Aspects of celestial amplitude and flat-space limit of AdS/CFT

Sarthak Duary

1 Introduction

The holographic principle stands out as a highly effective tool for understanding quantum gravity. It says that all information about a dynamic system with gravity is stored in a lower-dimensional boundary of the system. Initially proposed by 't Hooft [1] and later expanded upon in the context of string theory by Susskind [2], this principle is exemplified notably in the case of black hole entropy. In this instance, the entropy of a black hole is linked to the area of its horizon, yet it counts the microscopic details [3, 4].

Now, Physics at different energy scales, from particle accelerators to gravitational wave experiments, can be effectively described by flat spacetime. Though we understand the holographic principle in asymptotically Anti-de Sitter (AdS) spacetime, flat-space holography is still largely unexplored. Celestial holography is a bottom-up approach to flat-space holography. Celestial holography proposes that quantum gravity, or quantum field theory (QFT) in asymptotically flat spacetime can be represented by a dual theory that lives on the co-dimension 2 sphere at the boundary. The theory is referred to as celestial conformal field theory (CCFT) [5, 6]. The fundamental observable in asymptotically flat-space is the $\mathcal{S}$-matrix, exhibiting inherent holographic features by being defined with on-shell momenta at the boundary at infinity. Celestial holography redefines QFT scattering by transforming the $\mathcal{S}$-matrix into a celestial amplitude using boost eigenstates, which enables correlator with operators on points on the celestial sphere. This approach involves considering conformal primary wavefunctions that transform as conformal primaries under the Lorentz group, resulting in $\mathcal{S}$-matrix that covariantly transforms as conformal correlator, effectively trading the plane wave basis with a basis of conformal primary wave function.

Recent insights of celestial amplitude mainly come from a bottom-up approach, translating properties of scattering amplitude into properties of celestial amplitude. The analyses so far involves taking perturbative scattering amplitude as input and producing celestial amplitude. The analyses leave the non-perturbative defining properties of CCFT unknown, lacking an intrinsic definition. Recent works [7–9] explores 4d models with explicit CCFT duals. In the end, our goal is to develop a top-down stringy construction of CCFT. Alternatively, it can be seen as a CFT-inspired approach to analyzing amplitudes. With an intrinsic description of consistent CCFT, we aim to utilize CFT-bootstrap techniques for computing and constraining scattering.

Now, evaluating the celestial amplitude for a massive scalar in dimensions higher than $2d$ is technically challenging. In $2d$, we have the integrable $\mathcal{S}$-matrices, and it would be interesting to compute the celestial amplitudes. In the second section of the synopsis, we initiate the exploration of the celestial amplitude for the $2d$ bulk $\mathcal{S}$-matrix. As a first step towards bootstrapping celestial amplitude, we focus on $2d$ scattering. We show that the celestial amplitude is essentially the Fourier transform of the $2d$ $\mathcal{S}$-matrix in terms of rapidity variables. Translating the crossing and unitarity conditions, we establish their counterparts for the celestial amplitude. Perturbatively in the coupling constant for the $2d$ Sinh-Gordon model, we verify that the celestial amplitude satisfies the crossing and unitarity conditions. For Sinh-Gordon model, due to the presence of the pole in the origin of the complex rapidity-plane there are two types of celestial amplitude, the retarded and the advanced corresponding to $\pm i\epsilon$ prescriptions. We aim to apply the bootstrap idea, utilizing the crossing and unitarity conditions, to derive higher-order celestial amplitudes from lower-order ones.

Now, we will move to the other aspect of flat-space holography. A key insight here is that the flat-space limit of AdS provides a holographic representation of the flat-space $\mathcal{S}$-matrix through the correlator of a boundary CFT [11–14]. When examining AdS geometry, deep into the center, an observer limited to physics below the AdS length scale perceives it as flat. Essentially, this shows that flat-space is part of AdS spacetime, and so flat-space physics must be encoded in AdS spacetime. Since physics within AdS corresponds to CFT, it is reasonable to infer that CFTs encapsulate flat-space physics in an additional dimension. This reasoning forms the fundamental logic behind the flat-space limit of AdS/CFT.

This parallels our everyday experience: although the Earth's surface is spherical, we perceive it as flat. This discrepancy arises because the scale of our activities and the distance of our horizon are minuscule compared to the immense radius of the Earth. Just as the physicist perceives flatness within the depths of AdS geometry, our perception of flatness amidst the spherical Earth mirrors this phenomenon, albeit on a macroscopic scale.

In the third section of the synopsis, we explore infrared flat-space physics from the flat-space limit of AdS/CFT. The appearance of infrared (IR) divergence in the $\mathcal{S}$-matrix is linked to the assumption of asymptotic decoupling. This assumption treats the asymptotic Hamiltonian as free, implying that the asymptotic states are Fock space states and the fields behave as free fields in the asymptotic region of flat spacetime. However, by relaxing this assumption, Faddeev-Kulish state can be introduced, leading to an $\mathcal{S}$-matrix that is IR finite. Faddeev-Kulish state incorporates soft photon modes to dress the asymptotic scattering state in the Fock space. In the third chapter of the synopsis, we develop a framework for constructing the Faddeev-Kulish state in the context of AdS/CFT. AdS spacetime serves as a built-in infrared regulator, and when we take the flat-space limit, infrared divergences will show up unless the asymptotic dynamics of fields is examined appropriately. We construct the AdS correction to the Faddeev-Kulish dressed state, and the guiding principle in studying this is the equivalence established between Wilson line dressing and Faddeev-Kulish dressing. We construct modes for the massive scalar field dressed by the Wilson line using the bulk operator reconstruction. We establish a mapping between AdS corrected soft photon modes and CFT current operators. This mapping allows us to express the AdS corrected Faddeev-Kulish dressed state and utilizing it in the Wilson line dressing.

We elaborate on these in the subsequent sections.

