Target Space Entanglement and Space-Time Geometry

Sandip Trivedi
Tata Institute of Fundamental Research,
Mumbai, India.

Outline

- Introduction
- Target Space Entanglement: Definition
- Space-Time Entanglement: Proposal
- Conclusions

Entanglement

Quantum correlations have features which make them essentially different from their classical counterparts.

Entanglement Entropy is a measure of these quantum aspects.

It quantifies properties of the correlations which cannot arise in classical systems even when we allow for extra hidden local variables. (Bell 1960s)

Entanglement

It is increasingly felt that the entanglement properties of gravitational systems will play a key role in our quest to understand quantum gravity.

For instance, the emergence of a smooth spacetime geometry is tied to the entanglement properties.

(Swingle, Van Raamsdonk, …)

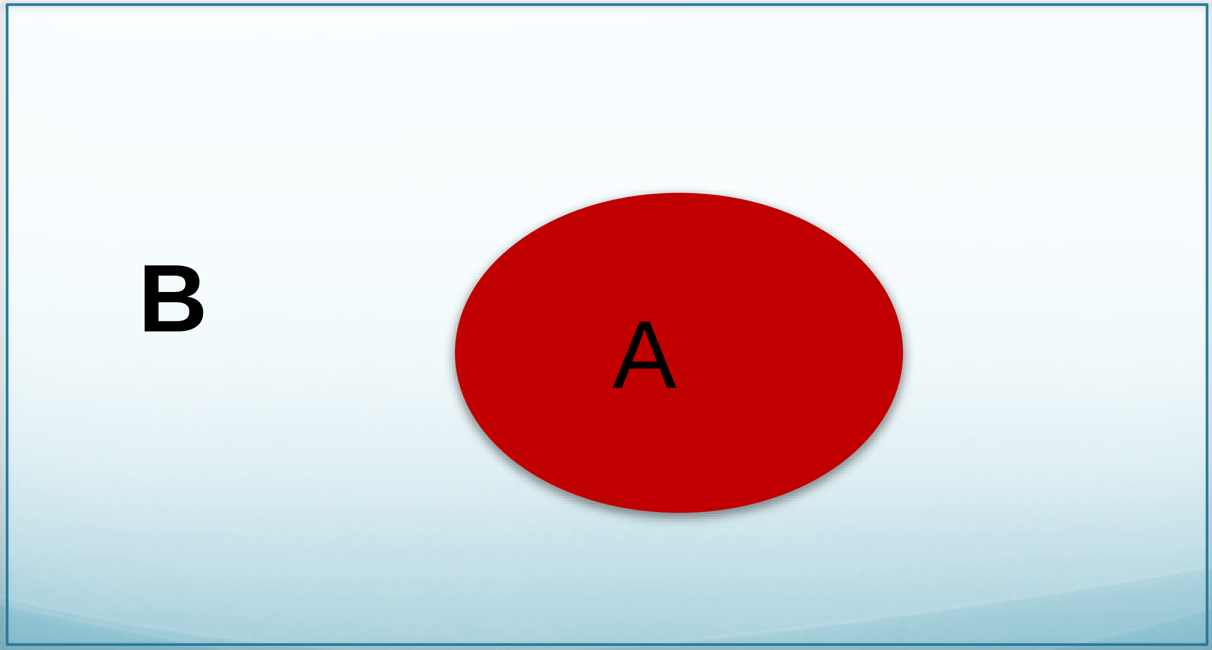
Entanglement in Gravity

How is Entanglement to be defined precisely in space-time?

This is not a straightforward question

In a field theory (without gravity),
we divide space (at say a given instance of time) into two parts.

Trace out the degrees in B to obtain the Entanglement for A.



More precisely:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

$$\rho = |\psi >< \psi|$$

$$\rho_A = Tr_B(\rho)$$

$$S_{EE} = -Tr_A(\rho_A \log \rho_A)$$

Von Neumann Entropy of Density Matrix

(Some subtleties for gauge theories,etc.
Casini Huerta Rosabal; Ghosh, Soni, ST,..)

With gravity present specifying a region of space-time itself is not straightforward.

Not easy to do in a gauge invariant manner.

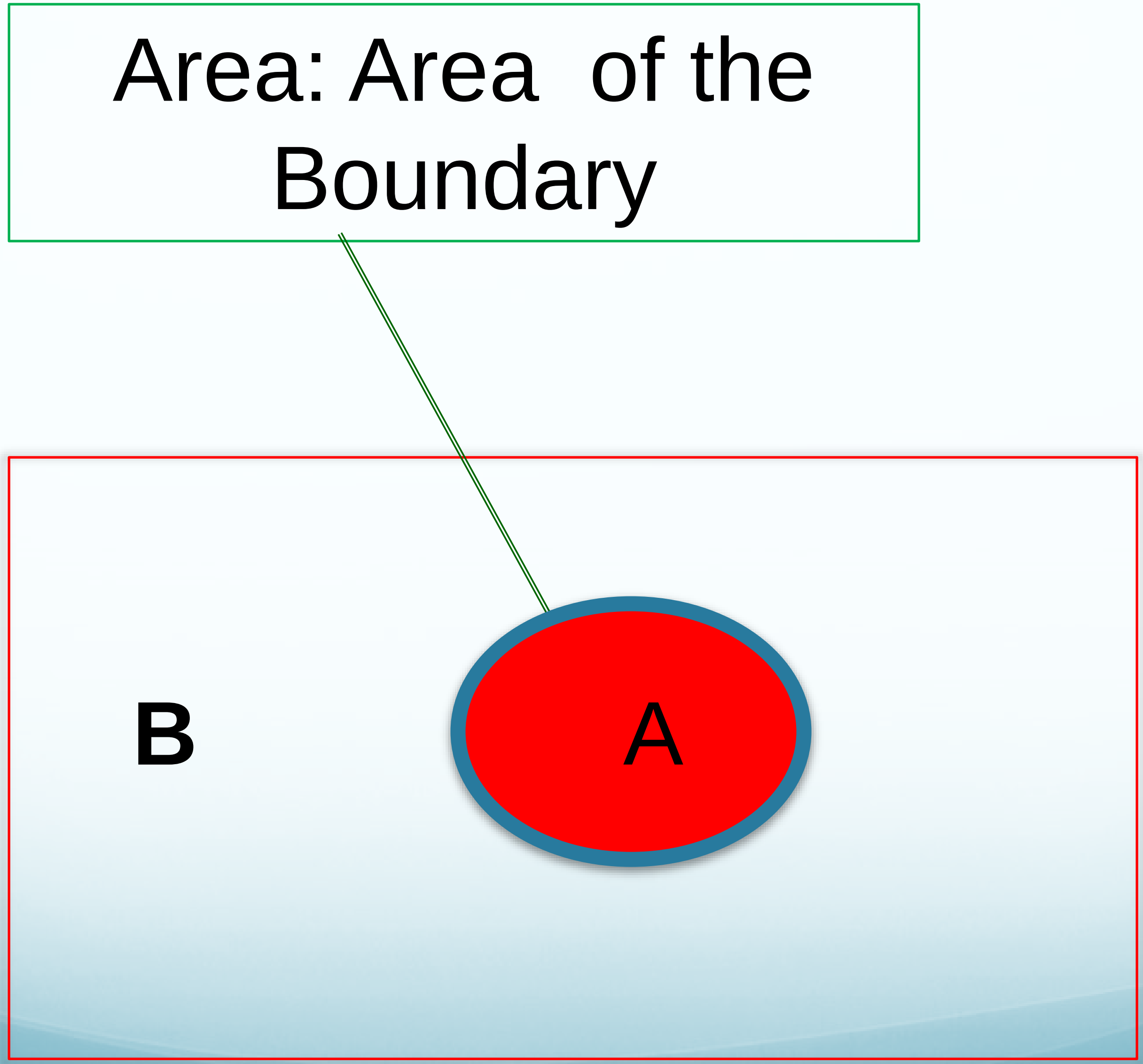
This makes it difficult to define such a notion of entanglement when gravity is present.

