Universal distribution of the number of minima for random walks and Lévy flights

Satya N. Majumdar

Laboratoire de Physique Théorique et Modèles Statistiques,CNRS,
Université Paris-Saclay, France

Collaborators:

Anupam Kundu (ICTS, Bangalore)

Gregory Schehr (LPTHE, Univ. Sorbonne)

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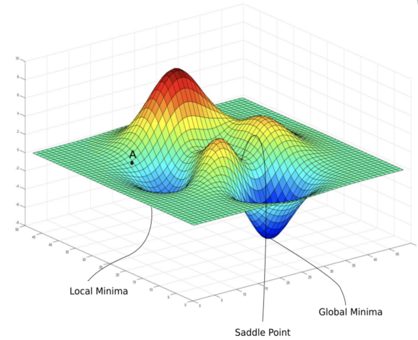
No. of local minima in a random landscape



liquids, glassy dynamics, disordered systems, string theory, fitness landscapes, optics, data science, ...

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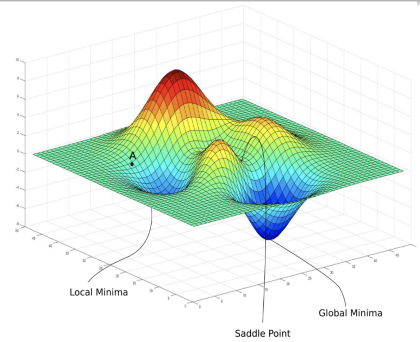

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Most studies $\Longrightarrow$ Average no. of minima $\langle m \rangle$

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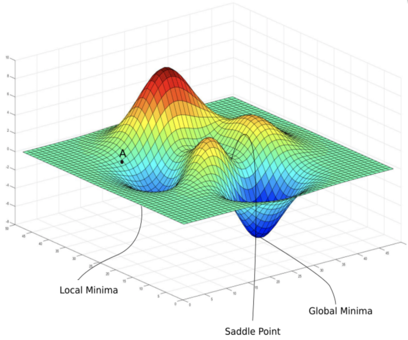

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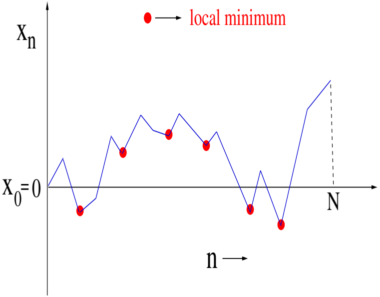
Full distribution of m $\Longrightarrow$ formidable challenge

Halperin & Lax 1966, Bray & Moore 1980, Weinrib & Halperin 1982, Cavagna et. al. 2000, Broderix et, al. 2000, Susskind 2003, Fyodorov 2004, Barton 2005, Aazami & Easther 2006, S.M. & Martin 2006, Bray & Dean 2007, Auffinger et. al. 2013, Dauphin et. al. 2014, Ben Arous et. al. 2021, Park et. al. 2020, Crona et. al. 2023, Ros & Fyodorov 2023

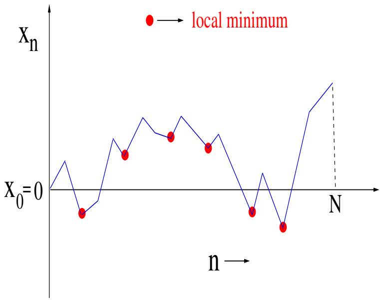
Random walk/Lévy landscape → Sinai model



discrete-time random walk:

$x_n = x_{n-1} + \eta_n$ from $x_0 = 0$

Random walk/Lévy landscape $\to$ Sinai model



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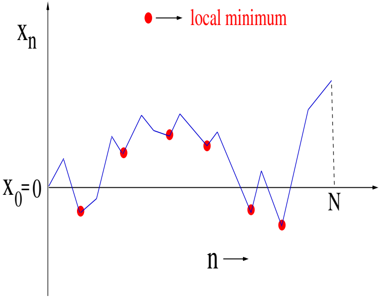

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$0 < \mu \leq 2 \Rightarrow$ Lévy index

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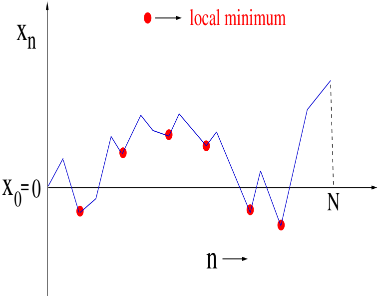

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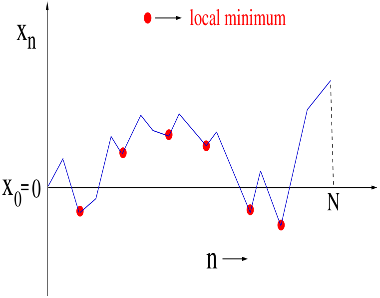

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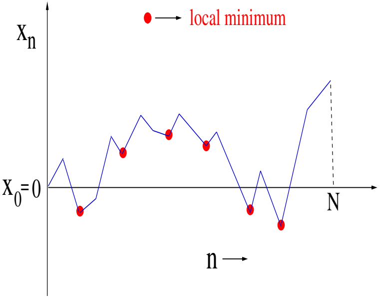

