Noninteracting particles in a harmonic trap with a stochastically driven center

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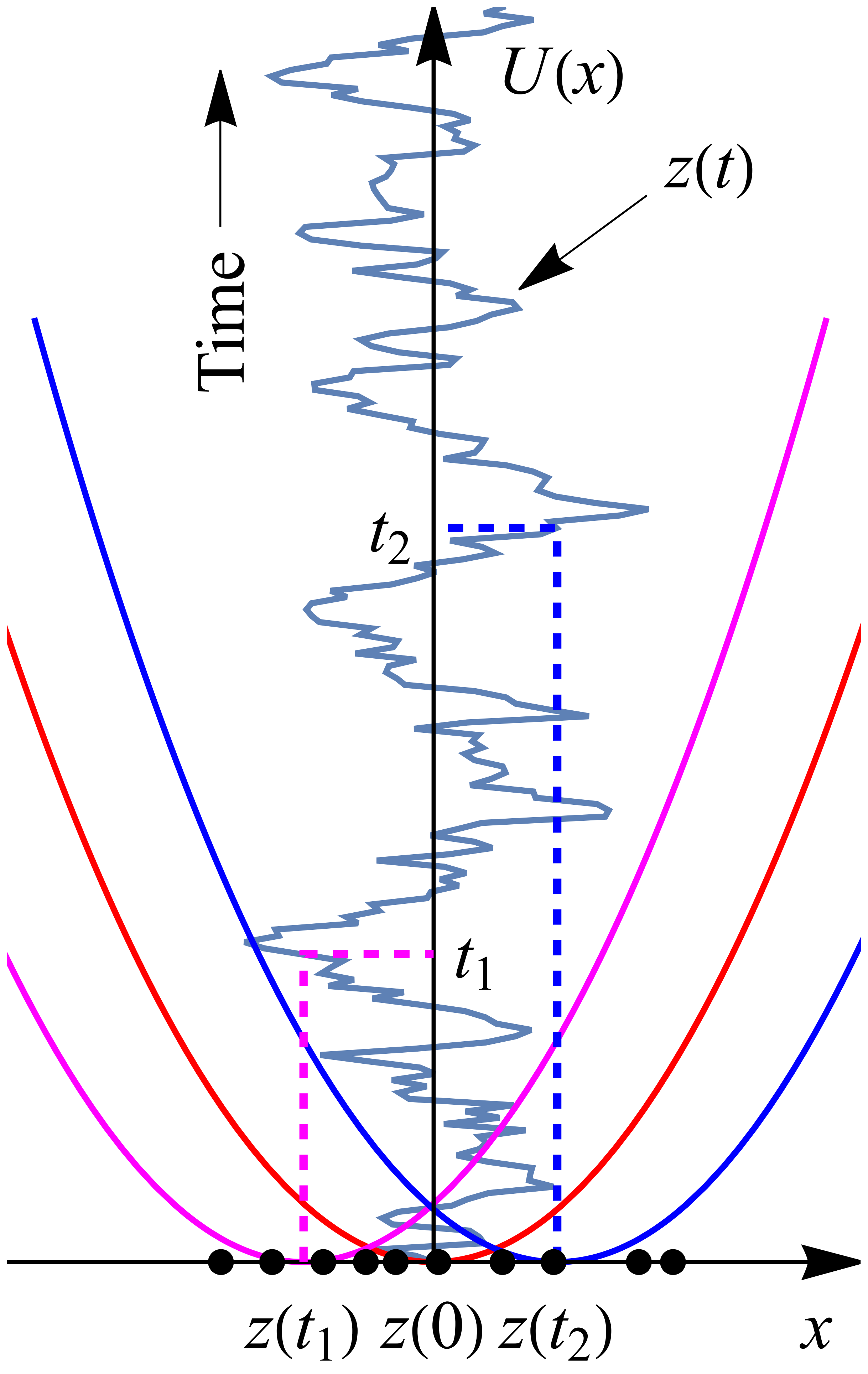
Main message of the talk

Stochastic modulation of a trap center generates strong correlations between independent particles in a harmonic trap.



System of interacting particles in equilibrium

- Energy function associated with a given configuration $\{x_1, x_2, \ldots, x_N\}$

$$E[\{x_i\}] = \sum_{i=1}^{N} U(x_i) + \sum_{i \neq j} U_2(x_i, x_j) + \sum_{i \neq j \neq k} U_3(x_i, x_j, x_k) + \cdots$$

- The joint probability density function (JPDF)

$$P_{\mathrm{eq}}[\{x_i\}] = \frac{1}{Z_N} e^{-\beta E[\{x_i\}]}$$

Micro and Macro observables

- Average density profile
- Correlation function: $C_{i,j} = \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle$
- Extreme value statistics and order statistics
- Spacing/gap distribution (between successive positions)
- Full counting statistics: # particles in a given interval, e.g., in $[-L, L]$

Noninteracting Limit: $U_2 = U_3 = \cdots = 0$

- The JPDF: $P_{\text{eq}}[\{x_i\}] = \prod_{i=1}^{n} p(x_i)$ where $p(x) = \dfrac{e^{-\beta U(x)}}{\int_{-\infty}^{\infty} e^{-\beta U(x')}\, dx'}$

- $\{x_1, x_2, \ldots, x_N\}$ are IID random variables.