2 Celestial amplitude for 2d theory

We define a massive scalar conformal primary wavefunction $\phi^{\pm}_{\Delta}(X^\mu;\vec{w})$ as a solution to the massive Klein-Gordon equation of mass m in $\mathbb{R}^{1,d-1}$ that transforms covariantly as a scalar conformal primary operator under a Lorentz group $SO(1,d-1)$ transformation. The massive scalar conformal primary wavefunction $\phi^{\pm}_{\Delta}(X^\mu;\vec{w})$ in $\mathbb{R}^{1,d-1}$ admits the Fourier expansion on the plane waves

$$\phi^{\pm}_{\Delta}(X^\mu;\vec{w}) = \int_{H_{d-1}} [d\hat{p}] \;\; G_{\Delta}(\hat{p};\vec{w}) \exp[\pm\, im\, \hat{p}.X] \quad , \tag{1}$$

where the on-shell momenta per mass, a unit timelike vector $\hat{p}(y,\vec{z})$ satisfying $\hat{p}^2 = -1$ can be parametrized using the H_{d-1} coordinates y ($y > 0$) and $z \in \mathbb{R}^{d-2}$, and, $[d\hat{p}]$ is the $SO(1,d-1)$ invariant measure on H_{d-1} written in terms of the hyperbolic coordinates.

Here, the Fourier coefficient $G_{\Delta}(\hat{p};\vec{w})$ is the scalar bulk-to-boundary propagator in H_{d-1} and $\vec{w} \in \mathbb{R}^{d-2}$ lies on the boundary of H_{d-1}.

Employing the mapping from the plane wave to the conformal primary wavefunction given by eq.(1). The $\mathcal{S}$-matrix elements in the conformal prinary basis is given in terms of an integral transform, and we define $\tilde{\mathcal{A}}(\Delta_i,\vec{w}_i)$ by the massive celestial amplitude

$$\tilde{\mathcal{A}}(\Delta_i,\vec{w}_i) = \prod_{k=1}^{n} \int_{H_{d-1}} [d\hat{p}_k] G_{\Delta_k}(\hat{p}_k;\vec{w}_k) \mathcal{A}(\pm m_i \hat{p}^{\mu}_i) \quad . \tag{2}$$

Massive Celestial amplitude for $2 \to 2$ scattering: The Celestial point

In $2d$, the celestial 4-point amplitude of the massive conformal primary wavefunction is

$$\tilde{\mathcal{A}} = \left(\prod_{i=1}^{4} \int \frac{d\hat{p}^1_i}{\hat{p}^0_i} \right) \times \prod_{i=1}^{4} G_{\Delta_i}(\hat{p}_i)\; \mathcal{S}_{2\to 2} \tag{3}$$

where, $\hat{p}^\mu_i \equiv \hat{p}^\mu(\theta_i) = \frac{p^\mu_i}{m} = (\cosh\theta_i, \sinh\theta_i)$ and $G_{\Delta_i}(\hat{p}_i)$ is the bulk-to-boundary propagator in H_1.

The $2d$ celestial amplitude is the Fourier transform of the $\mathcal{S}$-matrix with respect to rapidity given by

$$\mathcal{A}(\omega) \equiv \int_{-\infty}^{\infty} d\theta e^{i\omega\theta} S(\theta) \quad . \tag{4}$$

The variable θ is the rapidity, the conjugate to boosts, and the boost eigenstates are Fourier transforms with respect to rapidity. In this context, with no celestial coordinates, the dual theory at the "celestial point" is zero-dimensional, represented by a zero-dimensional CFT correlation function or an operator algebra without coordinates.

Celestial amplitude in 2d Sinh-Gordon model

Now, we evaluate the celestial amplitude in $2d$ Sinh-Gordon model by perturbatively expanding the $\mathcal{S}$-matrix of the Sinh-Gordon model. The Largangian of the Sinh-Gordon model is given by

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{b^2}(\cosh(b\phi) - 1) \quad . \tag{5}$$

The $\mathcal{S}$-matrix for Sinh-Gordon model with the parameter α related to the coupling b is

$$S(\theta) = \frac{\sinh\theta - i\sin\alpha}{\sinh\theta + i\sin\alpha} \quad , \quad \alpha = \frac{\pi b^2}{8\pi + b^2} \quad . \tag{6}$$

The perturbative expansion of the $\mathcal{S}$-matrix with respect to b^2 is given by

$$\begin{aligned} S^{(0)}(\theta) &= 1 \\ S^{(1)}(\theta) &= -\frac{1}{4}ib^2\text{csch}\theta \\ S^{(2)}(\theta) &= -\frac{b^4\text{csch}\theta(\pi\text{csch}\theta - i)}{32\pi}. \end{aligned} \tag{7}$$

Here, in eq.(7), we note only few terms, for details of the perturbative expansion we refer to the paper [33].

Now, the perturbative expansion of the $\mathcal{S}$-matrix $S(\theta)$ contains poles at $\theta = 0$, we define the celestial amplitude using $\pm i\epsilon$ prescription as

$$\mathcal{A}^{\pm}(\omega) \equiv \int_{-\infty}^{\infty} d\theta e^{i\omega\theta} S(\theta \pm i\epsilon)$$

where

$$\mathcal{A}^{\pm}(\omega) = 2\pi\Big[\delta(\omega) + b^2 f_1^{\pm}(\omega) + b^4 f_2^{\pm}(\omega) + \cdots\Big] \quad , \tag{8}$$

$$f_n^{\pm}(\omega) = \frac{1}{2\pi(b^2)^n}\int_{-\infty}^{\infty} d\theta e^{i\omega\theta} S^{(n)}(\theta \pm i\epsilon) \quad . \tag{9}$$

In the perturbative expansion of the $\mathcal{S}$-matrix $S(\theta)$, there are poles at $\theta = 0$. We use the $\pm i\epsilon$ prescription, and we define two celestial amplitudes based on these two prescriptions: $\mathcal{A}^{+}(\omega)$ is called the retarded celestial amplitude and $\mathcal{A}^{-}(\omega)$ is called the advanced celestial amplitude. Eq. (8) represents the perturbative expansion of both the retarded and advanced celestial amplitudes. We call $f_n^{+}(\omega)$ and $f_n^{-}(\omega)$ as the perturbative retarded and advanced celestial amplitude.

