On the solutions of certain Diophantine equations over totally real fields

Satyabrat Sahoo

Joint work with
Dr. Narasimha Kumar

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

RATIONAL POINTS ON MODULAR CURVES
ICTS, Bengaluru

September 20, 2023

1. Some known results for Diophantine equations.

Over $\mathbb{Z}$		
$x^n + y^n = z^n$	Wiles [Wil94]	$n \geq 3$ (no solutions)
$x^n + y^n = 2z^n$	Darmon-Merel [DM97]	$n \geq 3$ (no solutions)
$x^p + y^p = 2^r z^p$	Ribet [Rib97]	$2 \leq r < p$ (no solutions)
$x^n + y^n = z^2$	Darmon-Merel [DM97]	$n \geq 4$ (no solutions)
$x^2 = y^p + 2^r z^p$	Siksek [Sik03]	$r \geq 2, p \geq 7$ (finite solutions)
$2x^2 = y^p + 2^r z^p$	Ivorra [Ivo03]	$p \geq 7, 0 \leq r \leq p$ (finite solutions)

1. Some known results for Diophantine equations.

Over $\mathbb{Z}$		
$x^n + y^n = z^n$	Wiles [Wil94]	$n \geq 3$ (no solutions)
$x^n + y^n = 2z^n$	Darmon-Merel [DM97]	$n \geq 3$ (no solutions)
$x^p + y^p = 2^r z^p$	Ribet [Rib97]	$2 \leq r < p$ (no solutions)
$x^n + y^n = z^2$	Darmon-Merel [DM97]	$n \geq 4$ (no solutions)
$x^2 = y^p + 2^r z^p$	Siksek [Sik03]	$r \geq 2, p \geq 7$ (finite solutions)
$2x^2 = y^p + 2^r z^p$	Ivorra [Ivo03]	$p \geq 7, 0 \leq r \leq p$ (finite solutions)

- Let K be a totally real field of degree n.

1. Some known results for Diophantine equations.

Over $\mathbb{Z}$		
$x^n + y^n = z^n$	Wiles [Wil94]	$n \geq 3$ (no solutions)
$x^n + y^n = 2z^n$	Darmon-Merel [DM97]	$n \geq 3$ (no solutions)
$x^p + y^p = 2^r z^p$	Ribet [Rib97]	$2 \leq r < p$ (no solutions)
$x^n + y^n = z^2$	Darmon-Merel [DM97]	$n \geq 4$ (no solutions)
$x^2 = y^p + 2^r z^p$	Siksek [Sik03]	$r \geq 2, p \geq 7$ (finite solutions)
$2x^2 = y^p + 2^r z^p$	Ivorra [Ivo03]	$p \geq 7, 0 \leq r \leq p$ (finite solutions)

- Let K be a totally real field of degree n.
- Let $\mathcal{O}_K$ and p represent the ring of integers of K and a rational prime, respectively.

1. Some known results for Diophantine equations.

Over $\mathbb{Z}$		
$x^n + y^n = z^n$	Wiles [Wil94]	$n \geq 3$ (no solutions)
$x^n + y^n = 2z^n$	Darmon-Merel [DM97]	$n \geq 3$ (no solutions)
$x^p + y^p = 2^r z^p$	Ribet [Rib97]	$2 \leq r < p$ (no solutions)
$x^n + y^n = z^2$	Darmon-Merel [DM97]	$n \geq 4$ (no solutions)
$x^2 = y^p + 2^r z^p$	Siksek [Sik03]	$r \geq 2, p \geq 7$ (finite solutions)
$2x^2 = y^p + 2^r z^p$	Ivorra [Ivo03]	$p \geq 7, 0 \leq r \leq p$ (finite solutions)

- Let K be a totally real field of degree n.
- Let $\mathcal{O}_K$ and p represent the ring of integers of K and a rational prime, respectively.

Definition (1.1)

We say that a Diophantine equation $Ax^p + By^p = Cz^p$ of exponent p has no asymptotic solution in $S \subseteq K^3$,

1. Some known results for Diophantine equations.

Over $\mathbb{Z}$		
$x^n + y^n = z^n$	Wiles [Wil94]	$n \geq 3$ (no solutions)
$x^n + y^n = 2z^n$	Darmon-Merel [DM97]	$n \geq 3$ (no solutions)
$x^p + y^p = 2^r z^p$	Ribet [Rib97]	$2 \leq r < p$ (no solutions)
$x^n + y^n = z^2$	Darmon-Merel [DM97]	$n \geq 4$ (no solutions)
$x^2 = y^p + 2^r z^p$	Siksek [Sik03]	$r \geq 2, p \geq 7$ (finite solutions)
$2x^2 = y^p + 2^r z^p$	Ivorra [Ivo03]	$p \geq 7, 0 \leq r \leq p$ (finite solutions)

- Let K be a totally real field of degree n.
- Let $\mathcal{O}_K$ and p represent the ring of integers of K and a rational prime, respectively.

Definition (1.1)

We say that a Diophantine equation $Ax^p + By^p = Cz^p$ of exponent p has no asymptotic solution in $S \subseteq K^3$, if there exists a constant $V_{K,A,B,C}$ (depending on K, A, B, C) such that for primes $p > V_{K,A,B,C}$,

1. Some known results for Diophantine equations.

Over $\mathbb{Z}$		
$x^n + y^n = z^n$	Wiles [Wil94]	$n \geq 3$ (no solutions)
$x^n + y^n = 2z^n$	Darmon-Merel [DM97]	$n \geq 3$ (no solutions)
$x^p + y^p = 2^r z^p$	Ribet [Rib97]	$2 \leq r < p$ (no solutions)
$x^n + y^n = z^2$	Darmon-Merel [DM97]	$n \geq 4$ (no solutions)
$x^2 = y^p + 2^r z^p$	Siksek [Sik03]	$r \geq 2, p \geq 7$ (finite solutions)
$2x^2 = y^p + 2^r z^p$	Ivorra [Ivo03]	$p \geq 7, 0 \leq r \leq p$ (finite solutions)

- Let K be a totally real field of degree n.
- Let $\mathcal{O}_K$ and p represent the ring of integers of K and a rational prime, respectively.