- For this ideal gas, all the observables can be computed exactly.

Interacting systems in equilibrium

- The joint distribution is not factorable $\Longrightarrow$ the observables are very hard to compute.

- One example, where the observables can be computed is the Riesz gas:

$$E[\{x_i\}] = \frac{1}{2}\sum_i x_i^2 + \frac{J\mathrm{sgn(k)}}{2}\sum_{j\neq i}\frac{1}{|x_i - x_j|^k}$$ where $k > -2$

$k = 2$: Calogero-Moser model, $k \to 0^+$: log-gas, $k = -1$: Jellium model

[Agarwal, Dhar, Kulkarni, Kundu, Kethepalli, Santra, SS, Majumdar, and other collaborators]

Nonequilibrium systems

- When a many-body system is subjected to an external stochastic drive that breaks the time-reversal symmetry, one may reach a NESS.

- The stationary joint probability distribution $P_{\mathrm{st}}(x_1, x_2, \ldots, x_N)$ is not *a priori* given and is often difficult to obtain explicitly.

- Even when this stationary joint distribution is known explicitly, computing the observables is usually extremely hard for strongly interacting out-of-equilibrium systems — there is no general prescription.

A class of models (CIID structure)

- The JPDF: $P_{\rm st}(x_1, x_2, \ldots, x_N) = \int_{-\infty}^{\infty} du\, h(u) \prod_{j=1}^{N} p(x_j \,|\, u)$

- **Model-I**: Simultaneous resetting of independent Brownian motions:

  $h(u) = re^{-ru}$ and $p(x \,|\, u) = \dfrac{e^{-x^2/(4Du)}}{\sqrt{4\pi D u}}$ [Biroli, Larralde, Majumdar, Schehr (2023, 2024)]

- **Model-II**: Independent Brownian particles in a harmonic trap where the stiffness undergoes a dichotomous process, $\mu = \mu_1 \leftrightarrow \mu_2$: [Biroli, Kulkarni, Majumdar, Schehr (2024)]

  $h(u) = C\, u^{R_1 - 1}(1-u)^{R_2 - 1}\, V(u)$ and $p(x \,|\, u) = \dfrac{e^{-x^2/(2V(u))}}{\sqrt{2\pi V(u)}}$ with $V(u) = D\left[\dfrac{1-u}{\mu_1} + \dfrac{u}{\mu_2}\right]$

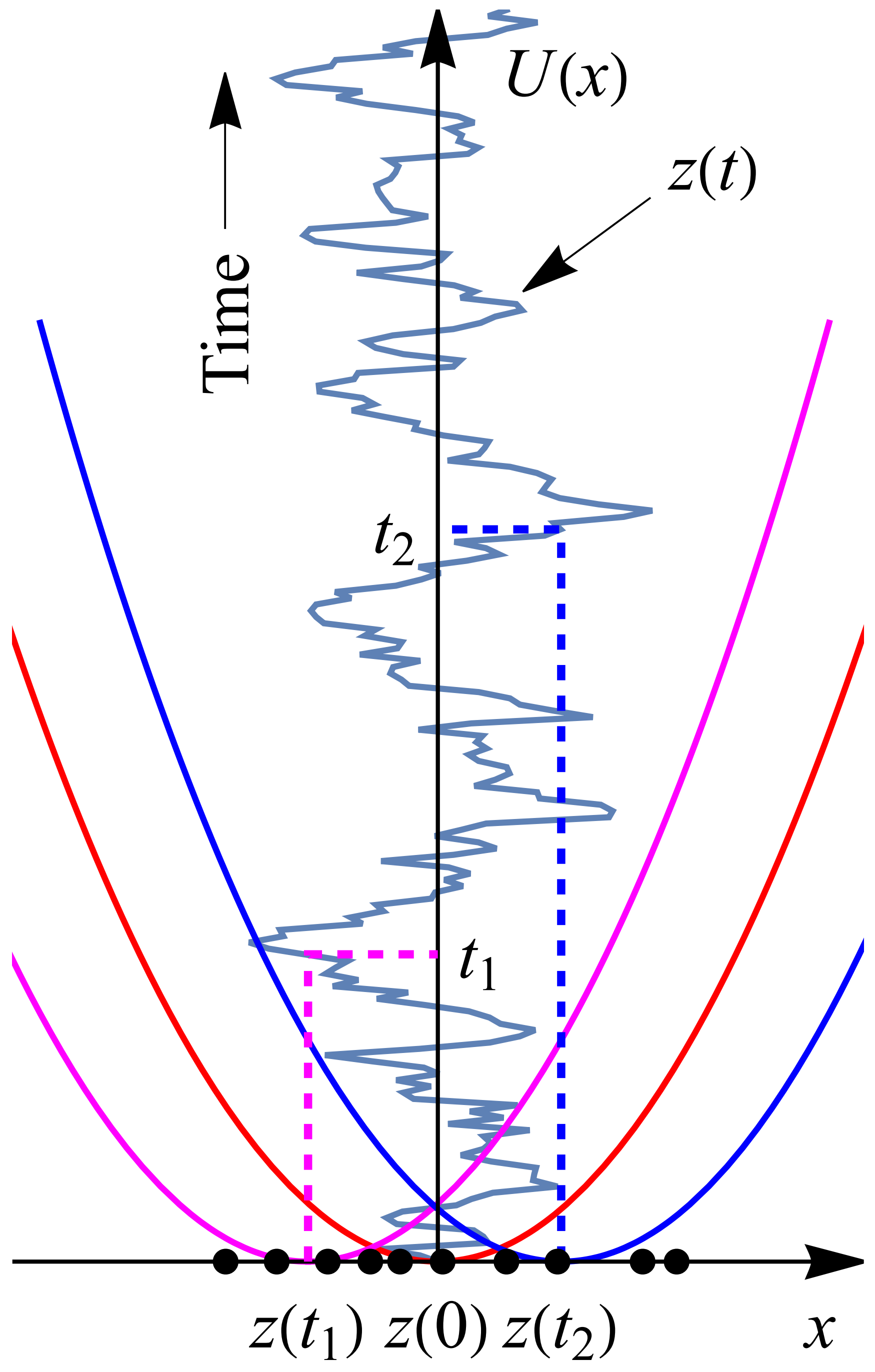
What are the necessary and sufficient conditions for a CIID structure?