Evaluating the Fourier transform using $+i\epsilon$ prescription by enclosing the contour in the upper half-plane, we get the perturbative retarded celestial amplitude

$$
\begin{aligned}
f_1^+(\omega) &= -\frac{1}{4(1+e^{\pi\omega})} \\
f_2^+(\omega) &= \frac{\left(\pi\omega + e^{\pi\omega}(\pi\omega+1)-1\right)\left(\coth(\pi\omega)-1\right)}{64\pi}.
\end{aligned}
\tag{10}
$$

Evaluating the Fourier transform using $-i\epsilon$ prescription by enclosing the contour in the lower half-plane, we get the perturbative advanced celestial amplitude

$$
\begin{aligned}
f_1^-(\omega) &= \frac{1}{4(e^{-\pi\omega}+1)} \\
f_2^-(\omega) &= \frac{e^{\pi\omega}\left(\pi\omega + e^{\pi\omega}(\pi\omega-1)+1\right)\left(\coth(\pi\omega)-1\right)}{64\pi}.
\end{aligned}
\tag{11}
$$

We note only few terms, for details of the perturbative celestial amplitude we refer to the paper [33].

Crossing and unitarity conditions in celestial space

In terms of rapidity, θ we express the crossing and the unitarity conditions as

$$
\begin{aligned}
&S(\theta) = S(i\pi - \theta) \\
&|S(\theta)|^2 = 1 \quad .
\end{aligned}
\tag{12}
$$

The crossing condition in celestial space is obtained by taking the Fourier transform of both sides of the crossing condition in the rapidity variable. Since, the perturbative expansion of the $\mathcal{S}$-matrix $S(\theta)$ contains poles at $\theta = 0$, therefore, perturbatively if we expand upto a given order we should take the fourier transform of $S(\theta \pm i\epsilon)$.

The crossing condition in celestial space is obtained by taking the fourier transform of both sides

$$
\mathcal{A}^{\pm}(\omega) = e^{\mp\omega\pi}\mathcal{A}^{\pm}(-\omega) \quad . \tag{13}
$$

The crossing condition relates the retarded (advanced) celestial amplitude of positive ω to the retarded (advanced) celestial amplitude of negative ω and vice-versa.

Perturbatively expanding $f_n^{\pm}(\omega)$ satisfy the crossing condition

$$
f_n^{\pm}(\omega) = e^{\mp\omega\pi} f_n^{\pm}(-\omega) \quad . \tag{14}
$$

The unitarity condition becomes

$$
\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' \mathcal{A}^{+}(\omega+\omega')\mathcal{A}^{-}(\omega') = 2\pi\delta(\omega) \quad . \tag{15}
$$

We have the unitarity condition order by order in perturbation theory

$$f_1^+(\omega) + f_1^-(-\omega) = 0 \quad ,$$

$$f_n^+(\omega) + f_n^-(-\omega) + \sum_{j=1}^{n-1} \int_{-\infty}^{\infty} d\omega' f_{n-j}^+(\omega+\omega') f_j^-(\omega') = 0 \quad (n > 1) \quad . \tag{16}$$

We check the crossing and unitarity conditions for the $2d$ Sinh-model using celestial amplitudes. Now, we see that using the crossing and unitarity conditions, how much we can get for the higher order celestial amplitude from the lower order celestial amplitude.

Crossing condition translated in celestial space gives

$$f_n^{\pm}(\omega) = e^{\mp\pi\omega} f_n^{\pm}(-\omega) \quad . \tag{17}$$

Unitarity condition translated in celestial space gives

$$f_n^+(\omega) + f_n^-(-\omega) + \sum_{j=1}^{n-1} \int_{-\infty}^{\infty} d\omega' f_{n-j}^+(\omega+\omega') f_j^-(\omega') = 0 \quad . \tag{18}$$

Now, the relation between $f_n^+(\omega)$ and $f_n^-(\omega)$ is

$$f_n^+(\omega) - f_n^-(\omega) + \frac{1}{2\pi(b^2)^n}\, 2\pi i \,\mathrm{Res}\Big[e^{i\omega\theta} S^{(n)}(\theta)\Big]_{\theta=0} = 0 \quad . \tag{19}$$

Using crossing and unitarity conditions in eq.(17) and eq.(18) along with eq.(19) we get

$$f_n^+(\omega) = \frac{1}{(1+e^{-\pi\omega})}\Bigg[-\underbrace{\frac{e^{-\pi\omega}}{2\pi(b^2)^n}\, 2\pi i \,\mathrm{Res}\Big[e^{i\omega\theta} S^{(n)}(\theta)\Big]_{\theta=0}}_{\text{not fixed by crossing and unitarity conditions}} - \underbrace{\sum_{j=1}^{n-1} \int_{-\infty}^{\infty} d\omega' f_{n-j}^+(\omega+\omega') f_j^-(\omega')}_{\text{fixed by crossing and unitarity conditions}} \Bigg] \quad . \tag{20}$$

Conclusions and future directions

In summary, we explore celestial amplitude for the $2d$ bulk $\mathcal{S}$-matrix, showing that for massive scalar particles, it is the Fourier transform of the $\mathcal{S}$-matrix in the rapidity variable. In the case of the Sinh-Gordon $\mathcal{S}$-matrix, perturbative analysis identifies two celestial amplitude types–retarded and advanced–attributed to a pole at the origin of the complex rapidity-plane. This pole plays a crucial role in perturbation theory, and we elaborate extensively on how to address it within perturbation theory by defining two celestial amplitudes corresponding to two different $i\epsilon$ prescriptions. In the celestial space, we translate the crossing and unitarity conditions and check these conditions for the Sinh-Gordon model. From the celestial CFT perspective, we ask about determining the higher order celestial amplitude from the lower order i.e., bootstrapping celestial amplitude by imposing the crossing and unitarity conditions. There are several promising future directions.