In this talk, for spacetimes related via Holography to gauge theories, we will provide such a definition.

It will use the idea of target space entanglement.

We will conjecture that the answer in terms of the bulk is

$$S_{EE} = \frac{\text{Area}}{4G_N}$$


Area: Area of the Boundary
B
A

Important Caveat:

- The Definition of Target Space Entanglement we give is precise.
- The connection with Beckenstein-Hawking formula in bulk is a proposal which might need to be``fine tuned" further.

I will comment on how this definition is different from the Ryu-Takayanagi entanglement below.

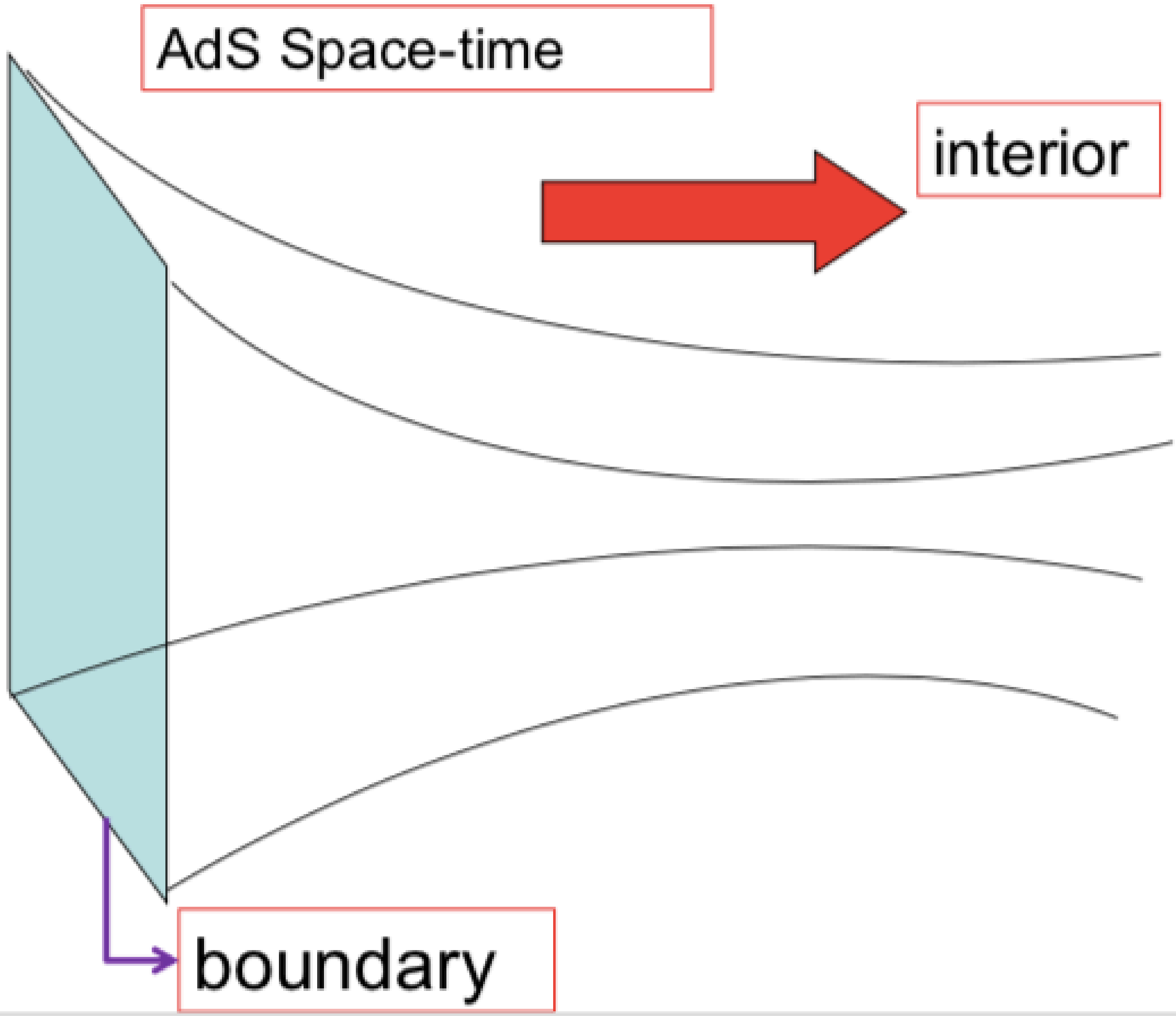
Some References:

1) Mazenc and Ranard (hep-th/1910.07449)

2) S. Das, A. Kashyap, G. Mandal, ST, (hep-th/2004.00613)

3) S. Das, S. Liu, A. Kashyap, G. Mandal, ST, (hep-th/2011.13857)

4) H. Hampapura, J. Harper, A. Lawrence, (hep-th/2012.15683)


AdS Space-time
interior
boundary

Smooth AdS: Large N, large coupling

Correspondence can be extended, by considering near horizon geometries of Dp-branes, to non-AdS spacetimes.

($p \neq 3$)

Bulk not AdS and boundary not CFT

We will in particular be interested in the case of D0 branes where the boundary is quantum mechanics.

A simple context.
Impressive numerical progress.

Caterall & Wiseman; *Hanada, Hyakutake, Ishiki & Nishimura*; *Berkowitz, Rinaldi, Hanada, Ishiki, Shimasaki & Vranas*

(Our results will be more general and also extend to field theories)

Matrix Theory Entanglement

$$S = \frac{N}{2(g_s N)l_s} \int dt [\sum_{i=1}^{9} Tr(D_t X^I)^2 - \frac{1}{l_s^4} \sum_{I \neq J=1}^{9} Tr([X^I, X^J]^2)]$$

+ fermions

$$D_t X^I = \partial_t X^I + i[A_t, X^I]$$

$X^1, X^2 \cdots, X^9$: 9 NXN matrices

$O(N^2)$: degrees of freedom.