A new class of analytically solvable models

- Consider a system of N noninteracting particles on a line in a harmonic trap $U(x) = \frac{1}{2}\mu\left[x - z(t)\right]^2 \implies$ Energy: $E\left[\{x_i\}, t\right] = \sum_{i=1}^{N} U(x_i)$

- Langevin equation: $\frac{dx_i}{dt} = -\mu\left[x_i - z(t)\right] + \sqrt{2D}\,\eta_i(t)$

- The trap center $z(t)$ undergoes a bounded stochastic motion, i.e., $\langle |z(t)| \rangle \sim O(1)$, does not grow with time.



Stationary joint probability density function

- For general stochastic drive $z(t)$, the stationary JPDF has the CIID structure:

$$P_{\rm st}(x_1, x_2, \ldots, x_N) = \int_{-\infty}^{\infty} du\, h(u) \prod_{j=1}^{N} p(x_j \,|\, u)$$

where $p(x_j \,|\, u) = \frac{\sqrt{\mu}}{\sqrt{2\pi D}} \exp\left(-\frac{\mu (x_j - u)^2}{2D} \right)$ and

$h(u)$ is the stationary PDF of a random variable u that evolves via the Langevin equation: $\frac{du}{dt} = -\mu\, u + \mu\, z(t)$

- Given this CIID structure, we can compute the asymptotic large N behavior of all the observables mentioned in terms of the single function $h(u)$.

Representative examples

Telegraphic drive

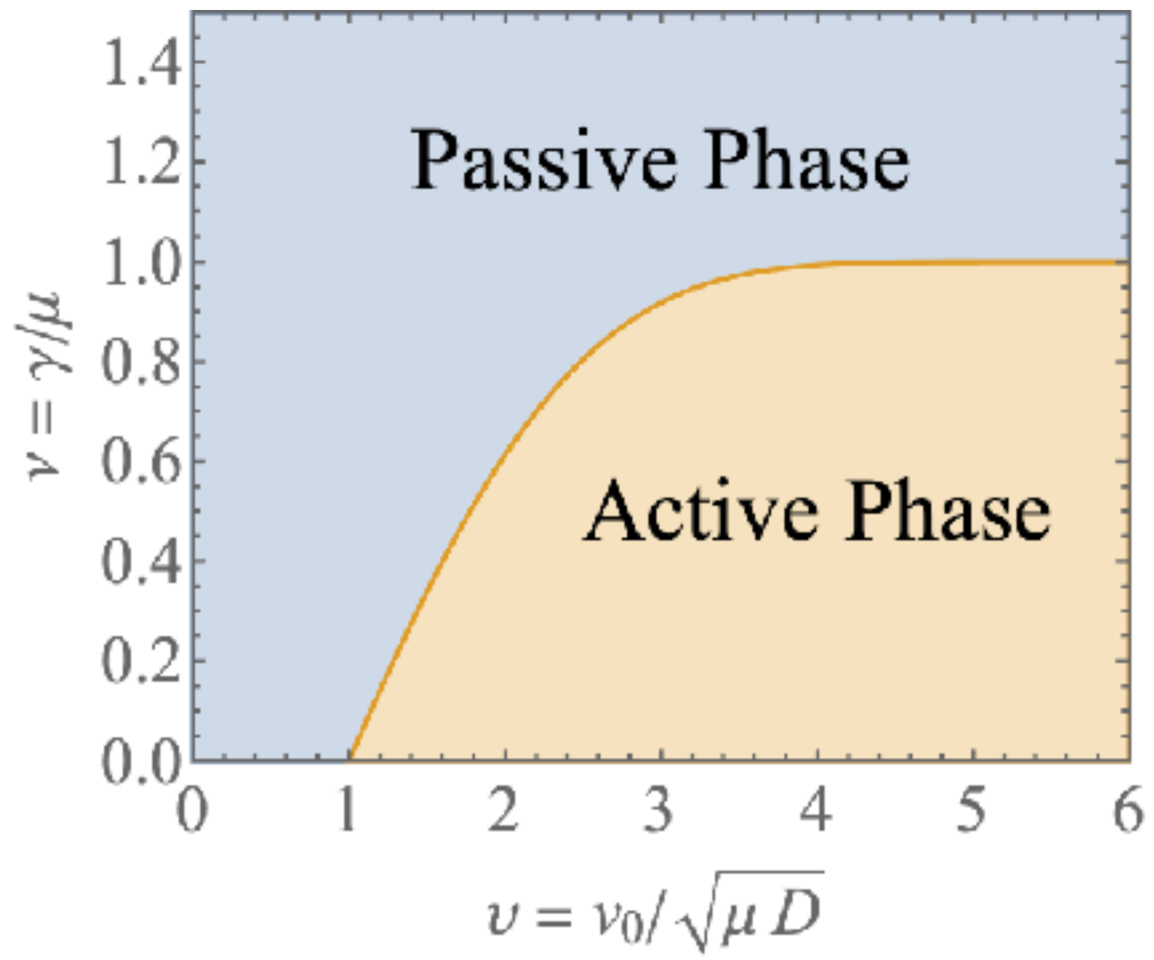
- $(\mu/v_0)\, z(t) = \sigma(t) = \pm 1$ is a dichotomous telegraphic noise that switches with a rate γ.
- Langevin equation: $\dfrac{du}{dt} = -\mu u + v_0\, \sigma(t)$ [RTP in a harmonic trap]
- Steady-state distribution: $h(u) = \dfrac{2^{1-2\nu}}{B(\nu,\nu)} \dfrac{\mu}{v_0} \left[1 - \left(\dfrac{\mu u}{v_0}\right)^2\right]^{\nu-1}, \quad u \in \left[-\dfrac{v_0}{\mu}, \dfrac{v_0}{\mu}\right]$ with $\nu = \dfrac{\gamma}{\mu}$.