Flat-space limit of massive scalar AdS amplitude and it's connection with massive scalar celestial amplitude. For the massive scalar celestial amplitude, it is nice to connect the celestial amplitude to the flat-space imit of AdS amplitude. For this, we can use the dictionary between the positions of operators at the boundary of AdS and the momenta of particles in the flat-space limit of AdS [22]. The celestial amplitude written in conformal basis which is an integral transform of the flat-space amplitude in momentum space [6] can be translated to an integral transform of positions of operators at the boundary of AdS. We can then try to see the interpretation of the integral transform in terms of CFT living on the boundary.

Conformal block expansion for the celestial amplitude. In this line of thinking, it would be useful to understand the analog of conformal block expansion for the celestial amplitude. In AdS, we can convolute the conformal block expansion of the boundary four-point function with the integral transform and see how the block gets modified. It would be nice to make connection with the conformal block expansion in the Celestial CFT [10].

3 AdS correction to the Faddeev-Kulish state: migrating from the flat peninsula

Scattering amplitudes in QED vanish in four spacetime dimensions in flat-space because of IR divergences. The soft photon interchange between the external legs due to long-range interactions is what causes these IR divergences. The soft contribution of each of the diagrams exponentiates after resumming the series [15]. The typical textbook solution to this IR divergence problem is to consider inclusive cross sections, involving a trace over soft photon modes in scattering states to eliminate the divergence and obtain a finite result. An upshot of the resolution of the IR divergence alternative to employing inclusive cross sections is to use dressed states as physical scattering states. The intuition is that because electromagnetic interactions are long-range interactions, soft modes of photons in the "in" and "out" scattering states are always present. These dressed states by soft modes of the photons is referred as Faddeev-Kulish dressed state [16]. The Faddeev-Kulish state is such that it precisely cancels the IR divergences in the $\mathcal{S}$-matrix, resulting in an IR finite $\mathcal{S}$-matrix [17–21].

We aim to construct the AdS correction to the Faddeev-Kulish state by using the equivalence of the Faddeev-Kulish dressing and the Wilson line dressing involving the soft modes of photons in flat-space [24–27]. We calculate modes for the massive scalar field dressed by the Wilson line utilizing bulk operator reconstruction. We use bulk operator reconstruction to create soft photon modes from CFT current operators. Incorporating AdS corrections, we map the CFT current operators to AdS corrected photon modes. This mapping facilitates the construction of the AdS corrected creation mode of the Wilson line dressed scalar field, resulting in the AdS corrected Faddeev-Kulish state.

Flat Peninsula inside AdS Lake: The flat-space limit

At the level of geometry, we take the large AdS radius limit such that the global AdS_4 metric becomes that of flat spacetime. The AdS_4 lorentzian metric in global coordinates is given by

$$ds^2 = \frac{L^2}{\cos^2\rho}(-d\tau^2 + d\rho^2 + \sin^2\rho\, d\Omega_2^2) \quad . \tag{21}$$

The coordinate replacement

$$\tau = \frac{t}{L} \;\; \text{and} \;\; \tan\rho = \frac{r}{L} \quad ,$$

develops the AdS_4 metric into that of flat spacetime metric upon taking the limit $L \to \infty$

$$ds^2 \xrightarrow[L\to\infty]{} -dt^2 + dr^2 + r^2 d\Omega_2^2 \quad . \tag{22}$$

Essentially, the flat peninsula can be thought of as being inside the AdS lake, and this operation can be thought of as flat-space limit.

Massive scalar field modes in the flat-space limit of AdS/CFT

The creation and annihilation modes for a free massive scalar field in flat spacetime can be constructed in terms of the CFT operator in the flat-space limit of AdS/CFT. For normalizable modes of the bulk AdS scalar field, the bulk AdS scalar field $\phi(\rho, x)$ is related to the dual boundary CFT operator $\mathcal{O}(x)$ through the fall-off condition

$$\phi(\rho, x) \xrightarrow[\rho\to\frac{\pi}{2}]{} (\cos\rho)^{\Delta}\mathcal{O}(x) \quad . \tag{23}$$

To extract the creation and annihilation modes in flat spacetime in terms of CFT operators, first we reconstruct bulk operators as operators in the CFT using the bulk operator reconstruction. Next, we extract the creation/annhilation modes using fourier transform and then we take a large AdS radius limit which is the flat-space limit. The flat-space creation mode for the free massive scalar field ϕ is given by [28]

$$\sqrt{2\omega_{\vec{p}}}\, a^{\dagger}_{\omega_{\vec{p}}} = \mathcal{C}\int d\tau\, e^{-i\omega_{\vec{p}}L\left[\tau - \frac{\pi}{2} - \frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]}\mathcal{O}\left(\tau, \hat{p}\right) \quad , \tag{24}$$

where

$$\mathcal{C} = \frac{1}{2\pi}\left(\frac{mL}{\pi^3}\right)^{\frac{1}{4}}\left(\frac{2m}{i|\vec{p}|}\right)^{mL+\frac{1}{2}} L \quad . \tag{25}$$

In the formula of eq.(24), the exponential part in the integrand is highly oscillatory as we take the zoomed in limit, $L \to \infty$, therefore the insertion points of the operators are in windows of size $\mathcal{O}(1/L)$ at the complex points

$$\mathrm{Re}(\tau) = \frac{\pi}{2} \;\; , \;\; \text{and} \;\; \mathrm{Im}(\tau) = \frac{1}{2}\log\left(\frac{\omega_{\vec{p}} + m}{\omega_{\vec{p}} - m}\right) \quad . \tag{26}$$

In fig.1, we illustrate the correspondence between asymptotic regions in flat-space and the boundary of AdS. The outgoing massive scattering state is depicted by the green arrow, piercing the boundary at a complex point in global time. Similarly, the outgoing massless scattering state is represented by the blue arrow, piercing the boundary when $\text{Re}(\tau) = \frac{\pi}{2}$. Consequently, two regions in the CFT play pivotal roles: the regions surrounding $\pm\frac{\pi}{2}$, termed null infinity, and the euclidean domes, denoting future/past timelike infinity. The fig. 1 on the boundary of AdS provides a holographic perspective of flat-space, particularly relevant in terms of particle propagation. Massless particles propagate along light-like geodesics, piercing the boundary of AdS at global time $\frac{\pi}{2}$ (outgoing). In contrast, massive particles follow time-like geodesics, therefore never pierce the boundary, or, to put it another way, they pierce the boundary at a complex point.