Definition (1.1)

We say that a Diophantine equation $Ax^p + By^p = Cz^p$ of exponent p has no asymptotic solution in $S \subseteq K^3$, if there exists a constant $V_{K,A,B,C}$ (depending on K, A, B, C) such that for primes $p > V_{K,A,B,C}$, the equation $Ax^p + By^p = Cz^p$ of exponent p has no non-trivial primitive solutions in S.

- In [FS15], Freitas and Siksek show that the equation $x^p + y^p = z^p$ of exponent p has no asymptotic solution in K^3, for a certain class of totally real fields K.

- In [FS15], Freitas and Siksek show that the equation $x^p + y^p = z^p$ of exponent p has no asymptotic solution in K^3, for a certain class of totally real fields K.
- In [Dec16], Deconinck extended the work in [FS15] to the generalized equation $Ax^p + By^p = Cz^p$ with $2 \nmid ABC$.

- In [FS15], Freitas and Siksek show that the equation $x^p + y^p = z^p$ of exponent p has no asymptotic solution in K^3, for a certain class of totally real fields K.
- In [Dec16], Deconinck extended the work in [FS15] to the generalized equation $Ax^p + By^p = Cz^p$ with $2 \nmid ABC$.
- In [IKO20, Theorem 1.1], Işik, Kara, and Ozman proved that the equation $x^p + y^p = z^2$ of exponent p has no asymptotic solution of certain type in $\mathcal{O}_K^3$,

- In [FS15], Freitas and Siksek show that the equation $x^p + y^p = z^p$ of exponent p has no asymptotic solution in K^3, for a certain class of totally real fields K.
- In [Dec16], Deconinck extended the work in [FS15] to the generalized equation $Ax^p + By^p = Cz^p$ with $2 \nmid ABC$.
- In [IKO20, Theorem 1.1], Işik, Kara, and Ozman proved that the equation $x^p + y^p = z^2$ of exponent p has no asymptotic solution of certain type in $\mathcal{O}_K^3$, whenever $h_K^+ = 1$ and $\mathbf{f}(\mathfrak{P}|2) = 1$ for some prime $\mathfrak{P}$ in $\mathcal{O}_K$.

- In [FS15], Freitas and Siksek show that the equation $x^p + y^p = z^p$ of exponent p has no asymptotic solution in K^3, for a certain class of totally real fields K.
- In [Dec16], Deconinck extended the work in [FS15] to the generalized equation $Ax^p + By^p = Cz^p$ with $2 \nmid ABC$.
- In [IKO20, Theorem 1.1], Işik, Kara, and Ozman proved that the equation $x^p + y^p = z^2$ of exponent p has no asymptotic solution of certain type in $\mathcal{O}_K^3$, whenever $h_K^+ = 1$ and $\mathbf{f}(\mathfrak{P}|2) = 1$ for some prime $\mathfrak{P}$ in $\mathcal{O}_K$.
- In [Moc22], Mocanu proved [IKO20, Theorem 1.1] by replacing the assumptions in [IKO20] to $2 \nmid h_K^+$ and 2 is inert in K.

- In [FS15], Freitas and Siksek show that the equation $x^p + y^p = z^p$ of exponent p has no asymptotic solution in K^3, for a certain class of totally real fields K.
- In [Dec16], Deconinck extended the work in [FS15] to the generalized equation $Ax^p + By^p = Cz^p$ with $2 \nmid ABC$.
- In [IKO20, Theorem 1.1], Işik, Kara, and Ozman proved that the equation $x^p + y^p = z^2$ of exponent p has no asymptotic solution of certain type in $\mathcal{O}_K^3$, whenever $h_K^+ = 1$ and $\mathbf{f}(\mathfrak{P}|2) = 1$ for some prime $\mathfrak{P}$ in $\mathcal{O}_K$.
- In [Moc22], Mocanu proved [IKO20, Theorem 1.1] by replacing the assumptions in [IKO20] to $2 \nmid h_K^+$ and 2 is inert in K.

We are interested to study the asymptotic solution of the following equations:

1. $x^p + y^p = 2^r z^p$ and $x^p + y^p = z^2$ over K;

- In [FS15], Freitas and Siksek show that the equation $x^p + y^p = z^p$ of exponent p has no asymptotic solution in K^3, for a certain class of totally real fields K.
- In [Dec16], Deconinck extended the work in [FS15] to the generalized equation $Ax^p + By^p = Cz^p$ with $2 \nmid ABC$.
- In [IKO20, Theorem 1.1], Işik, Kara, and Ozman proved that the equation $x^p + y^p = z^2$ of exponent p has no asymptotic solution of certain type in $\mathcal{O}_K^3$, whenever $h_K^+ = 1$ and $\mathbf{f}(\mathfrak{P}|2) = 1$ for some prime $\mathfrak{P}$ in $\mathcal{O}_K$.
- In [Moc22], Mocanu proved [IKO20, Theorem 1.1] by replacing the assumptions in [IKO20] to $2 \nmid h_K^+$ and 2 is inert in K.

We are interested to study the asymptotic solution of the following equations:

1. $x^p + y^p = 2^r z^p$ and $x^p + y^p = z^2$ over K;
2. $Bx^p + Cy^p = z^2$ and $Bx^p + Cy^p = 2z^2$ over K, where B is an odd integer, C is either an odd integer or a power of 2.

2. On the solutions of $x^p + y^p = 2^r z^p$ over K.

2. On the solutions of $x^p + y^p = 2^r z^p$ over K.

Consider the equation

$$x^p + y^p = 2^r z^p \tag{2.1}$$

of prime exponent p and $r \in \mathbb{N}$.

2. On the solutions of $x^p + y^p = 2^r z^p$ over K.

Consider the equation

$$x^p + y^p = 2^r z^p \tag{2.1}$$

of prime exponent p and $r \in \mathbb{N}$.

- Let K be a totally real field of degree n.