Ornstein-Uhlenbeck drive: $\dfrac{dz}{dt} = -\dfrac{z}{\tau_0} + \sqrt{2D_0}\, \xi(t)$

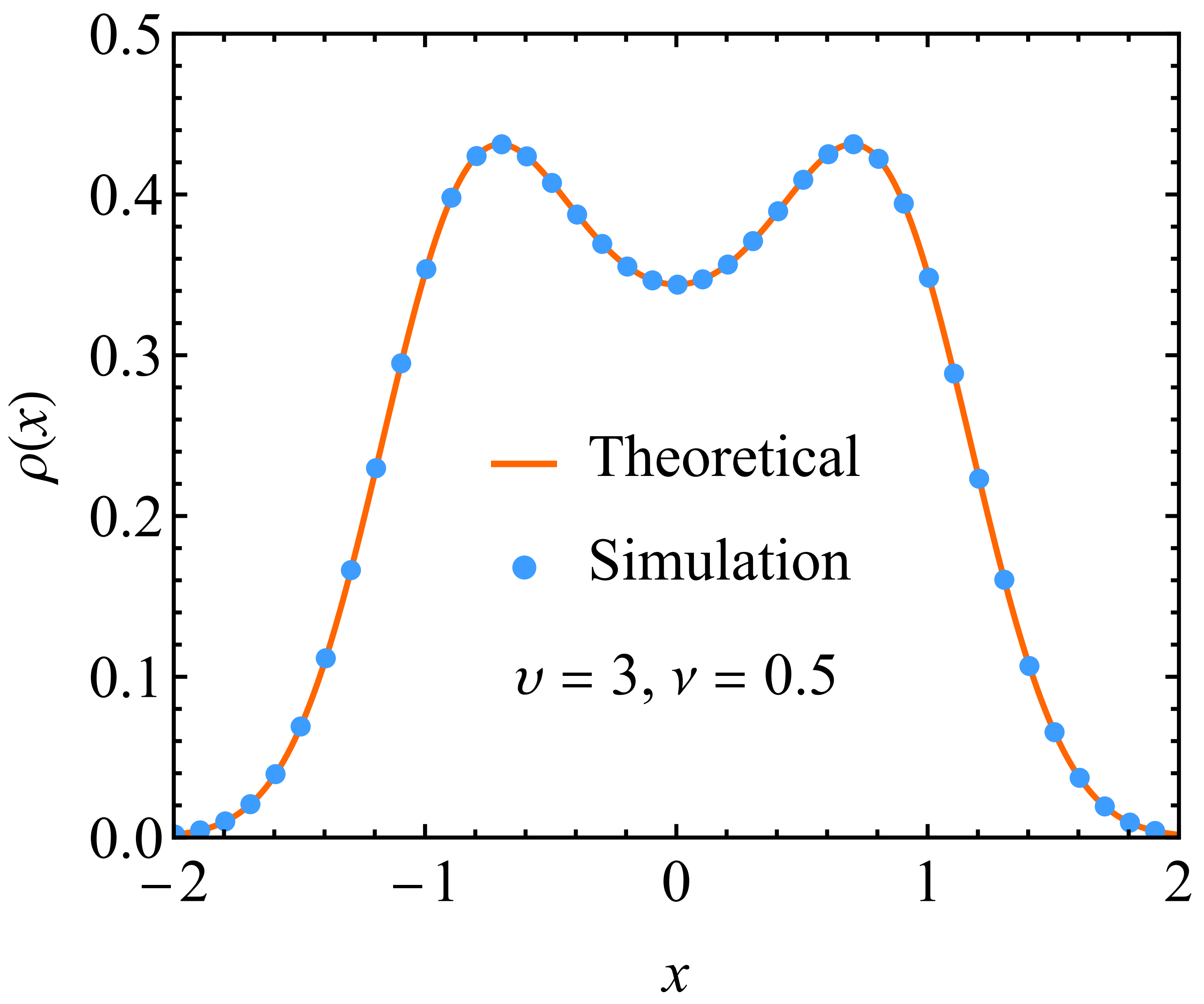
- Steady-state distribution: $h(u) = \sqrt{\dfrac{1+\mu\tau_0}{2\pi\mu\tau_0^2 D_0}} \exp\left(-\dfrac{(1+\mu\tau_0)\, u^2}{2\mu\tau_0^2 D_0}\right)$