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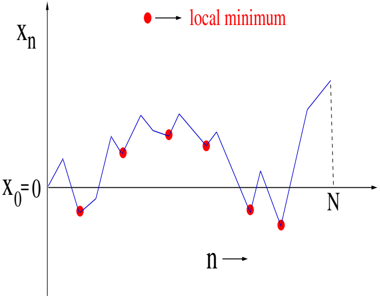

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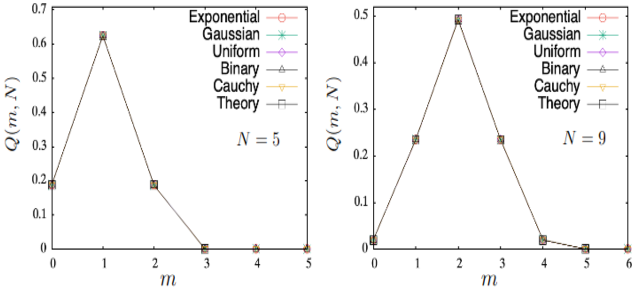
independent of $f(\eta) \Longrightarrow$ **Universal** for **all** m **and** N **!**

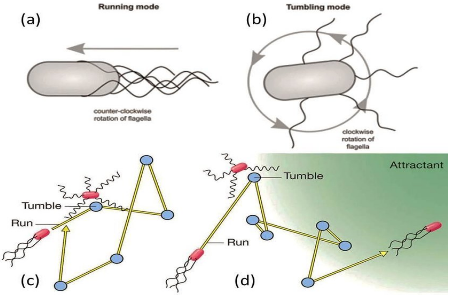
Universal distribution $Q(m, N)$



Distribution of the number of minima: $Q(m, N) = \frac{1}{2^N} \frac{(N+1)!}{(N-2m)!\,(2m+1)!}$

Mean: $\langle m \rangle = \frac{N-1}{4}$ and Variance: $\langle m^2 \rangle - \langle m \rangle^2 = \frac{N+1}{16}$

Application to run-and-tumble process (RTP)



Active RTP: random **autonomous** motion in two phases:

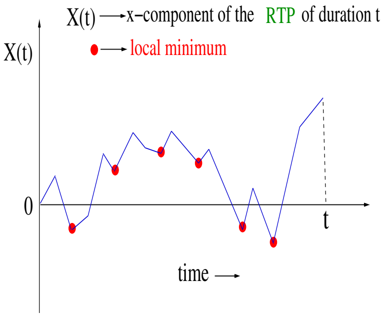
(i) **Run:** ballistic motion during exponential run-time (γ^{-1}) with a fixed velocity $\vec{v}$ drawn from $W(|\vec{v}|)$

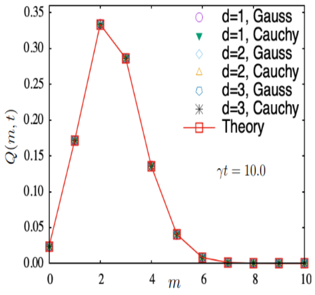
(ii) **Tumble:** a new run starts with a new velocity drawn from $W(|\vec{v}|)$ ....alternates...

$\gamma \longrightarrow$ **tumbling rate** $\quad W(|\vec{v}|) \longrightarrow$ velocity distribution

Ex: E. Coli bacteria in motion (Tailleur & Cates '08, Marchetti et. al. '13, Berg '14,..)

No. of minima in an RTP landscape of duration t



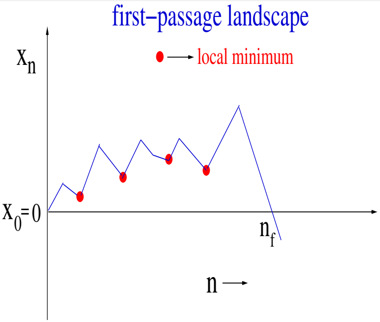


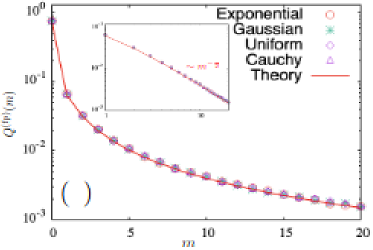
Prob. distr. of the number of minima m in an RTP of duration t

$$Q(m,t) = e^{-\gamma t/2}\, \frac{(\gamma t/2)^{2m-1}}{2\,(2m+1)!}\, \left[(2m+1)(2m+\gamma t) + \frac{\gamma^2 t^2}{4}\right]$$

independent of d and $W(|\vec{v}|) \Longrightarrow$ **Universal** for **all** m **and** t **!**

No. of minima in the first-passage landscape





Prob. distr. of the number of minima m in the first-passage landscape

$$Q^{\rm fp}(m) = \frac{1}{2^{2m+2}}\,\frac{(2m)!}{m!\,(m+1)!} \quad \text{for } m \geq 1$$

$$= 3/4 \quad \text{for } m = 0$$

independent of $f(\eta) \Longrightarrow$ **Universal** for **all** m **!**