2. On the solutions of $x^p + y^p = 2^r z^p$ over K.

Consider the equation
$$x^p + y^p = 2^r z^p \tag{2.1}$$
of prime exponent p and $r \in \mathbb{N}$.

- Let K be a totally real field of degree n.
- Denote $K^* := K \setminus \{0\}$. Let $P := \mathrm{Spec}(\mathcal{O}_K)$.

2. On the solutions of $x^p + y^p = 2^r z^p$ over K.

Consider the equation

$$x^p + y^p = 2^r z^p \tag{2.1}$$

of prime exponent p and $r \in \mathbb{N}$.

- Let K be a totally real field of degree n.
- Denote $K^* := K \setminus \{0\}$. Let $P := \mathrm{Spec}(\mathcal{O}_K)$.
- For any $S \subseteq P$, let $\mathcal{O}_S := \{\alpha \in K : v_{\mathfrak{P}}(\alpha) \geq 0 \text{ for all } \mathfrak{P} \in P \setminus S\}$ denote the ring of S-integers in K and $\mathcal{O}_S^*$ denote the units of $\mathcal{O}_S$.

2. On the solutions of $x^p + y^p = 2^r z^p$ over K.

Consider the equation

$$x^p + y^p = 2^r z^p \tag{2.1}$$

of prime exponent p and $r \in \mathbb{N}$.

- Let K be a totally real field of degree n.
- Denote $K^* := K \setminus \{0\}$. Let $P := \mathrm{Spec}(\mathcal{O}_K)$.
- For any $S \subseteq P$, let $\mathcal{O}_S := \{\alpha \in K : v_{\mathfrak{P}}(\alpha) \geq 0$ for all $\mathfrak{P} \in P \setminus S\}$ denote the ring of S-integers in K and $\mathcal{O}_S^*$ denote the units of $\mathcal{O}_S$.
- Let $S_K := \{\mathfrak{P} \in P : \mathfrak{P} | 2\}$, and $U_K := \{\mathfrak{P} \in S_K : (3, v_{\mathfrak{P}}(2)) = 1\}$.

2. On the solutions of $x^p + y^p = 2^r z^p$ over K.

Consider the equation

$$x^p + y^p = 2^r z^p \tag{2.1}$$

of prime exponent p and $r \in \mathbb{N}$.

- Let K be a totally real field of degree n.
- Denote $K^* := K \setminus \{0\}$. Let $P := \mathrm{Spec}(\mathcal{O}_K)$.
- For any $S \subseteq P$, let $\mathcal{O}_S := \{\alpha \in K : v_{\mathfrak{P}}(\alpha) \geq 0 \text{ for all } \mathfrak{P} \in P \setminus S\}$ denote the ring of S-integers in K and $\mathcal{O}_S^*$ denote the units of $\mathcal{O}_S$.
- Let $S_K := \{\mathfrak{P} \in P : \mathfrak{P} | 2\}$, and $U_K := \{\mathfrak{P} \in S_K : (3, v_{\mathfrak{P}}(2)) = 1\}$.
- (Trivial solution)
  $abc = 0$, or $(a, b, c) \in \{(1,1,1), (-1,-1,-1)\}$ for $r = 1$.

2. On the solutions of $x^p + y^p = 2^r z^p$ over K.

Consider the equation

$$x^p + y^p = 2^r z^p \tag{2.1}$$

of prime exponent p and $r \in \mathbb{N}$.

- Let K be a totally real field of degree n.
- Denote $K^* := K \setminus \{0\}$. Let $P := \mathrm{Spec}(\mathcal{O}_K)$.
- For any $S \subseteq P$, let $\mathcal{O}_S := \{\alpha \in K : v_{\mathfrak{P}}(\alpha) \geq 0 \text{ for all } \mathfrak{P} \in P \setminus S\}$ denote the ring of S-integers in K and $\mathcal{O}_S^*$ denote the units of $\mathcal{O}_S$.
- Let $S_K := \{\mathfrak{P} \in P : \mathfrak{P}|2\}$, and $U_K := \{\mathfrak{P} \in S_K : (3, v_{\mathfrak{P}}(2)) = 1\}$.
- (Trivial solution)
  $abc = 0$, or $(a, b, c) \in \{(1,1,1), (-1,-1,-1)\}$ for $r = 1$.

Definition (2.1)

Let W_K be the set of all non-trivial primitive solution $(a, b, c) \in \mathcal{O}_K^3$ to the equation (2.1) with $\mathfrak{P}|abc$ for every $\mathfrak{P} \in S_K$.

Main results.

Main results.

Theorem (2.2) (Kumar-Sahoo, 2023)

Let K be a totally real field.

Main results.

Theorem (2.2) (Kumar-Sahoo, 2023)

Let K be a totally real field. Suppose, for every solution (λ, μ) to the S_K-unit equation

$$\lambda + \mu = 1, \ \lambda, \mu \in \mathcal{O}_{S_K}^*, \tag{2.2}$$

Main results.

Theorem (2.2) (Kumar-Sahoo, 2023)

Let K be a totally real field. Suppose, for every solution (λ, μ) to the S_K-unit equation

$$\lambda + \mu = 1, \ \lambda, \mu \in \mathcal{O}^*_{S_K}, \tag{2.2}$$

there exists some $\mathfrak{P} \in S_K$ that satisfies

$$\max\{|v_{\mathfrak{P}}(\lambda)|, |v_{\mathfrak{P}}(\mu)|\} \leq 4 v_{\mathfrak{P}}(2). \tag{2.3}$$

Main results.

Theorem (2.2) (Kumar-Sahoo, 2023)

Let K be a totally real field. Suppose, for every solution (λ, μ) to the S_K-unit equation

$$\lambda + \mu = 1, \ \lambda, \mu \in \mathcal{O}_{S_K}^*, \tag{2.2}$$

there exists some $\mathfrak{P} \in S_K$ that satisfies

$$\max \{|v_{\mathfrak{P}}(\lambda)|, |v_{\mathfrak{P}}(\mu)|\} \leq 4 v_{\mathfrak{P}}(2). \tag{2.3}$$

Then the equation $x^p + y^p = 2^r z^p$ has no asymptotic solution in W_K.