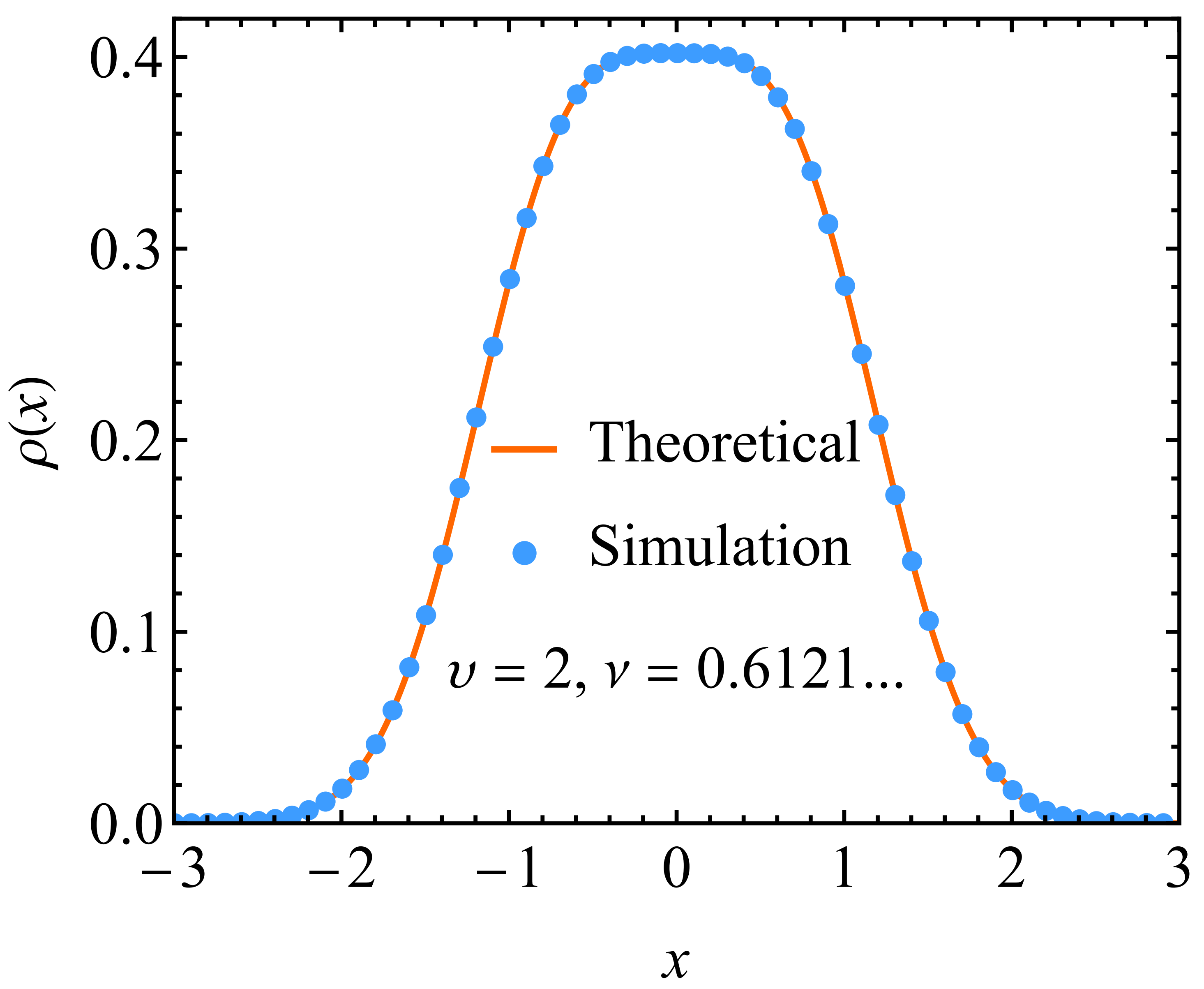
Average density profile

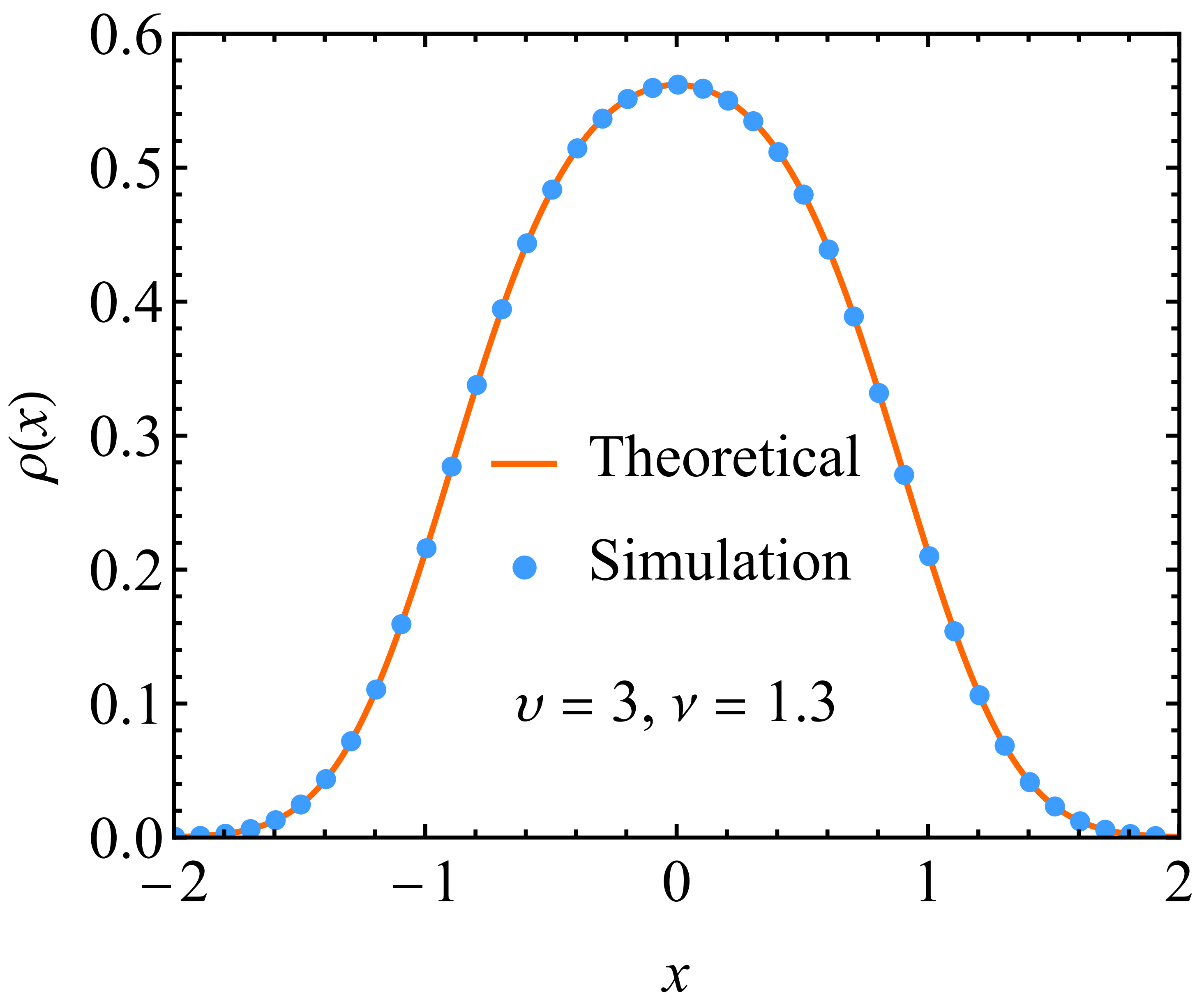
$$\rho(x) = \int_{-\infty}^{\infty} du\, h(u)\, p(x\,|\,u) = \frac{\sqrt{\mu}}{\sqrt{2\pi D}} \int_{-\infty}^{\infty} du\, h(u)\, \exp\left(-\frac{\mu(x-u)^2}{2D}\right)$$



Example: $(\mu/v_0)\, z(t) = \sigma(t) = \pm 1$ is a dichotomous noise







Correlation function

$$C_{i,j} = \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle = \mathrm{Var}(u) + \delta_{i,j} \frac{D}{\mu}$$

where

$$\mathrm{Var}(u) = \langle u^2 \rangle - \langle u \rangle^2 = \int_{-\infty}^{\infty} u^2 \, h(u) \, du - \left[\int_{-\infty}^{\infty} u \, h(u) \, du \right]^2$$