Matrix Theory

$U(N)$ Gauge Theory

A_t : Gauge Field

Action Invariant under $X^I \to U X^I U^{-1}$

$$A_t \to iU\partial_t U^{-1} + U A_t U^{-1}$$

$O(N^2)$ Colour degrees of freedom

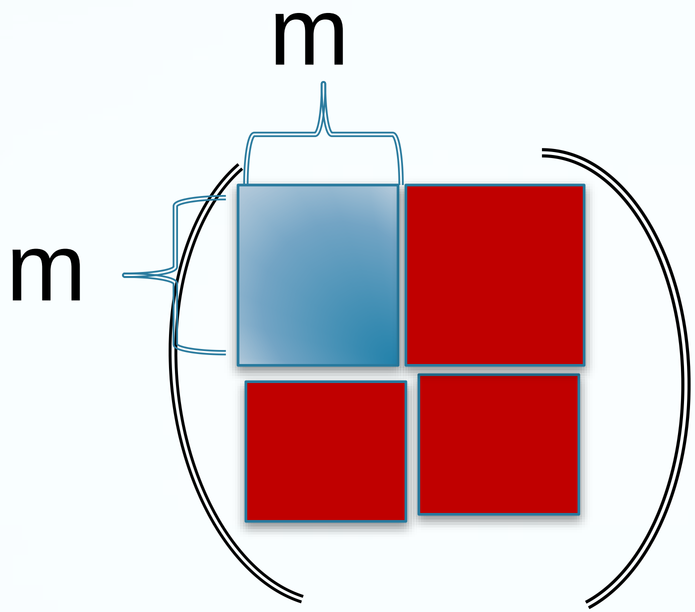
Matrix Theory

Spatial directions of bulk are NXN matrices in boundary.

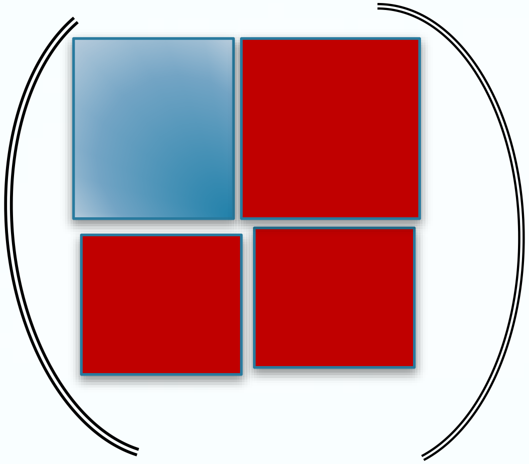
A bulk region, say $x^1 > 0$ corresponds to keeping some of the matrix degrees of freedom and tracing over the rest.

Roughly speaking, we want to ask how entangled are some of the degrees of freedom of the matrices with others?

Gauge Symmetry makes this question complicated!



Keep $m \times m$ block for all matrices and ask what is the entanglement entropy of the resulting density matrix?



NXN Matrix

U(N) gauge transformation will mix the blue block with the rest.

So what is the gauge invariant way to ask this question?

This is the question we will address.

Entanglement Entropy in Matrix Theory

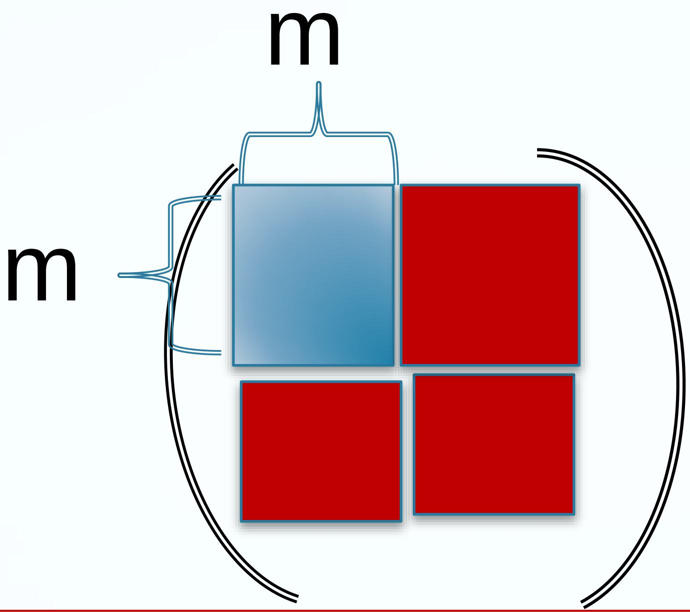
Let us go to a gauge where X^1 is diagonal

$$X^1 = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix}$$

Arrange eigenvalues in descending order

$$\lambda_1 > \lambda_2 \cdots > \lambda_{N-1} > \lambda_N$$

Let's say first m eigenvalues meet the condition $\lambda > a$



Keep $m \times m$ block for all matrices and compute the density matrix for these degrees of freedom.

Note: in general the other matrices, $X^2, \cdots X^9$ will not be diagonal in this gauge.

But the Hilbert space (in this sector) still admits a tensor product decomposition for all matrix degrees of freedom.

By tracing over the remaining matrix elements we can then obtain a density matrix ρ_m

(This is version 1 of the proposal)

Entanglement in the m th sector:

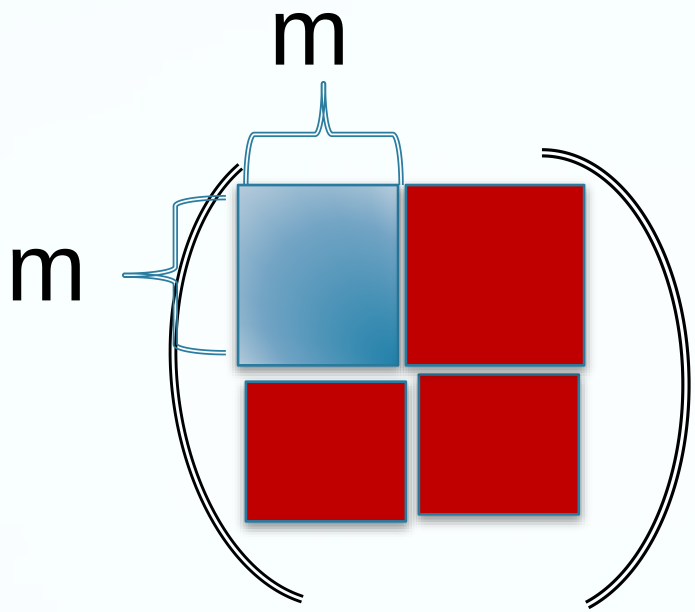
$$S_m = -Tr_{(m)}\rho_m log(\rho_m)$$

Full entanglement given by a sum over all sectors

Target space entanglement

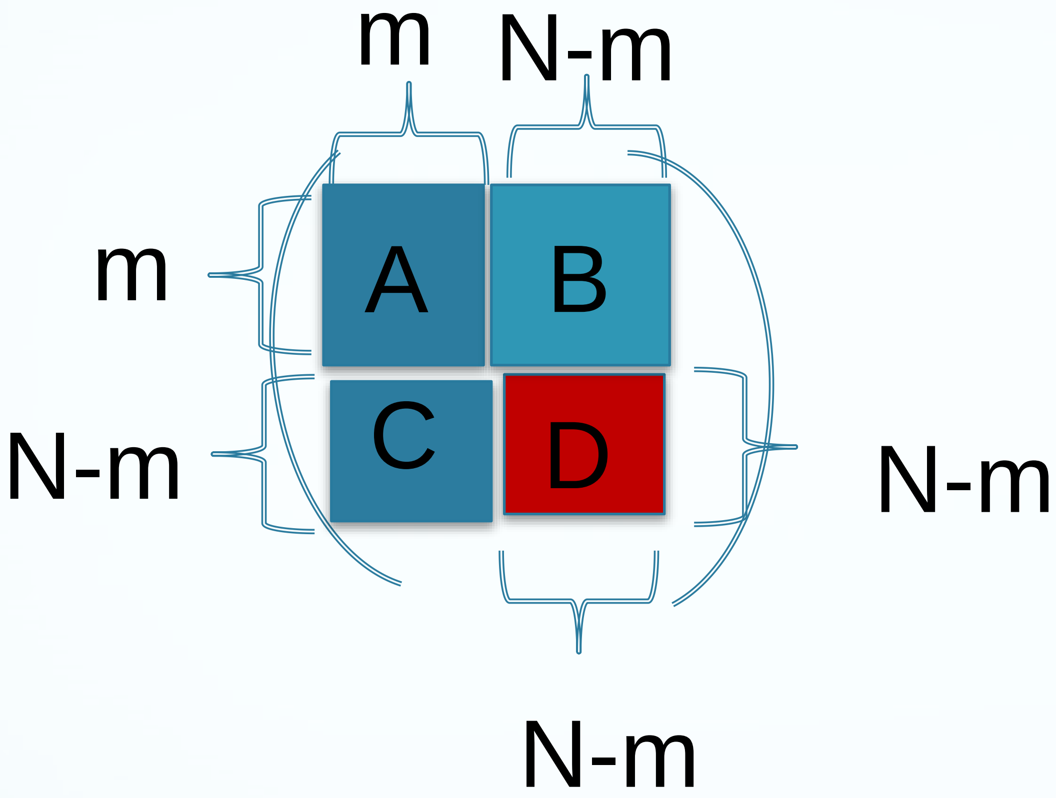
$$S_T = \sum_{m=0}^{m=N} S_m$$

Include possibility that no eigenvalues lies in region of interest



Version 1

Keep $m \times m$ block for all matrices and compute its entanglement entropy.



Version 2

Keep blocks A, B, C and remove D – which is of size (N-m)X(N-m)- for all matrices.

For either version we now have to sum over the various sectors: $m = 0,1, \ldots, N$

Full Entanglement is the sum over the contributions from the various sectors.

Matrix Theory Entanglement

We have used a particular gauge to define the entropy.

But in fact it can be given a gauge invariant meaning using suitable projection operators and the resulting gauge invariant subalgebra.

(For both versions)

By mapping the constraint specifying the region ($x^1 > 0$) to the matrix theory we can make the idea of entanglement precise.

The boundary theory is quantum mechanics. The matrices, $X^1, X^2 \cdots X^9$ are target space degrees.

This is therefore a kind of target space entanglement.

The Ryu-Takayanagi entanglement refers to the entanglement of the boundary theory along the spatial directions in which the boundary theory extends (in higher dimensional cases).

This is an example of base space entanglement.

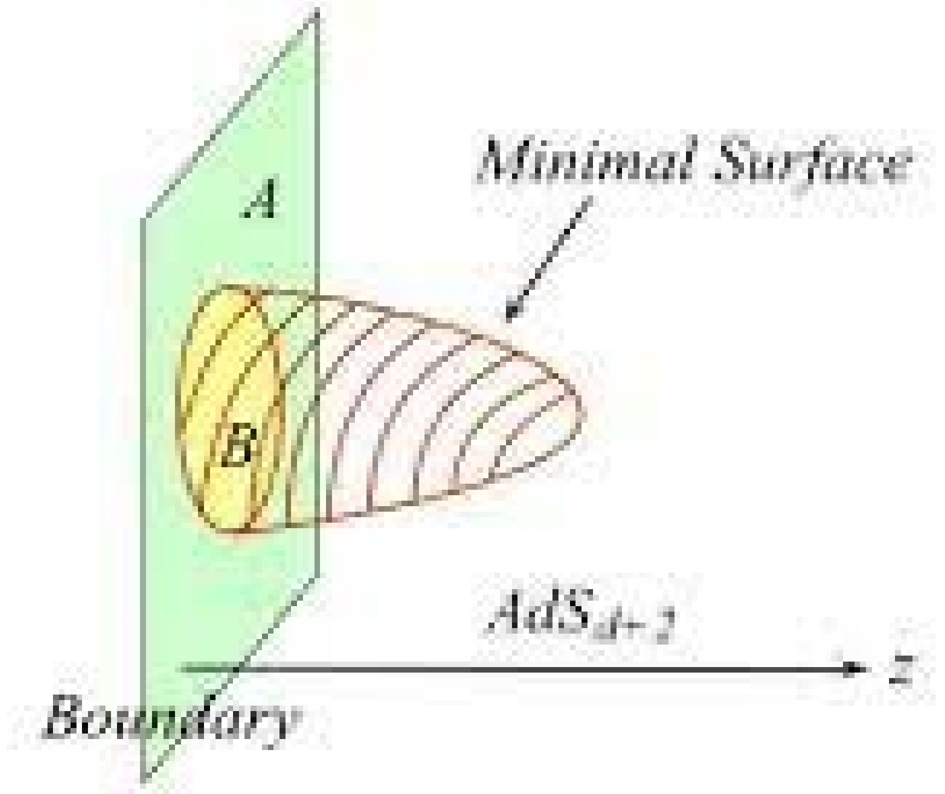
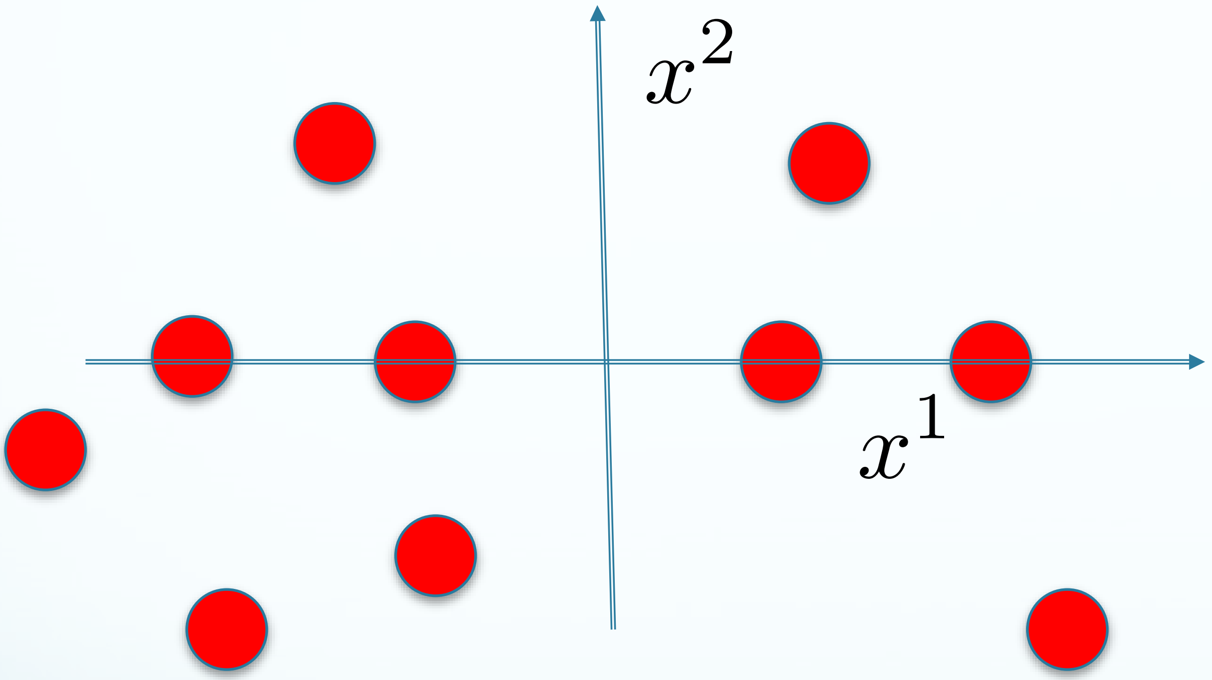


Figure 3: The holographic calculation of entanglement entropy via AdS/CFT.