Main results.

Theorem (2.2) (Kumar-Sahoo, 2023)

Let K be a totally real field. Suppose, for every solution (λ, μ) to the S_K-unit equation

$$\lambda + \mu = 1, \ \lambda, \mu \in \mathcal{O}_{S_K}^*, \tag{2.2}$$

there exists some $\mathfrak{P} \in S_K$ that satisfies

$$\max \{|v_{\mathfrak{P}}(\lambda)|, |v_{\mathfrak{P}}(\mu)|\} \leq 4 v_{\mathfrak{P}}(2). \tag{2.3}$$

Then the equation $x^p + y^p = 2^r z^p$ has no asymptotic solution in W_K.

We write (ES) for "either $[K : \mathbb{Q}]$ is odd or Eichler-Shimura Conjecture holds for K."

Main results.

Theorem (2.2) (Kumar-Sahoo, 2023)

Let K be a totally real field. Suppose, for every solution (λ, μ) to the S_K-unit equation

$$\lambda + \mu = 1, \ \lambda, \mu \in \mathcal{O}_{S_K}^*, \tag{2.2}$$

there exists some $\mathfrak{P} \in S_K$ that satisfies

$$\max \{|v_{\mathfrak{P}}(\lambda)|, |v_{\mathfrak{P}}(\mu)|\} \leq 4 v_{\mathfrak{P}}(2). \tag{2.3}$$

Then the equation $x^p + y^p = 2^r z^p$ has no asymptotic solution in W_K.

We write (ES) for "either $[K : \mathbb{Q}]$ is odd or Eichler-Shimura Conjecture holds for K."

Theorem (2.3) (Kumar-Sahoo, 2023)

Let K be a totally real field satisfying the condition (ES).

Main results.

Theorem (2.2) (Kumar-Sahoo, 2023)

Let K be a totally real field. Suppose, for every solution (λ, μ) to the S_K-unit equation

$$\lambda + \mu = 1, \ \lambda, \mu \in \mathcal{O}_{S_K}^*, \tag{2.2}$$

there exists some $\mathfrak{P} \in S_K$ that satisfies

$$\max\{|v_{\mathfrak{P}}(\lambda)|, |v_{\mathfrak{P}}(\mu)|\} \leq 4v_{\mathfrak{P}}(2). \tag{2.3}$$

Then the equation $x^p + y^p = 2^r z^p$ has no asymptotic solution in W_K.

We write (ES) for "either $[K : \mathbb{Q}]$ is odd or Eichler-Shimura Conjecture holds for K."

Theorem (2.3) (Kumar-Sahoo, 2023)

Let K be a totally real field satisfying the condition (ES). Suppose, for every solution (λ, μ) to the S_K-unit equation (2.2) there exists some $\mathfrak{P} \in U_K$ that satisfies

Main results.

Theorem (2.2) (Kumar-Sahoo, 2023)

Let K be a totally real field. Suppose, for every solution (λ, μ) to the S_K-unit equation

$$\lambda + \mu = 1, \ \lambda, \mu \in \mathcal{O}_{S_K}^*, \tag{2.2}$$

there exists some $\mathfrak{P} \in S_K$ that satisfies

$$\max\{|v_{\mathfrak{P}}(\lambda)|, |v_{\mathfrak{P}}(\mu)|\} \leq 4v_{\mathfrak{P}}(2). \tag{2.3}$$

Then the equation $x^p + y^p = 2^r z^p$ has no asymptotic solution in W_K.

We write (ES) for "either $[K : \mathbb{Q}]$ is odd or Eichler-Shimura Conjecture holds for K."

Theorem (2.3) (Kumar-Sahoo, 2023)

Let K be a totally real field satisfying the condition (ES). Suppose, for every solution (λ, μ) to the S_K-unit equation (2.2) there exists some $\mathfrak{P} \in U_K$ that satisfies

$$\max\{|v_{\mathfrak{P}}(\lambda)|, |v_{\mathfrak{P}}(\mu)|\} \leq 4v_{\mathfrak{P}}(2) \text{ and } v_{\mathfrak{P}}(\lambda\mu) \equiv v_{\mathfrak{P}}(2) \pmod 3. \tag{2.4}$$

Main results.

Theorem (2.2) (Kumar-Sahoo, 2023)

Let K be a totally real field. Suppose, for every solution (λ, μ) to the S_K-unit equation

$$\lambda + \mu = 1, \ \lambda, \mu \in \mathcal{O}_{S_K}^*, \tag{2.2}$$

there exists some $\mathfrak{P} \in S_K$ that satisfies

$$\max\{|v_{\mathfrak{P}}(\lambda)|, |v_{\mathfrak{P}}(\mu)|\} \leq 4v_{\mathfrak{P}}(2). \tag{2.3}$$

Then the equation $x^p + y^p = 2^r z^p$ has no asymptotic solution in W_K.

We write (ES) for "either $[K : \mathbb{Q}]$ is odd or Eichler-Shimura Conjecture holds for K."

Theorem (2.3) (Kumar-Sahoo, 2023)

Let K be a totally real field satisfying the condition (ES). Suppose, for every solution (λ, μ) to the S_K-unit equation (2.2) there exists some $\mathfrak{P} \in U_K$ that satisfies

$$\max\{|v_{\mathfrak{P}}(\lambda)|, |v_{\mathfrak{P}}(\mu)|\} \leq 4v_{\mathfrak{P}}(2) \text{ and } v_{\mathfrak{P}}(\lambda\mu) \equiv v_{\mathfrak{P}}(2) \pmod 3. \tag{2.4}$$

Then the equation $x^p + y^p = 2^r z^p$ with $r = 2, 3$ has no asymptotic solution in $\mathcal{O}_K^3$.

Key steps to prove Theorem (2.2) and Theorem (2.3).

Key steps to prove Theorem (2.2) and Theorem (2.3).