Order statistics

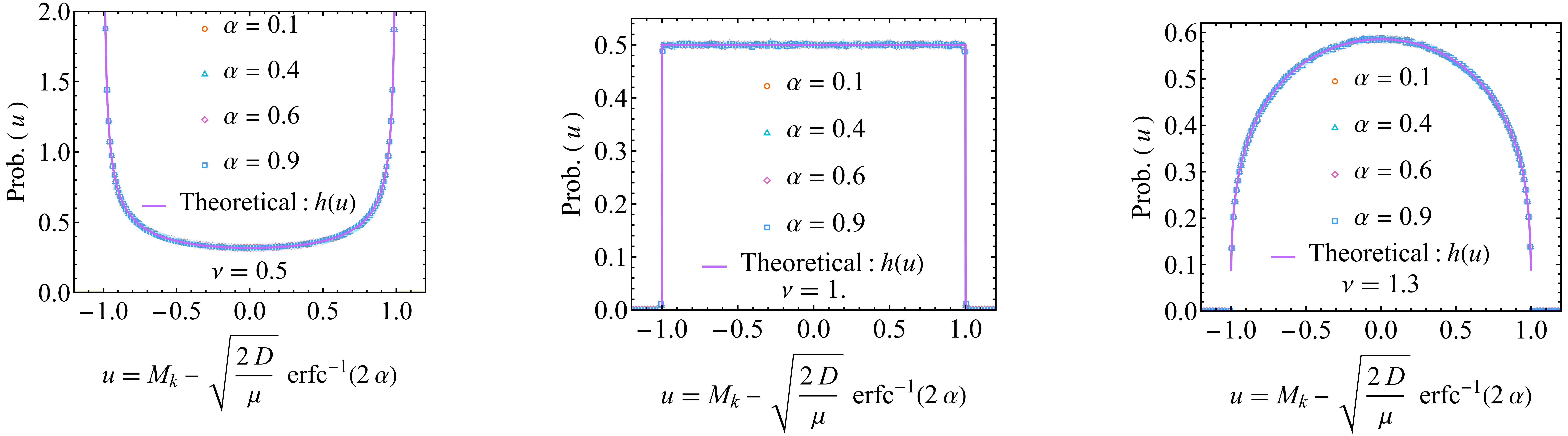
We first arrange the positions $\{x_1, x_2, \ldots, x_N\}$ in descending order $\{M_1 > M_2 > \cdots > M_N\}$ such that

$M_1 = \max\{x_1, x_2, \ldots, x_N\}, \;\; M_N = \min\{x_1, x_2, \ldots, x_N\},$ and
M_k represents the position of the k-th particle from the right.

$$\mathrm{Prob.}[M_k = w] \simeq h(w - l_k) \quad \text{where} \quad l_k \simeq \begin{cases} \sqrt{\dfrac{2D}{\mu}}\,\mathrm{erfc}^{-1}(2\alpha) & \text{when } \dfrac{k}{N} = \alpha \sim O(1) \\ \sqrt{\dfrac{2D}{\mu}\ln N} & \text{when } k \sim O(1) \end{cases}$$

