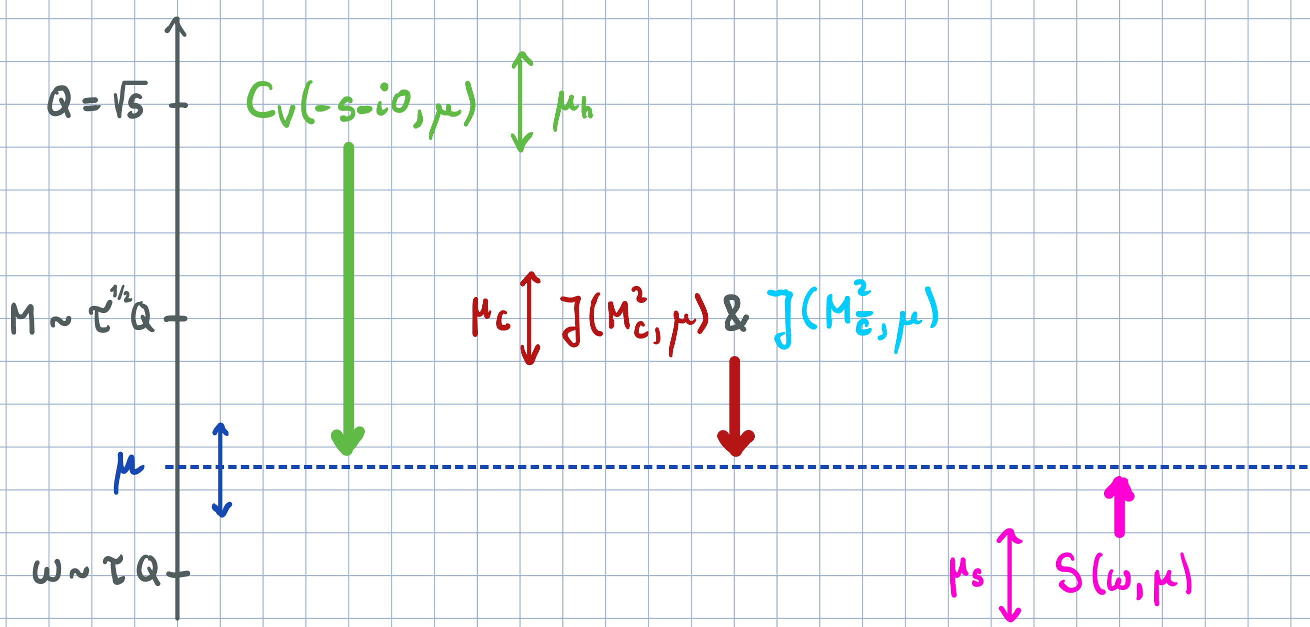
$$\boxed{\begin{aligned}\frac{d^2\sigma}{d\tau\, d\cos\theta} &= \sum_q \frac{N_c \pi\, Q_q^2\, \alpha^2}{2s}\,(1+\cos^2\theta)\,\left|C_V(-s-i0,\mu)\right|^2 \\ &\times \int_0^\infty dM_c^2 \int_0^\infty dM_{\bar c}^2 \int_0^\infty d\omega\; \delta\!\left(\tau - \frac{M_c^2 + M_{\bar c}^2 + \sqrt{s}\,\omega}{s}\right) \\ &\times J(M_c^2,\mu)\; J(M_{\bar c}^2,\mu)\; S(\omega,\mu)\end{aligned}}$$



This is a paradigmatic example of the derivation of a <u>QCD factorization formula</u> using SCET. The scale dependence of the various functions arises after renormalization of the SCET current operator, see the last lecture.

Resummation of large logarithms:

The theoretical prediction for the cross section is independent of the renormalization scale μ. However, for each fixed choice of μ there are large logs in at least some of the component functions C_V, J and S. The strategy is therefore to calculate these functions at their "natural" scales, and then evolve ("run") them to a common (and arbitrary) scale μ by solving their RG equations:



For the hard matching coefficient we have discussed the solution of the RG equation in lecture 8 (see pages 3-5). The RG equations for the jet and soft functions are more complicated, e.g.:

(Becher, MN: hep-ph/0603140)

$$\mu \frac{d}{d\mu} J(p^2,\mu) = \left[-2\Gamma_{cusp}(\alpha_s) \ln\frac{p^2}{\mu^2} - 2\gamma_J(\alpha_s) \right] J(p^2,\mu) + 2\Gamma_{cusp}(\alpha_s) \int_0^\infty dp'^2 \, \frac{J(p^2,\mu) - J(p'^2,\mu)}{p^2 - p'^2}$$

and similarly for the soft function.