Entanglement of region B on boundary with A is calculated from the Area of the Minimal surface.

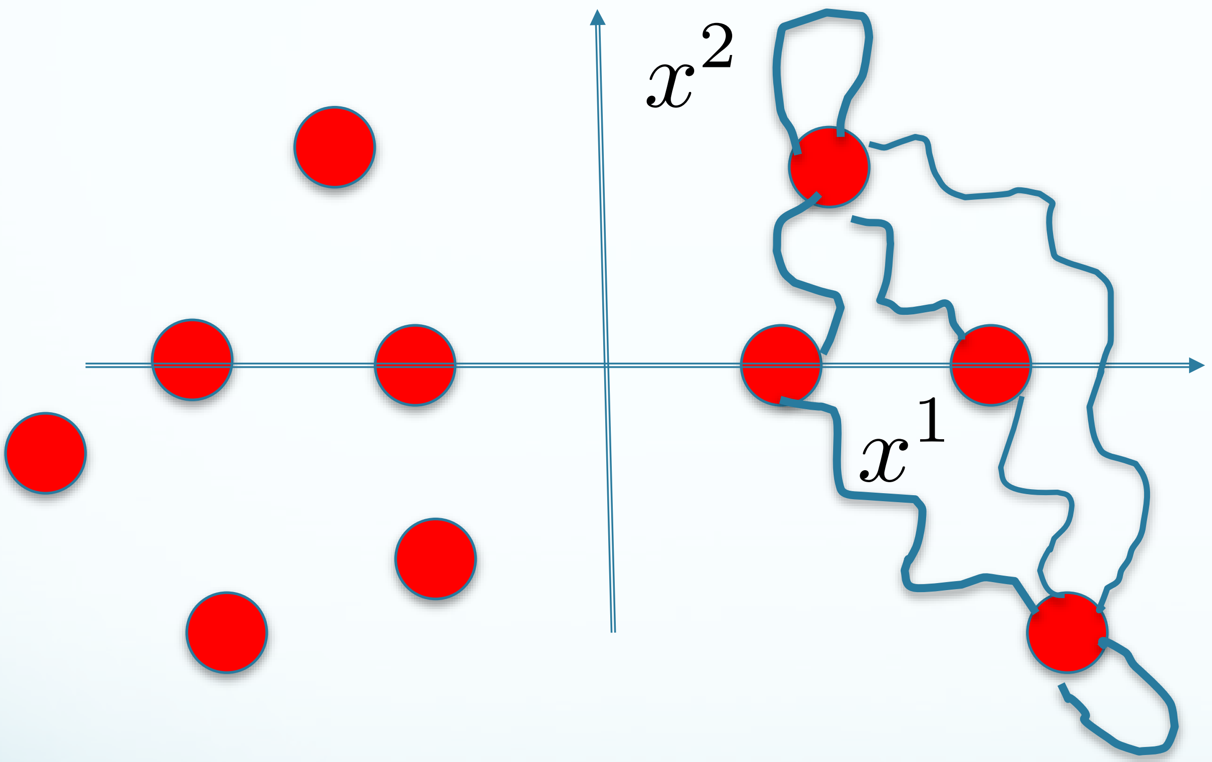
Physical Picture: $x^1 > 0$



4 D0 branes to meet constraint.
(m=4 sector)

Version 1:

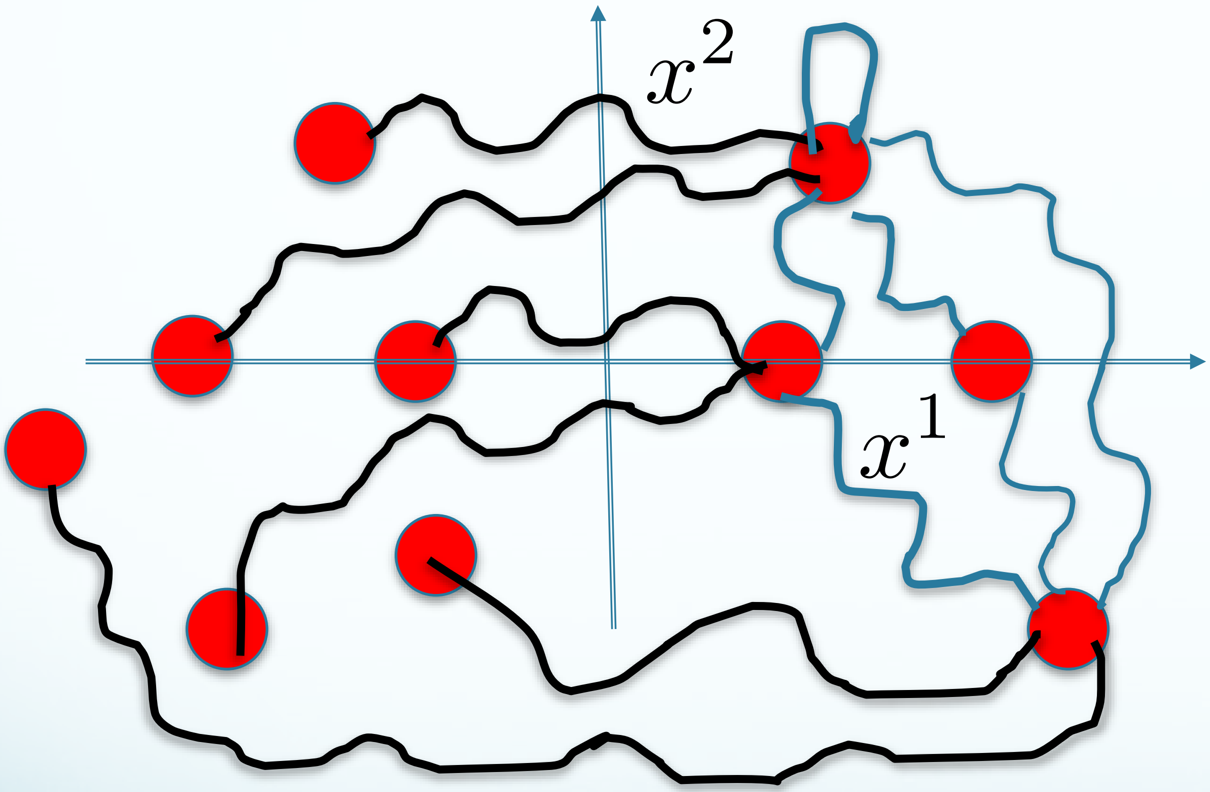
$$x^1 > 0$$



4 D0 branes meet constraint. Keep all degrees of freedom associated with them denoted by blue strings.

Version 2:

$$x^1 > 0$$



In addition also keep black strings running between the 4 D0 branes and the rest.