Example of order statistics for dichotomous drive

$$(\mu/v_0)\, z(t) = \sigma(t) = \pm 1$$



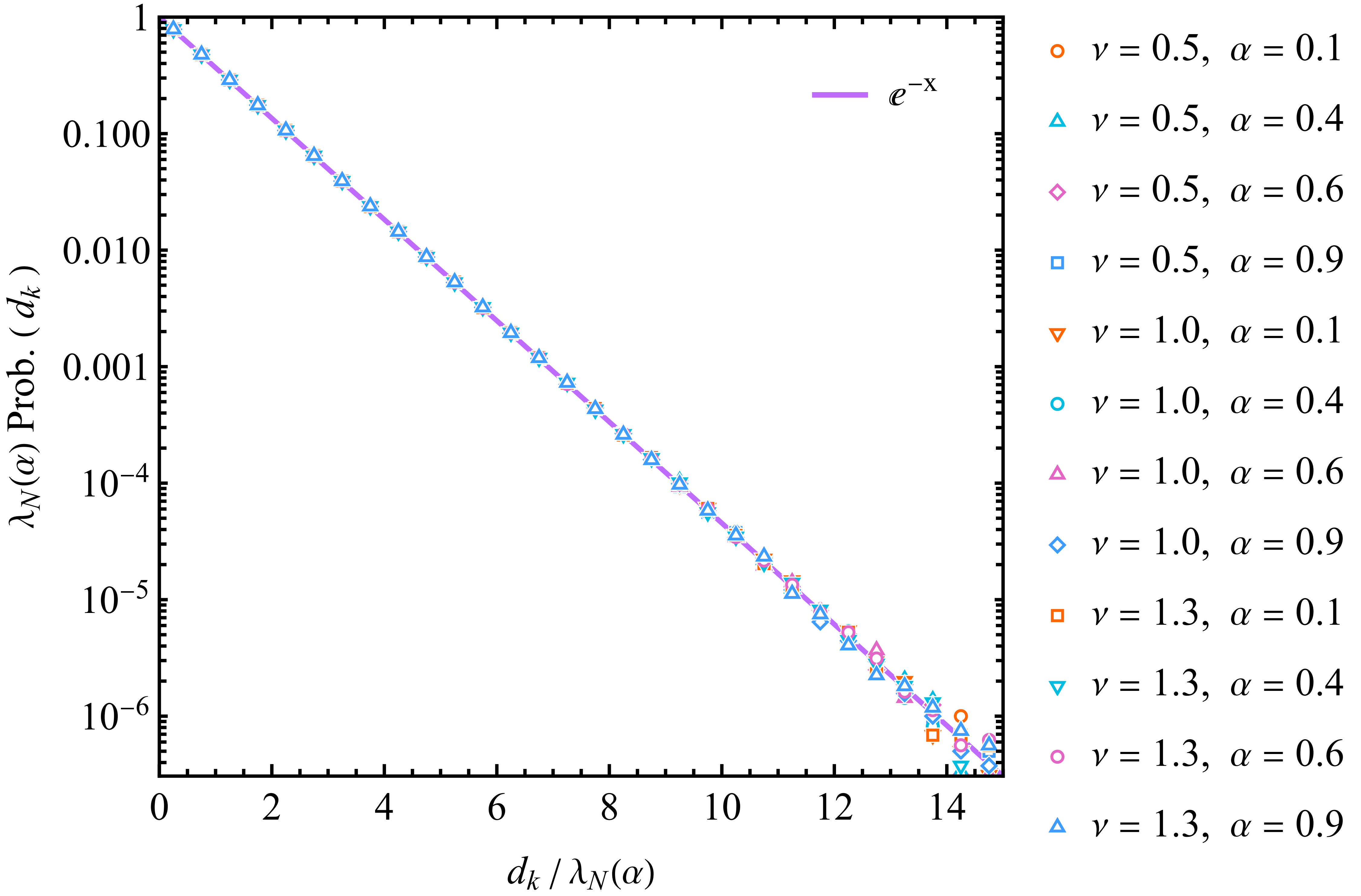
Gap statistics

$$\text{Prob}.\,(d_k = g) = \int_{-\infty}^{\infty} du\, h(u)\, \text{Prob}.\,\big(M_k(u) - M_{k+1}(u) = g\big)$$

$$\text{Prob}.\,(d_k = g) \simeq \frac{1}{\lambda_N} \exp\left(-\frac{g}{\lambda_N}\right)$$

$$\lambda_N \simeq \begin{cases} \left[\dfrac{N\sqrt{\mu}}{\sqrt{2\pi D}} \exp\big(-[\text{erfc}^{-1}(2\alpha)]^2\big)\right]^{-1} & \text{when } \dfrac{k}{N} = \alpha \sim O(1) \\ \sqrt{\dfrac{D}{2\mu k^2}} \dfrac{1}{\sqrt{\ln N}} & \text{when } k \sim O(1) \end{cases}$$

For dichotomous drive

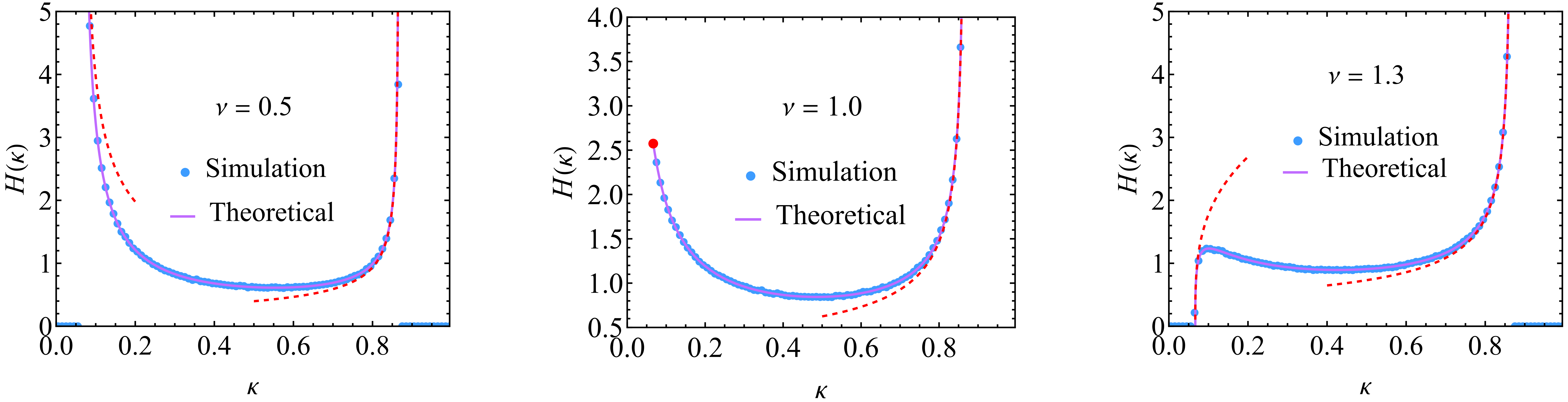


Full counting statistics

$$P(N_L, N) \simeq \frac{1}{N} H\left(\frac{N_L}{N}\right) \text{ where } H(\kappa) = \sqrt{\frac{2\pi D}{\mu}}\, h[u(\kappa)] \frac{\exp\left(\frac{\mu}{2D}[L^2 + [u(\kappa)]^2]\right)}{\sinh\left(\frac{\mu L}{D} u(\kappa)\right)} \sim \frac{1}{\sqrt{\kappa_{\max} - \kappa}} \text{ as } \kappa \to \kappa_{\max} = q_L(0)$$

$$u(\kappa) = q_L^{-1}(\kappa) \text{ with } q_L(u) = \frac{1}{2}\left(\operatorname{erf}\left[\frac{\sqrt{\mu}(L-u)}{\sqrt{2D}}\right] + \operatorname{erf}\left[\frac{\sqrt{\mu}(L+u)}{\sqrt{2D}}\right]\right)$$

For dichotomous drive:



Conclusions

- Stochastic modulation of a trap center generates strong correlations between independent particles in a harmonic trap.

- The stationary state:

$$P_{\mathrm{st}}(x_1, x_2, \ldots, x_N) = \int_{-\infty}^{\infty} du\, h(u) \prod_{j=1}^{N} p(x_j \,|\, u)$$

- CIID structure allows the analytical computation of several observables in a strongly correlated system.