For any non-trivial primitive solution $(a, b, c) \in \mathcal{O}_K^3$ to the equation $x^p + y^p = 2^r z^p$, consider the Frey elliptic curve as

Key steps to prove Theorem (2.2) and Theorem (2.3).

For any non-trivial primitive solution $(a, b, c) \in \mathcal{O}_K^3$ to the equation $x^p + y^p = 2^r z^p$, consider the Frey elliptic curve as

$$E = E_{a,b,c} : Y^2 = X(X - a^p)(X + b^p), \qquad (2.5)$$

Key steps to prove Theorem (2.2) and Theorem (2.3).

For any non-trivial primitive solution $(a, b, c) \in \mathcal{O}_K^3$ to the equation $x^p + y^p = 2^r z^p$, consider the Frey elliptic curve as

$$E = E_{a,b,c} : Y^2 = X(X - a^p)(X + b^p), \tag{2.5}$$

where $\Delta_E = 2^{4+2r}(abc)^{2p},\ c_4 = 2^4(a^{2p} + 2^r b^p c^p)$ and $j_E = 2^{8-2r}\frac{(a^{2p}+2^r b^p c^p)^3}{(abc)^{2p}}$.

Key steps to prove Theorem (2.2) and Theorem (2.3).

For any non-trivial primitive solution $(a, b, c) \in \mathcal{O}_K^3$ to the equation $x^p + y^p = 2^r z^p$, consider the Frey elliptic curve as

$$E = E_{a,b,c} : Y^2 = X(X - a^p)(X + b^p), \tag{2.5}$$

where $\Delta_E = 2^{4+2r}(abc)^{2p}$, $c_4 = 2^4(a^{2p} + 2^r b^p c^p)$ and $j_E = 2^{8-2r} \frac{(a^{2p}+2^r b^p c^p)^3}{(abc)^{2p}}$.

- There exists a constant $A_{K,r}$ (depending on K, r) such that for primes $p > A_{K,r}$, the Frey curve E/K is modular.

Key steps to prove Theorem (2.2) and Theorem (2.3).

For any non-trivial primitive solution $(a, b, c) \in \mathcal{O}_K^3$ to the equation $x^p + y^p = 2^r z^p$, consider the Frey elliptic curve as

$$E = E_{a,b,c} : Y^2 = X(X - a^p)(X + b^p), \tag{2.5}$$

where $\Delta_E = 2^{4+2r}(abc)^{2p}$, $c_4 = 2^4(a^{2p} + 2^r b^p c^p)$ and $j_E = 2^{8-2r} \frac{(a^{2p}+2^r b^p c^p)^3}{(abc)^{2p}}$.

- There exists a constant $A_{K,r}$ (depending on K, r) such that for primes $p > A_{K,r}$, the Frey curve E/K is modular.
- Then at all primes $\mathfrak{q} \in P$ away from S_K, the Frey curve E is minimal, semi-stable and satisfies $p | v_{\mathfrak{q}}(\Delta_E)$.

Key steps to prove Theorem (2.2) and Theorem (2.3).

For any non-trivial primitive solution $(a, b, c) \in \mathcal{O}_K^3$ to the equation $x^p + y^p = 2^r z^p$, consider the Frey elliptic curve as

$$E = E_{a,b,c} : Y^2 = X(X - a^p)(X + b^p), \tag{2.5}$$

where $\Delta_E = 2^{4+2r}(abc)^{2p}$, $c_4 = 2^4(a^{2p} + 2^r b^p c^p)$ and $j_E = 2^{8-2r} \frac{(a^{2p}+2^r b^p c^p)^3}{(abc)^{2p}}$.

- There exists a constant $A_{K,r}$ (depending on K, r) such that for primes $p > A_{K,r}$, the Frey curve E/K is modular.
- Then at all primes $\mathfrak{q} \in P$ away from S_K, the Frey curve E is minimal, semi-stable and satisfies $p | v_{\mathfrak{q}}(\Delta_E)$.
- There exists a constant C_K (depending on K) such that for primes $p > C_K$, $\bar{\rho}_{E,p}$ is irreducible, where $\bar{\rho}_{E,p}$ is the residual Galois representation of G_K acting on the p-torsion points of E.

Key steps to prove Theorem (2.2) and Theorem (2.3).

For any non-trivial primitive solution $(a, b, c) \in \mathcal{O}_K^3$ to the equation $x^p + y^p = 2^r z^p$, consider the Frey elliptic curve as

$$E = E_{a,b,c} : Y^2 = X(X - a^p)(X + b^p), \tag{2.5}$$

where $\Delta_E = 2^{4+2r}(abc)^{2p}$, $c_4 = 2^4(a^{2p} + 2^r b^p c^p)$ and $j_E = 2^{8-2r} \frac{(a^{2p}+2^r b^p c^p)^3}{(abc)^{2p}}$.

- There exists a constant $A_{K,r}$ (depending on K, r) such that for primes $p > A_{K,r}$, the Frey curve E/K is modular.
- Then at all primes $\mathfrak{q} \in P$ away from S_K, the Frey curve E is minimal, semi-stable and satisfies $p | v_{\mathfrak{q}}(\Delta_E)$.
- There exists a constant C_K (depending on K) such that for primes $p > C_K$, $\bar{\rho}_{E,p}$ is irreducible, where $\bar{\rho}_{E,p}$ is the residual Galois representation of G_K acting on the p-torsion points of E.

Proof continues...

Lemma (2.4)

Let $\mathfrak{P} \in S_K$*. Suppose* $(a, b, c) \in \mathcal{O}_K^3$ *is a non-trivial primitive solution to the equation* $x^p + y^p = 2^r z^p$ *of exponent* $p > \max\{|(4-r)v_{\mathfrak{P}}(2)|, [K:\mathbb{Q}]r\}$ *and let* $E := E_{a,b,c}$ *be the associated Frey curve.*

Proof continues...