On the bulk side the entanglement corresponds to strings stretching between the D branes.

On the boundary side it corresponds to ``colour” degrees of freedom of $O(N^2)$.

Version 1

m values

N-m values

Wave function $\Psi(x^a, x^\alpha, Y_{ab}, Y_{a\alpha}, Y_{\alpha a}, Y_{\alpha\beta})$

$$\rho_m(x^a, Y_{ab}, x'^a, Y'_{ab}) =$$

$$\int_{\bar{A}} \Psi^*(x^a, x^\alpha, Y_{ab}, Y_{a\alpha}, Y_{\alpha a}, Y_{\alpha\beta})\Psi(x'^a, x^\alpha, Y'_{ab}, Y_{a\alpha}, Y_{\alpha a}, Y_{\alpha\beta}) dx^\alpha dY_{a\alpha} dY_{\alpha a} dY_{\alpha\beta}$$

m th sector density matrix

$\bar{A}$ Complement of the region of interest, $x^1 < 0$

Target Space Entanglement

Definition can be generalised for any constraint involving the matrices

$$X^I, I = 1 \cdots 9$$

As operators X^I commute, any constraint of the form $F(X^I) > 0$ can be dealt with by diagonalising the constraint and then obtaining the entanglement sector by sector.

E.g. $$F(X^I) = \sum_{I=1}^{9} (X^I)^2 > R^2$$

Target Space Entanglement

Definition can be extended to a general density matrix (instead of a pure state)

In particular to thermal density matrix

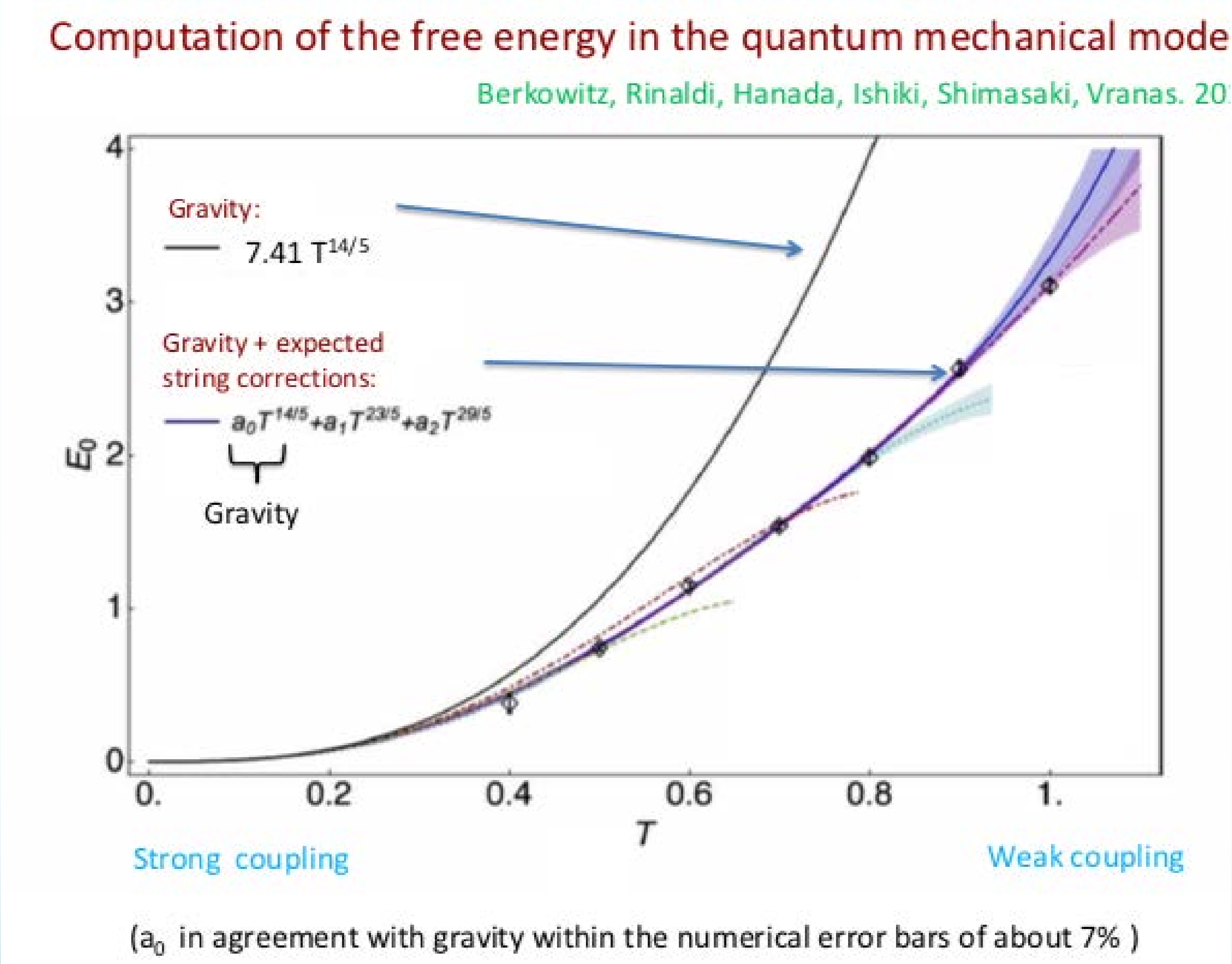
$$\rho = \frac{e^{-\beta H}}{Tr(e^{-\beta H})}$$

It is tempting to speculate that the answer in the matrix theory will agree with the bulk area (in Planck units)

$$S_{TS} = \frac{A}{4G_N} \qquad G_N \rightarrow N^2$$

Connection with bulk area is a proposal. Remains to be checked.

Impressive advances in numerical work on D0 brane system make one hopeful…



(slide taken from J. Maldacena, ICTS lecture 2018)

At finite temperature: Black Hole

$$\frac{E}{\Lambda} = 7.41 N^2 \left(\frac{T}{\Lambda}\right)^{14/5}$$

Agrees with Numerics:

Berkowiz, Rinaldi, Hanada, Ishiki, Shimasaki, Vranas, 2016

Boundary Quantum Mechanics
Characterised by one energy scale:
Λ

$$H = \frac{(g_s N)^{1/3}}{2 l_s} \mathrm{Tr}\left[\frac{1}{N}\sum_{I=1}^{9}(\tilde{P}^I)^2 + N\sum_{I\neq J=1}^{9}[\tilde{X}^I, \tilde{X}^J]^2\right] + \text{fermions}$$

$$X^I = (g_s N)^{1/3} l_s \tilde{X}^I \qquad P^I = \frac{1}{(g_s N)^{1/3} l_s}\tilde{P}^I$$

$$\Lambda = \frac{(g_s N)^{1/3}}{l_s}$$

Summary

- Entanglement is a key to understanding the strangeness in quantum mechanics, and might be a key to understanding quantum gravity.