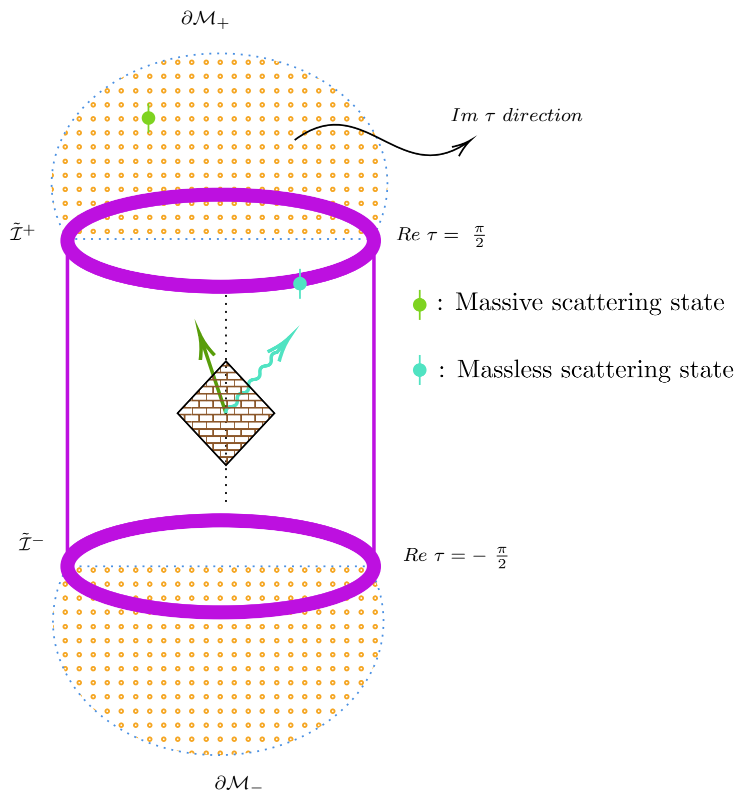

Figure 1: A schematic picture linking the boundary of AdS and the asymptotic regions of flat spacetime. The euclidean domes $\partial\mathcal{M}_\pm$, which play the roles of future/past timelike infinity $i^\pm$, are analytic continuations of the boundary CFT in the imaginary global time direction. Regions around, real part of global time at $\pm\frac{\pi}{2}$, Re $\tau = \pm\frac{\pi}{2}$ in an $\mathcal{O}(1/L)$ window, play the role of null infinity $\widetilde{\mathcal{I}}^\pm$.

CFT and Mixed representations of the Faddeev-Kulish dressed state

The bulk massive scalar field dressed by the bulk-to-boundary Wilson line $U_{\mathfrak{B}\partial}(y, x)$ is given by

$$\widetilde{\phi}(y) = U_{\mathfrak{B}\partial}(y, x)\phi(y) \quad , \tag{27}$$

where the bulk-to-boundary Wilson line is

$$U_{\mathfrak{B}\partial}(y,x) = \mathcal{P}\left\{ e^{iq\int_y^x dx^M A_M} \right\} \quad . \tag{28}$$

We dress the scalar field with soft modes of the photon in the Wilson line and in AdS, the minimum frequency of photon is of $\mathcal{O}\left(\frac{1}{L}\right)$. This dressed scalar field we refer as soft Wilson line dressed scalar field. As a consequence of this dressing, the field $\widetilde{\phi}$ is free field.

The CFT operator of the dressed scalar field $\widetilde{\mathcal{O}}\,(\tau,\hat{p})$ is related to the CFT operator $\mathcal{O}\,(\tau,\hat{p})$ as

$$\widetilde{\mathcal{O}}\,(\tau,\hat{p}) = e^{iq\int_{\Gamma(\tau,\hat{p})} j_a dx^a}\,\mathcal{O}\,(\tau,\hat{p}) \quad . \tag{29}$$

Using eq.(29), we get the creation mode of the dressed massive scalar field in terms of $\mathcal{O}\,(\tau,\hat{p})$

$$\sqrt{2\omega_{\vec{p}}}\,\widetilde{a}^{\dagger}_{\omega_{\vec{p}}} = \widetilde{c}\int d\tau e^{-i\omega_{\vec{p}}L\left[\tau-\frac{\pi}{2}-\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]} e^{iq\int_{\Gamma(\tau,\hat{p})} j_a dx^a}\,\mathcal{O}\,(\tau,\hat{p}) \quad . \tag{30}$$

We choose the path to be global time coordinate varies from 0 to τ maintaining fixed coordinates in the angular direction, and then compute the integral over global time. We denote $\omega_m = kL$ and take the flat-space imit, $L\to\infty$, and then to take soft limit with $k\to 0$. The mode is given by

$$\omega_m = Lk \quad , \quad \frac{\omega_m}{L} = k \to 0 \quad . \tag{31}$$

The Wilson line dressed creation mode of the massive scalar field at $\mathcal{O}(q)$ is given by

$$\sqrt{2\omega_{\vec{p}}}\,\widetilde{a}^{\dagger}_{\omega_{\vec{p}}} = \frac{q}{2}\widetilde{c}\int d\tau\int\frac{dk}{k}e^{-i\omega_{\vec{p}}L\left[\tau-\frac{\pi}{2}-\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]}\left(e^{iLk\tau}-1\right)\hat{\chi}_k(\hat{p})\,\mathcal{O}\,(\tau,\hat{p}) \quad . \tag{32}$$

Here, $\hat{\chi}$ is the angular dependent part of the the global time component of the CFT current operator. This expression eq.(32) for the dressed creation mode acting on vacuum state gives the “CFT representation” of the Faddeev-Kulish dressed state.