The form of the cross section on p. 11 and the RG equations simplify in Laplace space, where one defines:

$$\tilde{\sigma}(\nu) \equiv \int_0^\infty d\tau \, e^{-\nu\tau} \frac{d\sigma}{d\tau}$$

$$= \frac{4\pi Q_q^2 \alpha^2}{s} \left| C_V(-s-i0,\mu) \right|^2 \int_0^\infty dM_c^2 \, e^{-\frac{\nu M_c^2}{s}} J(M_c^2,\mu) \times \int_0^\infty dM_{\bar{c}}^2 \, e^{-\frac{\nu M_{\bar{c}}^2}{s}} J(M_{\bar{c}}^2,\mu) \int_0^\infty d\omega \, e^{-\frac{\nu\omega}{\sqrt{s}}} S(\omega,\mu)$$

$$\equiv \frac{4\pi Q_q^2 \alpha^2}{s} \left| C_V(-s-i0,\mu) \right|^2 \underbrace{\tilde{J}\left(\frac{\nu}{s},\mu\right)}_{\sim (\tau s)^{-1}} \tilde{J}\left(\frac{\nu}{s},\mu\right) \underbrace{\tilde{S}\left(\frac{\nu}{\sqrt{s}},\mu\right)}_{\sim (\tau\sqrt{s})^{-1}}$$

In Laplace space we obtain a product rather than a convolution. Also, the RG equations take on a local form (Becher, MN: hep-ph/0605050) and one obtains:

$$\mu \frac{d}{d\mu} \tilde{J}\left(\frac{\nu}{s}, \mu\right) = \left[-2\Gamma_{cusp}(\alpha_s)\left(\ln\frac{s}{\nu\mu^2} - \gamma_E\right) - 2\gamma_J(\alpha_s)\right] \tilde{J}\left(\frac{\nu}{s}, \mu\right)$$

$$\mu \frac{d}{d\mu} \tilde{S}\left(\frac{\nu}{\sqrt{s}}, \mu\right) = \left[2\Gamma_{cusp}(\alpha_s)\left(\ln\frac{s}{\nu^2\mu^2} - 2\gamma_E\right) + 2\gamma_S(\alpha_s)\right] \tilde{S}\left(\frac{\nu}{\sqrt{s}}, \mu\right)$$

They can be solved using the same techniques as for the hard function. The fact that the cross section is RG invariant, $\mu \frac{d}{d\mu}\tilde{\sigma}(\nu) = 0$, implies:

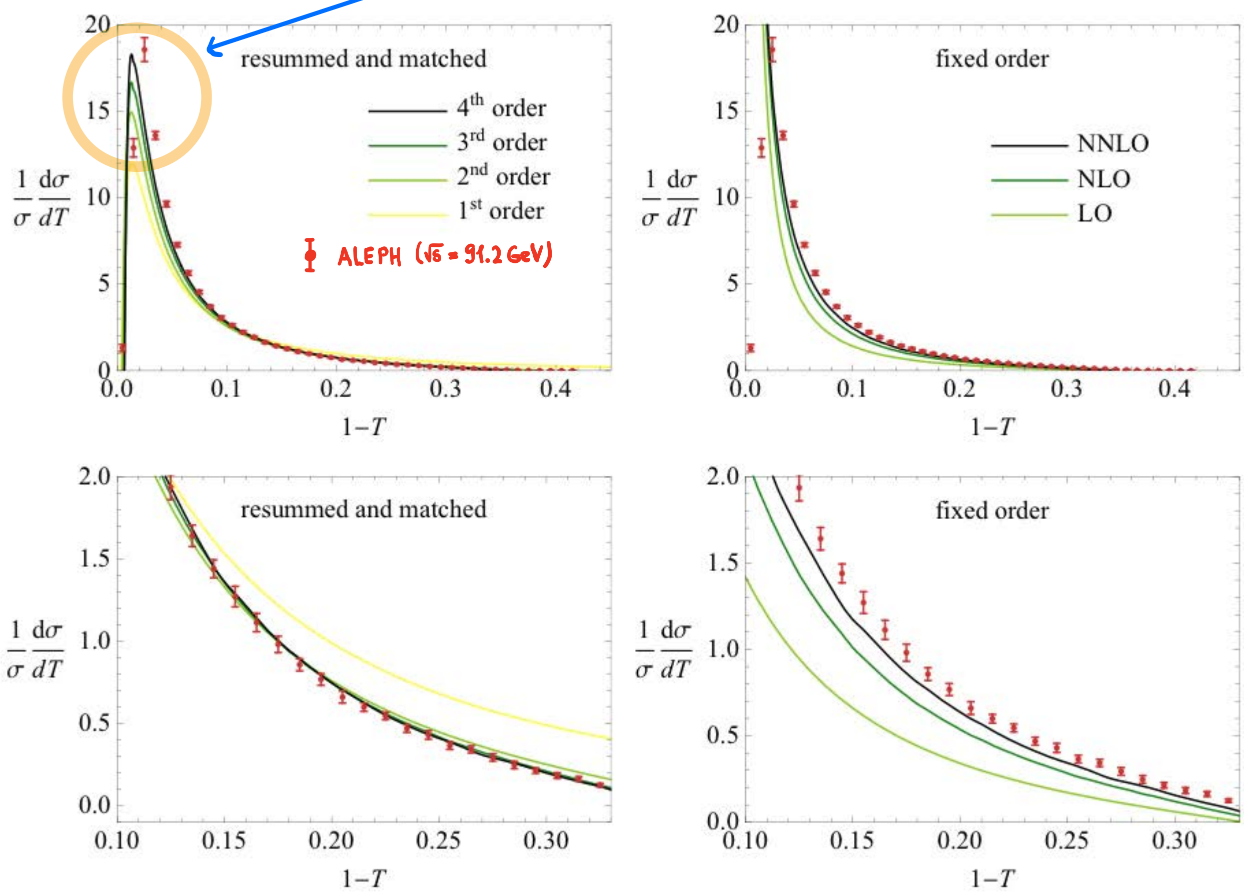
$$\begin{aligned} & 2\Gamma_{cusp}(\alpha_s)\ln\frac{s}{\mu^2} + 2\gamma_V(\alpha_s) && |C_V|^2 \\ & - 4\Gamma_{cusp}(\alpha_s)\left(\ln\frac{s}{\nu\mu^2} - \gamma_E\right) - 4\gamma_J(\alpha_s) && \tilde{J}\ \tilde{J} \\ & + 2\Gamma_{cusp}(\alpha_s)\left(\ln\frac{s}{\nu^2\mu^2} - 2\gamma_E\right) + 2\gamma_S(\alpha_s) \overset{!}{=} 0 && \tilde{S} \end{aligned}$$

$$\Rightarrow \quad \gamma_S(\alpha_s) = 2\gamma_J(\alpha_s) - \gamma_V(\alpha_s)$$

This consistency condition is satisfied to all orders in α_s.

Comparison with data:

region where nonperturbative effects can be important



(Becher, Schwartz: 0803.0342)

The "matching" of a resummed SCET prediction (valid for $\tau \ll 1$) onto a prediction obtained using fixed-order perturbation theory (valid for $\tau = \mathcal{O}(1)$) is performed as follows:

$$\left(\frac{d\sigma}{d\tau}\right)_{\text{matched}} = \left(\frac{d\sigma}{d\tau}\right)^{\text{SCET}}_{\text{resummed}}$$

optimal prediction for $\tau \ll 1$

$$+ \left[\left(\frac{d\sigma}{d\tau}\right)_{\text{fixed order}} - \left(\frac{d\sigma}{d\tau}\right)^{\text{SCET}}_{\text{resummed}} \Bigg|_{\substack{\text{reexpanded} \\ \text{in powers of } \alpha_s}} \right]$$

subtraction to avoid double counting

optimal prediction for $\tau = \mathcal{O}(1)$ (vanishes for $\tau \to 0$)