- We discussed how to define target space entanglement in Matrix theories which can be boundary holograms.

Summary

- This should teach us something interesting about entanglement in the bulk

- The Matrix theory entanglement we have defined could be useful also in condensed matter etc systems (Frenkel, Hartnoll)

Much excitement lies ahead!

Thank You!

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Ryu Takayanagi Formula

For the higher dimensional cases, where the boundary theory lives along some spatial directions (eg along the 3 spatial directions for 3+1 dimensional CFT):

One can ask how entangled are the degrees of freedom in some region, along these spatial directions, with the rest?

$$S_{EE} = \frac{A_{min}}{4G_N}$$

Remarkable Result of Ryu and Takayanagi!

This is ``base space” entanglement.

We have been interested here in entanglement along the Target Space, in which the boundary theory does not extend.

The D0 branes (or Matrix Mechanics) only lives in time and no spatial directions. So the 9 matrices, $X^1, X^2, \cdots X^9$ are target space directions.

Connection to the RT formula remains to be understood.

Near-horizon D0 branes (10 dim)

$$ds^2_{string} = -H_0(r)^{-1/2} dt^2 + H_0(r)^{1/2} [dx_1^2 + \cdots + dx_9^2]$$

$$H_0(r) = \frac{R^7}{r^7}$$

$$r^2 = x_1^2 + x_2^2 + \cdots x_9^2$$

$$e^{-2\phi} = g_s^{-2} H_0(r)^{-3/2}$$

$$R^7 = \frac{(2\pi)^7}{7\Omega_8} l_s^7 (g_s N).$$

IIA Supergravity solution valid when

$$(N \gg 1)$$

$$g_s^{1/3} N^{1/7} \ll r/l_s \ll (g_s N)^{1/3}$$

$$\frac{g_s^{1/3} N^{1/7}}{l_s} \ll E \ll \frac{(g_s N)^{1/3}}{l_s}$$

$$(\quad E \sim r/l_s^2)$$

Duality

Matrix theory weakly coupled when

$$E \gg \Lambda$$

Gravity theory is highly curved in this region

Similarly when gravity theory weakly coupled, matrix theory strongly coupled, $E \ll \Lambda$.

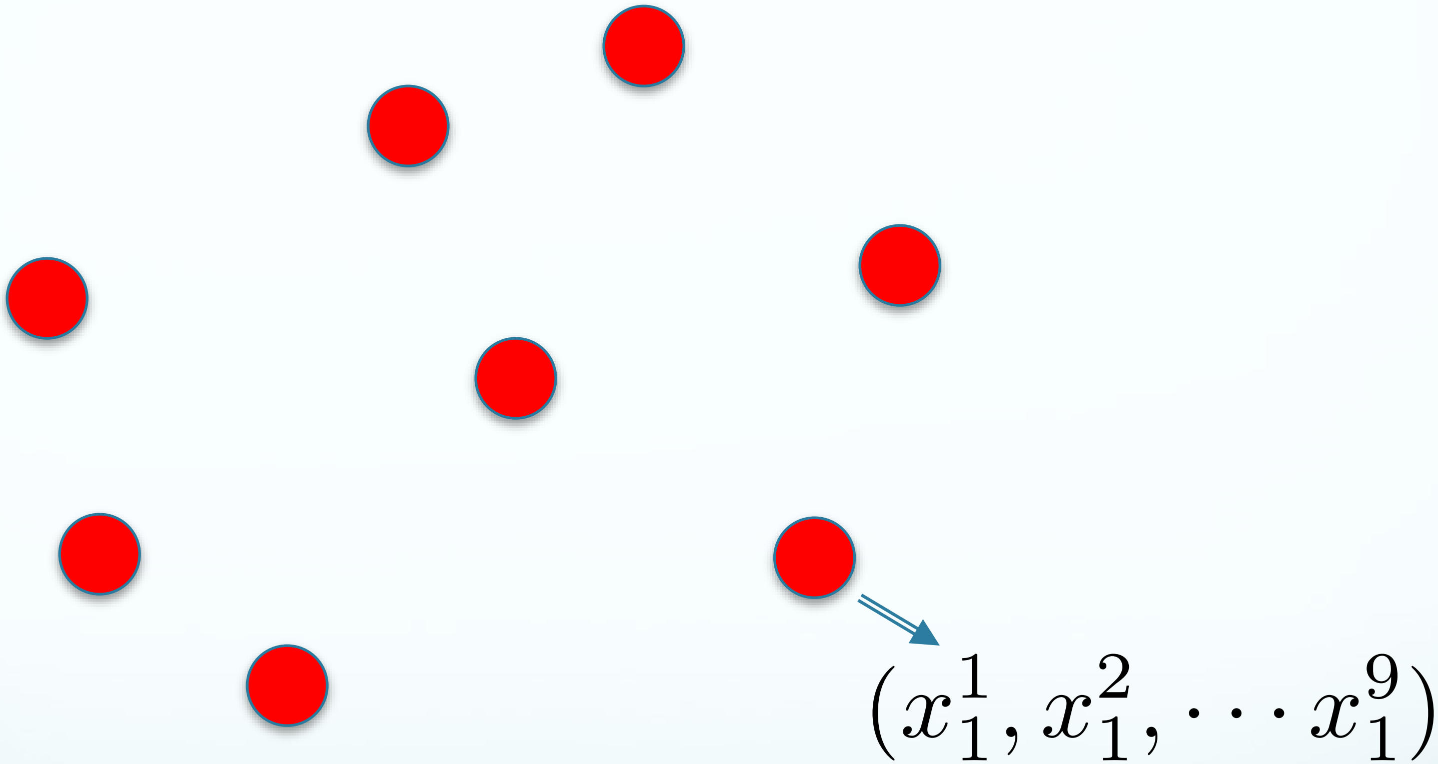
Boundary Quantum Mechanics

16 supersymmetries

Moduli space of vacua (Coulomb branch) : All 9 matrices commute.

$$X^1 = diag(x^1_1, x^1_2, \cdots x^1_N),$$
$$X^2 = diag(x^2_1, x^2_2, \cdots x^2_N),$$