Lemma (2.4)

Let $\mathfrak{P} \in S_K$*. Suppose* $(a, b, c) \in \mathcal{O}_K^3$ *is a non-trivial primitive solution to the equation* $x^p + y^p = 2^r z^p$ *of exponent* $p > \max\{|(4-r)v_{\mathfrak{P}}(2)|, [K:\mathbb{Q}]r\}$ *and let* $E := E_{a,b,c}$ *be the associated Frey curve.*

1. *If* $(a, b, c) \in W_K$*, then* $v_{\mathfrak{P}}(j_E) < 0$ *and* $p \nmid v_{\mathfrak{P}}(j_E)$*, equivalently* $p | \#\bar{\rho}_{E,p}(I_{\mathfrak{P}})$*.*

Proof continues...

Lemma (2.4)

Let $\mathfrak{P} \in S_K$. *Suppose* $(a, b, c) \in \mathcal{O}_K^3$ *is a non-trivial primitive solution to the equation* $x^p + y^p = 2^r z^p$ *of exponent* $p > \max\{|(4-r)v_{\mathfrak{P}}(2)|, [K:\mathbb{Q}]r\}$ *and let* $E := E_{a,b,c}$ *be the associated Frey curve.*

1. *If* $(a, b, c) \in W_K$, *then* $v_{\mathfrak{P}}(j_E) < 0$ *and* $p \nmid v_{\mathfrak{P}}(j_E)$, *equivalently* $p | \#\bar{\rho}_{E,p}(I_{\mathfrak{P}})$.
2. *Let* $r = 2, 3$. *If* $\mathfrak{P} \in U_K$, *then either* $p | \#\bar{\rho}_{E,p}(I_{\mathfrak{P}})$ *or* $3 | \#\bar{\rho}_{E,p}(I_{\mathfrak{P}})$.

Proof continues...

Lemma (2.4)

Let $\mathfrak{P} \in S_K$. Suppose $(a, b, c) \in \mathcal{O}_K^3$ is a non-trivial primitive solution to the equation $x^p + y^p = 2^r z^p$ of exponent $p > \max\{|(4-r)v_{\mathfrak{P}}(2)|, [K : \mathbb{Q}]r\}$ and let $E := E_{a,b,c}$ be the associated Frey curve.

1. *If $(a, b, c) \in W_K$, then $v_{\mathfrak{P}}(j_E) < 0$ and $p \nmid v_{\mathfrak{P}}(j_E)$, equivalently $p | \#\bar{\rho}_{E,p}(I_{\mathfrak{P}})$.*
2. *Let $r = 2, 3$. If $\mathfrak{P} \in U_K$, then either $p | \#\bar{\rho}_{E,p}(I_{\mathfrak{P}})$ or $3 | \#\bar{\rho}_{E,p}(I_{\mathfrak{P}})$.*

- Let $(a, b, c) \in W_K$ be a non-trivial primitive solution to the equation $x^p + y^p = 2^r z^p$ of exponent $p \gg 0$ and let $E := E_{a,b,c}$ be the associated Frey curve.

Proof continues...

Lemma (2.4)

Let $\mathfrak{P} \in S_K$. Suppose $(a,b,c) \in \mathcal{O}_K^3$ is a non-trivial primitive solution to the equation $x^p + y^p = 2^r z^p$ of exponent $p > \max\{|(4-r)v_{\mathfrak{P}}(2)|, [K:\mathbb{Q}]r\}$ and let $E := E_{a,b,c}$ be the associated Frey curve.

1. *If $(a,b,c) \in W_K$, then $v_{\mathfrak{P}}(j_E) < 0$ and $p \nmid v_{\mathfrak{P}}(j_E)$, equivalently $p | \#\bar{\rho}_{E,p}(I_{\mathfrak{P}})$.*
2. *Let $r = 2, 3$. If $\mathfrak{P} \in U_K$, then either $p | \#\bar{\rho}_{E,p}(I_{\mathfrak{P}})$ or $3 | \#\bar{\rho}_{E,p}(I_{\mathfrak{P}})$.*

- Let $(a,b,c) \in W_K$ be a non-trivial primitive solution to the equation $x^p + y^p = 2^r z^p$ of exponent $p \gg 0$ and let $E := E_{a,b,c}$ be the associated Frey curve.
- Then by using two key results in [FS15], there exists an elliptic curve E'/K such that E'/K has good reduction away from S_K and has full 2-torsion, $\bar{\rho}_{E,p} \sim \bar{\rho}_{E',p}$ and $v_{\mathfrak{P}}(j_{E'}) < 0$ for all $\mathfrak{P} \in S_K$.

Proof continues...

Lemma (2.4)

Let $\mathfrak{P} \in S_K$. Suppose $(a,b,c) \in \mathcal{O}_K^3$ is a non-trivial primitive solution to the equation $x^p + y^p = 2^r z^p$ of exponent $p > \max\{|(4-r)v_{\mathfrak{P}}(2)|, [K:\mathbb{Q}]r\}$ and let $E := E_{a,b,c}$ be the associated Frey curve.

1. *If $(a,b,c) \in W_K$, then $v_{\mathfrak{P}}(j_E) < 0$ and $p \nmid v_{\mathfrak{P}}(j_E)$, equivalently $p | \#\bar{\rho}_{E,p}(I_{\mathfrak{P}})$.*
2. *Let $r = 2, 3$. If $\mathfrak{P} \in U_K$, then either $p | \#\bar{\rho}_{E,p}(I_{\mathfrak{P}})$ or $3 | \#\bar{\rho}_{E,p}(I_{\mathfrak{P}})$.*

- Let $(a,b,c) \in W_K$ be a non-trivial primitive solution to the equation $x^p + y^p = 2^r z^p$ of exponent $p \gg 0$ and let $E := E_{a,b,c}$ be the associated Frey curve.
- Then by using two key results in [FS15], there exists an elliptic curve E'/K such that E'/K has good reduction away from S_K and has full 2-torsion, $\bar{\rho}_{E,p} \sim \bar{\rho}_{E',p}$ and $v_{\mathfrak{P}}(j_{E'}) < 0$ for all $\mathfrak{P} \in S_K$.
- Then we relate $j_{E'}$ in terms of solution of S_K-unit equation $\lambda + \mu = 1$ along with the condition $\max\{|v_{\mathfrak{P}}(\lambda)|, |v_{\mathfrak{P}}(\mu)|\} \leq 4v_{\mathfrak{P}}(2)$ to get $v_{\mathfrak{P}}(j_{E'}) \geq 0$ for some $\mathfrak{P} \in S_K$, which is a contradiction.