We rewrite the creation mode for the Wilson line dressed massive scalar field with photon soft modes in terms of undressed modes by inserting a delta function $\delta(\tau-\tau')$ and expressing it as an integral over frequency, $\Delta_{\vec{p}}$, corresponding to the undressed mode frequency of the massive scalar field

$$\int d\tau'\delta(\tau'-\tau) = \int d\tau'\int d\Delta_{\vec{p}}\,\frac{L}{2\pi}e^{-i\Delta_{\vec{p}}L(\tau'-\tau)} \quad . \tag{33}$$

Now, using the expression for the creation mode $a^{\dagger}_{\Delta_{\vec{p}}}$ corresponing to frequency of the undressed mode, $\Delta_{\vec{p}}$ in terms of the boundary CFT operator $\mathcal{O}\,(\tau',\hat{p})$

$$\sqrt{2\Delta_{\vec{p}}}\,a^{\dagger}_{\Delta_{\vec{p}}} = \mathcal{C}\int d\tau'\,e^{-i\Delta_{\vec{p}}L\left[\tau'-\frac{\pi}{2}-\frac{i}{2}\log\left(\frac{\Delta_{\vec{p}}+m}{\Delta_{\vec{p}}-m}\right)\right]}\mathcal{O}\,(\tau',\hat{p}) \quad , \tag{34}$$

the creation mode of the soft Wilson line dressed massive scalar field is expressed as

$$\sqrt{2\omega_{\vec{p}}}\,\widetilde{a}^{\dagger}_{\omega_{\vec{p}}} = \widetilde{\mathfrak{C}}\,L\int d\Delta_{\vec{p}}\,e^{-i\Delta_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\Delta_{\vec{p}}+m}{\Delta_{\vec{p}}-m}\right)\right]}\;e^{i\omega_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]}$$
$$\int d\tau\;e^{-iL\tau(\omega_{\vec{p}}-\Delta_{\vec{p}})}\;e^{iq\int_{\Gamma(\tau,\hat{p})}j_a dx^a}\;\sqrt{2\Delta_{\vec{p}}}\,\frac{a^{\dagger}_{\Delta_{\vec{p}}}}{\mathcal{C}}\quad. \tag{35}$$

In eq.(35), we express the creation mode of the soft Wilson line dressed massive scalar field in terms of the undressed creation mode. We choose a particular path for the Wilson line $\Gamma(\tau,\hat{p})$. We write eq.(35) by substituting $\omega_m = kL$ and replacing sum over modes by integration over k at $\mathcal{O}(q)$ and, then we can perform the $\Delta_{\vec{p}}$ integral to simplify the expression and finally the dreesed creation mode is given by

$$\sqrt{2\omega_{\vec{p}}}\,\widetilde{a}^{\dagger}_{\omega_{\vec{p}}} = \widetilde{\mathfrak{C}}\,\pi\,q\,\int_0^{\infty} dk\frac{1}{k}\Bigg(e^{-i(\omega_{\vec{p}}+k)L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+k+m}{\omega_{\vec{p}}+k-m}\right)\right]}\;e^{i\omega_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]}$$
$$\hat{\chi}_k\sqrt{2(\omega_{\vec{p}}+k)}\frac{a^{\dagger}_{\omega_{\vec{p}}+k}}{\mathcal{C}} - \sqrt{2\omega_{\vec{p}}}\,\hat{\chi}_k\,\frac{a^{\dagger}_{\omega_{\vec{p}}}}{\mathcal{C}}\Bigg)\quad. \tag{36}$$

The dressed creation mode acting on the vacuum is the Faddeev-Kulish dressed state. In this way of writing the expression, the frequency of the creation mode of the scalar field will get shifted from $\omega_{\vec{p}}$ to $\omega_{\vec{p}}+k$. The expression eq.(36) is written using the creation mode of the scalar field, whereas the Wilson line is defined in relation to the CFT current operator. Thus, this representation is considered "mixed representation", combining aspects of both the "CFT representation" and the "flat-space representation".

AdS correction to the Faddeev-Kulish dressed state

In the previous section (ref.eq.(35)), we found the expression for the dressed scalar field creation mode

$$\sqrt{2\omega_{\vec{p}}}\,\widetilde{a}^{\dagger}_{\omega_{\vec{p}}} = \widetilde{\mathfrak{C}}\,L\int d\Delta_{\vec{p}}\,e^{-i\Delta_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\Delta_{\vec{p}}+m}{\Delta_{\vec{p}}-m}\right)\right]}\;e^{i\omega_{\vec{p}}L\left[\frac{\pi}{2}+\frac{i}{2}\log\left(\frac{\omega_{\vec{p}}+m}{\omega_{\vec{p}}-m}\right)\right]}$$
$$\int d\tau\;e^{-iL\tau(\omega_{\vec{p}}-\Delta_{\vec{p}})}\;e^{iq\int_{\Gamma(\tau,\hat{p})}j_a dx^a}\;\sqrt{2\Delta_{\vec{p}}}\,\frac{a^{\dagger}_{\Delta_{\vec{p}}}}{\mathcal{C}}\quad. \tag{37}$$

This expression involves the CFT current operator within the Wilson line, resulting in a mixed representation formula. To fully represent this dressed creation mode in flat-space representation, we map the CFT current operators within the Wilson line into AdS corrected modes of photons. Acting on the vacuum, the dressed creation operator yields the AdS corrected Faddeev-Kulish dressed state.