$(x^1_1, x^2_1, \cdots x^9_1)$: location of first D0 brane etc

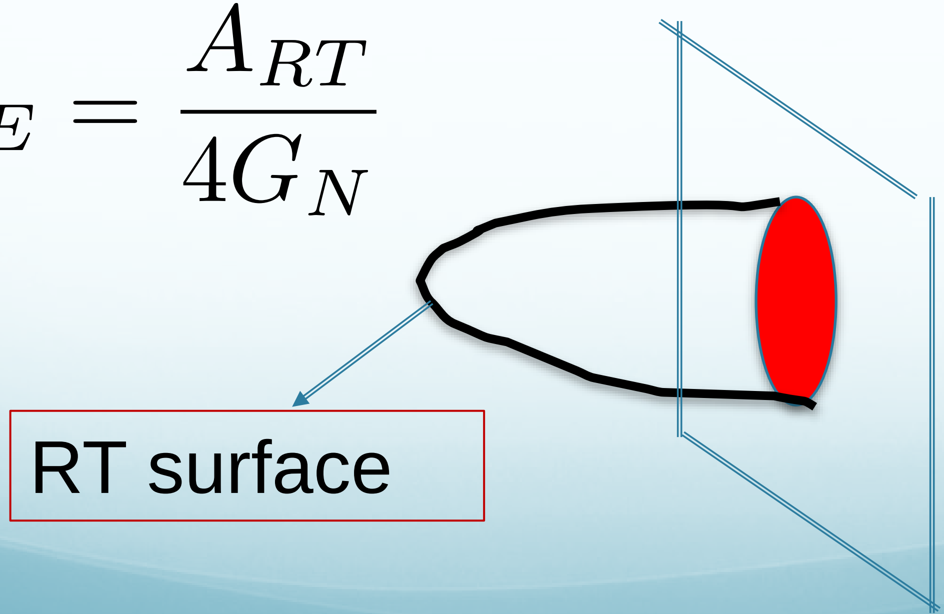


D0 branes in Bulk (in Coulomb branch) One to one correspondence with the vacuua

When field theory is a CFT on the boundary of AdS

This entanglement is given by the Ryu-Takayanagi formula

$$S_{EE} = \frac{A_{RT}}{4G_N}$$



Gauge Invariant Description

Another way to think about entangelment entropy:

Associated with a sub-algebra of observers.

Density matrix lies in subalgebra

Gives the correct expectation value of all operators in subalgebra.

Gauge Invariant Description

$$|\psi >= \frac{1}{\sqrt{2}}[|+- > -|-+ >]$$

Instead of tracing over 2nd spin.
Think of Subalgebra:

$$\sigma_i \otimes I \text{ , } \quad I \otimes I$$

$$\rho = (\frac{1}{2})I \otimes I)$$

For target space entanglement

Define projector:

$$P^1 = \int_{x>0} dx\ \delta(x\mathbf{I} - X^1)$$

For target space constraint $x^1 > 0$

Version 1:

Project all operators by acting with this projector.

Then take a trace to obtain gauge invariant operators.

$$X^I \to (X^I)^{P_1} = P^1 X^I P^1,\ \Pi_J \to (\Pi_J)^{P_1} = P_1 \Pi_J P^1$$

$$Tr((X^I)^{P_1}, ...(\Pi_J)^{P_1}, \cdots)$$

Version 1:

This gives rise to a sub algebra.

Entangelment entropy is associated with this sub algebra.

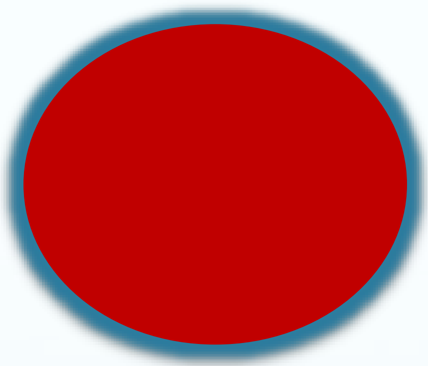
Similarly for version 2.

Subalgebra defined With an appropriately chosen projector.

Proposal for Bulk Entanglement

And $$S_{Bulk} = \frac{A_\partial}{4G_N} = S_T$$

Target space entanglement



Boundary with area A_∂

(In one of the two versions of our proposal for S_T)

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$$X^2 = diag(x^2_1, x^2_2, \cdots x^2_N),$$

$(x^1_1, x^2_1, \cdots x^9_1)$: location of first D0 brane etc

$$(x_1^1, x_1^2, \cdots x_1^9)$$

D0 branes in Bulk (in Coulomb branch) One to one correspondence with the vacuua

Testing The Proposal

Calculation in Matrix Quantum Mechanics should reproduce this result.

Including N^2 factor

(for $T_0, T_0' \ll 1$)

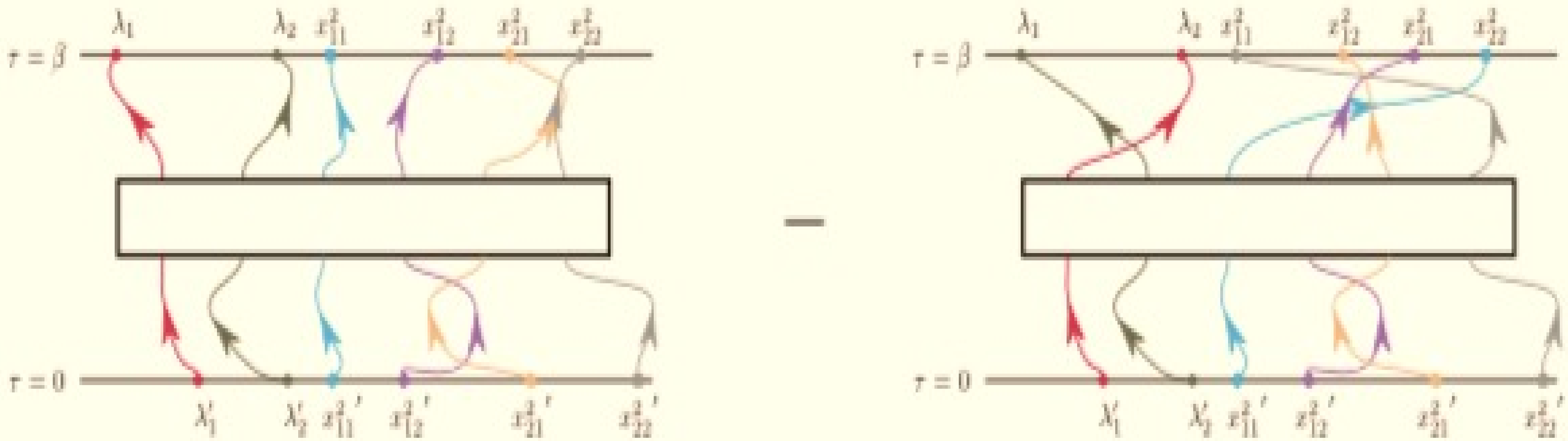
Note: Encouraging that gravity result is expressible in terms of quantities in gauge theory. Not true if $S_{Bulk} \sim A/l_s^8$

How can one calculate target space entanglement?

Path integral methods can be used to calculate density matrix (at Finite Temperature)

$$\langle \lambda_i, X^L_{ij} | \hat{\rho}_0 | \lambda'_i, (X^L_{ij})' \rangle$$

$$= \frac{1}{N!} \sum_{\sigma \in S(N)} (-)^\sigma \int_{\lambda_i(0)=\lambda'_i}^{\lambda_i(\beta)=\lambda_{\sigma(i)}} \mathcal{D}\lambda_i(\tau) \int_{X^L_{ij}(0)=(X^L_{ij})'}^{X^L_{ij}(\beta)=X^L_{\sigma(i)\sigma(j)}} \mathcal{D}X^L_{ij}(\tau) \exp[-S_\beta]$$



Trajectories contributing to density matrix

By tracing over appropriate degrees of freedom the reduced density matrix can then be calculated.

Thank You!

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Thank you!