Proof continues...

Lemma (2.4)

Let $\mathfrak{P} \in S_K$. Suppose $(a, b, c) \in \mathcal{O}_K^3$ is a non-trivial primitive solution to the equation $x^p + y^p = 2^r z^p$ of exponent $p > \max\{|(4-r)v_{\mathfrak{P}}(2)|, [K : \mathbb{Q}]r\}$ and let $E := E_{a,b,c}$ be the associated Frey curve.

1. *If $(a, b, c) \in W_K$, then $v_{\mathfrak{P}}(j_E) < 0$ and $p \nmid v_{\mathfrak{P}}(j_E)$, equivalently $p | \# \bar{\rho}_{E,p}(I_{\mathfrak{P}})$.*
2. *Let $r = 2, 3$. If $\mathfrak{P} \in U_K$, then either $p | \# \bar{\rho}_{E,p}(I_{\mathfrak{P}})$ or $3 | \# \bar{\rho}_{E,p}(I_{\mathfrak{P}})$.*

- Let $(a, b, c) \in W_K$ be a non-trivial primitive solution to the equation $x^p + y^p = 2^r z^p$ of exponent $p \gg 0$ and let $E := E_{a,b,c}$ be the associated Frey curve.
- Then by using two key results in [FS15], there exists an elliptic curve E'/K such that E'/K has good reduction away from S_K and has full 2-torsion, $\bar{\rho}_{E,p} \sim \bar{\rho}_{E',p}$ and $v_{\mathfrak{P}}(j_{E'}) < 0$ for all $\mathfrak{P} \in S_K$.
- Then we relate $j_{E'}$ in terms of solution of S_K-unit equation $\lambda + \mu = 1$ along with the condition $\max\{|v_{\mathfrak{P}}(\lambda)|, |v_{\mathfrak{P}}(\mu)|\} \leq 4v_{\mathfrak{P}}(2)$ to get $v_{\mathfrak{P}}(j_{E'}) \geq 0$ for some $\mathfrak{P} \in S_K$, which is a contradiction.
- A similar argument also holds for any non-trivial primitive solution $(a, b, c) \in \mathcal{O}_K^3$ to the equation $x^p + y^p = 2^r z^p$ of exponent $p \gg 0$ with $r = 2, 3$, but in this case we get either $v_{\mathfrak{P}}(j_{E'}) < 0$ or $3 \nmid v_{\mathfrak{P}}(j_{E'})$ for $\mathfrak{P} \in U_K$.

Local criteria

Local criteria

Now, we will provide several local criteria of K the equation $x^p + y^p = 2^r z^p$.

Local criteria

Now, we will provide several local criteria of K the equation $x^p + y^p = 2^r z^p$.

Proposition (Even degree)

Let $K = \mathbb{Q}(\sqrt{d})$. Suppose $d \geq 2$ is a square-free integer satisfy one of the following conditions:

1. $d \equiv 3 \pmod 8$;
2. $d \equiv 5 \pmod 8$;
3. $d \equiv 6$ *or* $10 \pmod{16}$;
4. $d \equiv 2 \pmod{16}$ *and* d *has some prime divisor* $q \equiv 5$ *or* $7 \pmod 8$;
5. $d \equiv 14 \pmod{16}$ *and* d *has some prime divisor* $q \equiv 3$ *or* $5 \pmod 8$.

Then the equation $x^p + y^p = 2^r z^p$ has no asymptotic solution in W_K.

Local criteria

Now, we will provide several local criteria of K the equation $x^p + y^p = 2^r z^p$.

Proposition (Even degree)

Let $K = \mathbb{Q}(\sqrt{d})$. Suppose $d \geq 2$ is a square-free integer satisfy one of the following conditions:

1. $d \equiv 3 \pmod 8$;
2. $d \equiv 5 \pmod 8$;
3. $d \equiv 6$ *or* $10 \pmod{16}$;
4. $d \equiv 2 \pmod{16}$ *and d has some prime divisor* $q \equiv 5$ *or* $7 \pmod 8$;
5. $d \equiv 14 \pmod{16}$ *and d has some prime divisor* $q \equiv 3$ *or* $5 \pmod 8$.

Then the equation $x^p + y^p = 2^r z^p$ has no asymptotic solution in W_K.

Proposition (Odd degree)

Assume 2 is either inert or totally ramified in K. Suppose one of the hypothesis holds:

1. *Suppose $n = [K : \mathbb{Q}]$ and $l > 5$ is a prime number such that $(n, l-1) = 1$ and l totally ramifies in K.*
2. *Suppose $[K : \mathbb{Q}]$ is odd and 3 totally splits in K.*

Then the equation $x^p + y^p = 2^r z^p$ has no asymptotic solution in W_K.

3. On the solutions of $x^p + y^p = z^2$ over K.

3. On the solutions of $x^p + y^p = z^2$ over K.

Consider the equation

$$x^p + y^p = z^2 \tag{3.1}$$

with prime exponent p.

3. On the solutions of $x^p + y^p = z^2$ over K.

Consider the equation

$$x^p + y^p = z^2 \tag{3.1}$$

with prime exponent p.

- (Trivial solution)
  $abc = 0$

3. On the solutions of $x^p + y^p = z^2$ over K.

Consider the equation

$$x^p + y^p = z^2 \tag{3.1}$$

with prime exponent p.

- (Trivial solution)
  $abc = 0$

Definition (3.1)

Let W'_K be the set of all non-trivial primitive solution $(a, b, c) \in \mathcal{O}_K^3$ to the equation (3.1) with $\mathfrak{P} | ab$ for every $\mathfrak{P} \in S_K$.