For details of the AdS correction to the Faddeev-Kulish state we refer to the paper [34].

Conclusions and future directions

In summary, we have constructed AdS correction to the Faddeev-Kulish dressed state. We follow the Wilson line dressing in the context of AdS/CFT to arrive at the Faddeev-Kulish dressed state. The issues of IR divergences and AdS as an IR cutoff are as follows. If we want to talk about bulk observables in a small enough region where flat spacetime physics applies, then of course there is no effect from being in AdS; the physics of the IR divergence is no different from being in flat spacetime. However, if we want to talk about CFT quantities like a correlator, the IR divergence will change the precise nature of the flat spacetime limit, but the correlator will still be finite.

Future directions.

Asymptotic charges and states from AdS/CFT. The asymptotic symmetries inevitably lead to the Faddeev-Kulish dressed state since the amplitudes that preserve the asymtotic charge are IR finite, the explicit construction was done in [29]. The study of generalizations of asymptotic symmetries in AdS was done in [30–32] using different boundary conditions. An excellent open problem is to apply them to create the physical asymptotic states. It would be fascinating to have the states from AdS/CFT and AdS corrections to it.

4 Conclusions

In this synopsis, we explore two key aspects of flat-space holography. First, we study the celestial amplitude, which takes a bottom-up approach. We consider conformal primary wavefunctions that transform as conformal primaries the Lorentz group. Consequently, the resulting $\mathcal{S}$-matrix transforms covariantly as conformal correlators, trading the plane wave basis with a basis of conformal primary wavefunction. We study celestial holography ideas in $2d$. This setting serves as an excellent testing ground, as we have have exact $\mathcal{S}$-matrices to play with in $2d$ and try to learn lessons from. Second, we study the flat-space limit of the AdS/CFT. We consider a scenario where we have a AdS geometry, and within this geometry, there is an observer who can only examine physical phenomena occurring at length scales smaller than the characteristic AdS length scale. From the perspective of this observer, confined to probing only these smaller length scales, the geometry they perceive and experience would appear to be flat, rather than the true underlying AdS geometry. In other words, if an observer is limited to studying physics within a certain distance scale in a AdS geometry, their observations and measurements would be indistinguishable from those made in a flat-space, as the curvature effects of the AdS geometry would be negligible at those small distance scales. The key idea is that flat spacetime is essentially a part of AdS spacetime. Consequently, the physics of flat spacetime must be encoded within the framework of AdS spacetime. Since the physics in AdS spacetime is known to be dual to CFTs via the AdS/CFT correspondence, it follows that CFTs must also encode the physics of flat spacetime. This reasoning forms the general logic behind the concept of flat-space limit of AdS/CFT, which establishes a connection between flat spacetime physics and

CFTs through the intermediate step of flat-space limit of AdS spacetime.

In the first part of the synopsis, we initiate the study of the celestial amplitude for the $2d$ bulk $\mathcal{S}$-matrix. We show that, in the case of massive scalar particles, the celestial amplitude is essentially the Fourier transform of the $\mathcal{S}$-matrix expressed in terms of rapidity. Considering the Sinh-Gordon $\mathcal{S}$-matrix, we compute the perturbative celestial amplitude, identifying the existence of two distinct types: the retarded and the advanced, attributed to a pole at the origin of the complex rapidity-plane. This pole holds significant importance within perturbation theory, and we elaborate extensively on how to address it within the framework of perturbation theory by introducing two celestial amplitudes that correspond to two different $i\epsilon$ prescriptions. By translating the crossing and unitarity conditions, we establish their equivalents for the celestial amplitude. Through perturbative analysis of the coupling constant in the $2d$ Sinh-Gordon model, we verify that the celestial amplitude satisfies the crossing and unitarity conditions. We employ the bootstrap idea to derive higher-order celestial amplitudes based on lower-order ones.

In the second part of the synopsis, we construct the AdS radius correction to the Faddeev-Kulish dressed state. The IR divergence in the $\mathcal{S}$-matrix arises due to the assumption of asymptotic decoupling. This assumption considers the asymptotic Hamiltonian as free, implying that the asymptotic states are described by Fock space states, and the fields behave like free fields in the asymptotic region of flat spacetime. However, by relaxing this assumption, the Faddeev-Kulish state can be introduced, leading to an IR-finite $\mathcal{S}$-matrix. The Faddeev-Kulish state incorporates soft photon modes that dress the scattering state within the Fock space, thereby accounting for the long-range effects of the electromagnetic interaction. Using the bulk operator reconstruction, we establish modes for the massive scalar field dressed by the Wilson line and examine both the CFT representation and the mixed representation of the Faddeev-Kulish dressed state. We construct soft photon modes in terms of CFT current operators. Then, after incorporating the AdS correction into the soft photon modes, we apply AdS radius correction to the Wilson line dressing. We invert the mapping of the AdS radius-corrected soft photon modes in terms of CFT current operators, essentially evaluating the CFT current operators in relation to the AdS radius-corrected photon modes. In the Wilson line dressing, we utilize this inverse mapping between the CFT current operators and soft photon modes to construct the AdS radius-corrected creation mode of the Wilson line dressed scalar field. The resulting dressed mode, acting on the vacuum, represents the desired Faddeev-Kulish dressed state, incorporating the AdS radius correction.

The ultimate aim is to understand quantum gravity in flat-space, either through the flat-space limit of the AdS/CFT framework or via celestial holography. There is still a long way to go ahead in this regard, and I really hope that my works will make a meaningful contribution.

References

[1] G. 't Hooft, *Dimensional reduction in quantum gravity*, Conf. Proc. C **930308** (1993), 284-296 [gr-qc/9310026].

[2] L. Susskind, *The World as a hologram*, J. Math. Phys. **36** (1995), 6377-6396 doi:10.1063/1.531249 [hep-th/9409089].

[3] J. D. Bekenstein, *Black holes and entropy*, Physical Review D **7** (1973) 2333.

[4] G. W. Gibbons and S. W. Hawking, *Action integrals and partition functions in quantum gravity*, Physical Review D **15** (1977) 2752.

[5] S. Pasterski, S. H. Shao and A. Strominger, *Conformal Symmetry of Celestial Amplitudes*, *Phys. Rev.* **D96** (2017) 065026 [1701.00049].

[6] S. Pasterski and S.-H. Shao, *Conformal basis for flat space amplitudes*, *Phys. Rev.* **D96** (2017) 065022 [1705.01027].

[7] K. Costello and N. M. Paquette, *Celestial holography meets twisted holography: 4d amplitudes from chiral correlators*, *JHEP* **10** (2022) 193, [2201.02595].

[8] K. Costello, N. M. Paquette and A. Sharma, *Top-Down Holography in an Asymptotically Flat Spacetime*, Phys. Rev. Lett. **130** (2023) no.6, 061602 doi:10.1103/PhysRevLett.130.061602 [2208.14233].

[9] A. Ball, S. A. Narayanan, J. Salzer and A. Strominger, *Perturbatively exact $w_{1+\infty}$ asymptotic symmetry of quantum self-dual gravity*, *JHEP* **01** (2022) 114, [2111.10392].

[10] A. Atanasov, W. Melton, A. M. Raclariu and A. Strominger, *Conformal block expansion in celestial CFT*, *Phys. Rev. D* **104** (2021) no.12, 126033, [2104.13432].

[11] J. Polchinski, *S matrices from AdS space-time*, [hep-th/9901076].

[12] L. Susskind, *Holography in the flat space limit*, *AIP Conf. Proc.* **493** (1999), no. 1 98–112, [hep-th/9901079].

[13] S. B. Giddings, *The Boundary S matrix and the AdS to CFT dictionary*, *Phys. Rev. Lett.* **83** (1999) 2707–2710, [hep-th/9903048].

[14] S. B. Giddings, *Flat space scattering and bulk locality in the AdS / CFT correspondence*, *Phys. Rev. D* **61** (2000) 106008, [hep-th/9907129].

[15] S. Weinberg, *Infrared photons and gravitons*, Phys. Rev. **140** (1965), B516-B524 doi:10.1103/PhysRev.140.B516.

[16] P. P. Kulish and L. D. Faddeev, *Asymptotic conditions and infrared divergences in quantum electrodynamics*, Theor. Math. Phys. **4** (1970), 745 doi:10.1007/BF01066485.

[17] V. Chung, *Infrared Divergence in Quantum Electrodynamics*, Phys. Rev. **140** (1965), B1110-B1122 doi:10.1103/PhysRev.140.B1110.

[18] T. W. B. Kibble, *Coherent soft-photon states and infrared divergences. ii. mass-shell singularities of green's functions*, Phys. Rev. **173** (1968), 1527-1535 doi:10.1103/PhysRev.173.1527.

[19] T. W. B. Kibble, *Coherent soft-photon states and infrared divergences. iii. asymptotic states and reduction formulas*, Phys. Rev. **174** (1968), 1882-1901 doi:10.1103/PhysRev.174.1882.

[20] T. W. B. Kibble, *Coherent soft-photon states and infrared divergences. iv. the scattering operator*, Phys. Rev. **175** (1968), 1624-1640 doi:10.1103/PhysRev.175.1624.

[21] J. Ware, R. Saotome and R. Akhoury, *Construction of an asymptotic S matrix for perturbative quantum gravity*, *JHEP* **10** (2013) 159, [1308.6285].

[22] S. Komatsu, M. F. Paulos, B. C. Van Rees and X. Zhao, *Landau diagrams in AdS and S-matrices from conformal correlators*, *JHEP* **11** (2020) 046, [2007.13745].

[23] T. Okuda and J. Penedones, *String scattering in flat space and a scaling limit of Yang-Mills correlators*, *Phys. Rev. D* **83** (2024) 086001, [1002.2641].

[24] S. Mandelstam, *Quantum electrodynamics without potentials*, *Annals Phys.* **19** (1962) 1–24.

[25] R. Jakob and N. G. Stefanis, *Path dependent phase factors and the infrared problem in QED*, *Annals Phys.* **210** (1991) 112–136.

[26] S. Choi and R. Akhoury, *Soft Photon Hair on Schwarzschild Horizon from a Wilson Line Perspective*, *JHEP* **12** (2018) 074, [1809.03467].

[27] S. Choi, S. Sandeep Pradhan and R. Akhoury, *Supertranslation Hair of Schwarzschild Black Hole: A Wilson Line Perspective*, *JHEP* **01** (2020) 013, [1910.05882].

[28] E. Hijano and D. Neuenfeld, *Soft photon theorems from CFT Ward identites in the flat limit of AdS/CFT*, *JHEP* **11** (2020) 009, [2005.03667].

[29] S. Choi and R. Akhoury, *BMS Supertranslation Symmetry Implies Faddeev-Kulish Amplitudes*, *JHEP* **02** (2018) 171, [1712.04551].

[30] G. Compère, A. Fiorucci and R. Ruzziconi, *The Λ-BMS$_4$ group of dS$_4$ and new boundary conditions for AdS$_4$*, *Class. Quant. Grav.* **36** (2019) no.19, 195017, [1905.00971].

[31] G. Compère, A. Fiorucci and R. Ruzziconi, *The Λ-BMS$_4$ charge algebra*, *JHEP* **10** (2020) 205, [2004.10769].

[32] A. Fiorucci and R. Ruzziconi, *Charge algebra in Al(A)dS$_n$ spacetimes*, *JHEP* **05** (2021) 210, [2011.02002].

[33] S. Duary, *Celestial amplitude for 2d theory*, *JHEP* **12** (2022) 060, [2209.02776].

[34] S. Duary, *AdS correction to the Faddeev-Kulish state: migrating from the flat peninsula*, *JHEP* **05** (2023) 079, [2212.09509].