3. On the solutions of $x^p + y^p = z^2$ over K.

Consider the equation

$$x^p + y^p = z^2 \tag{3.1}$$

with prime exponent p.

- (Trivial solution)
  $abc = 0$

Definition (3.1)

Let W'_K be the set of all non-trivial primitive solution $(a, b, c) \in \mathcal{O}_K^3$ to the equation (3.1) with $\mathfrak{P} | ab$ for every $\mathfrak{P} \in S_K$.

- We define the m-Selmer group K and S_K by
  $K(S_K, m) := \{x \in K^*/(K^*)^m : v_{\mathfrak{p}}(x) \equiv 0 \pmod{m}, \forall \mathfrak{p} \notin S_K\}$.

3. On the solutions of $x^p + y^p = z^2$ over K.

Consider the equation

$$x^p + y^p = z^2 \tag{3.1}$$

with prime exponent p.

- (Trivial solution)
  $abc = 0$

Definition (3.1)

Let W'_K be the set of all non-trivial primitive solution $(a, b, c) \in \mathcal{O}_K^3$ to the equation (3.1) with $\mathfrak{P}|ab$ for every $\mathfrak{P} \in S_K$.

- We define the m-Selmer group K and S_K by
  $K(S_K, m) := \{x \in K^*/(K^*)^m : v_{\mathfrak{p}}(x) \equiv 0 \pmod{m}, \forall \mathfrak{p} \notin S_K\}$.
- Let $L = K(\sqrt{a})$ for $a \in K(S_K, 2)$.

3. On the solutions of $x^p + y^p = z^2$ over K.

Consider the equation

$$x^p + y^p = z^2 \tag{3.1}$$

with prime exponent p.

- (Trivial solution)
  $abc = 0$

Definition (3.1)

Let W_K' be the set of all non-trivial primitive solution $(a, b, c) \in \mathcal{O}_K^3$ to the equation (3.1) with $\mathfrak{P}|ab$ for every $\mathfrak{P} \in S_K$.

- We define the m-Selmer group K and S_K by
  $K(S_K, m) := \{x \in K^*/(K^*)^m : v_{\mathfrak{p}}(x) \equiv 0 \pmod{m}, \forall \mathfrak{p} \notin S_K\}$.
- Let $L = K(\sqrt{a})$ for $a \in K(S_K, 2)$.
- Let S_L be the set of all prime ideals of L lying over primes of S_K.

Main result.

Main result.

Theorem (3.2) (Kumar-Sahoo, 2023)

Let K be a totally real field. For each $a \in K(S_K, 2)$, let $L = K(\sqrt{a})$.

Main result.

Theorem (3.2) (Kumar-Sahoo, 2023)

Let K be a totally real field. For each $a \in K(S_K, 2)$, let $L = K(\sqrt{a})$. Suppose, for every solution (λ, μ) to the S_K-unit equation $\lambda + \mu = 1$ with $\lambda, \mu \in \mathcal{O}_{S_K}^*$, there exists some $\mathfrak{P} \in S_K$ that satisfies

$$\max\{|v_{\mathfrak{P}}(\lambda)|, |v_{\mathfrak{P}}(\mu)|\} \leq 4v_{\mathfrak{P}}(2), \tag{3.2}$$

Main result.

Theorem (3.2) (Kumar-Sahoo, 2023)

Let K be a totally real field. For each $a \in K(S_K, 2)$, let $L = K(\sqrt{a})$. Suppose, for every solution (λ, μ) to the S_K-unit equation $\lambda + \mu = 1$ with $\lambda, \mu \in \mathcal{O}_{S_K}^*$, there exists some $\mathfrak{P} \in S_K$ that satisfies

$$\max\{|v_{\mathfrak{P}}(\lambda)|, |v_{\mathfrak{P}}(\mu)|\} \leq 4v_{\mathfrak{P}}(2), \tag{3.2}$$

and for every solution (λ, μ) to the S_L-unit equation $\lambda + \mu = 1$ with $\lambda, \mu \in \mathcal{O}_{S_L}^*$, there exists some $\mathfrak{P}' \in S_L$ that satisfies

$$\max\{|v_{\mathfrak{P}'}(\lambda)|, |v_{\mathfrak{P}'}(\mu)|\} \leq 4v_{\mathfrak{P}'}(2). \tag{3.3}$$

Main result.

Theorem (3.2) (Kumar-Sahoo, 2023)

Let K be a totally real field. For each $a \in K(S_K, 2)$, let $L = K(\sqrt{a})$. Suppose, for every solution (λ, μ) to the S_K-unit equation $\lambda + \mu = 1$ with $\lambda, \mu \in \mathcal{O}^*_{S_K}$, there exists some $\mathfrak{P} \in S_K$ that satisfies

$$\max\{|v_{\mathfrak{P}}(\lambda)|, |v_{\mathfrak{P}}(\mu)|\} \leq 4v_{\mathfrak{P}}(2), \tag{3.2}$$

and for every solution (λ, μ) to the S_L-unit equation $\lambda + \mu = 1$ with $\lambda, \mu \in \mathcal{O}^*_{S_L}$, there exists some $\mathfrak{P}' \in S_L$ that satisfies

$$\max\{|v_{\mathfrak{P}'}(\lambda)|, |v_{\mathfrak{P}'}(\mu)|\} \leq 4v_{\mathfrak{P}'}(2). \tag{3.3}$$

Then the equation $x^p + y^p = z^2$ has no asymptotic solution in W'_K.

4. On the solutions of $Bx^p + Cy^p = z^2$ and $Bx^p + 2^r y^p = 2z^2$ over K.

4. On the solutions of $Bx^p + Cy^p = z^2$ and $Bx^p + 2^r y^p = 2z^2$ over K.

Consider the equation

$$Bx^p + Cy^p = z^2, \tag{4.1}$$

4. On the solutions of $Bx^p + Cy^p = z^2$ and $Bx^p + 2^r y^p = 2z^2$ over K.

Consider the equation

$$Bx^p + Cy^p = z^2, \tag{4.1}$$

where B is an odd integer, C is either an odd integer or a power of 2.

4. On the solutions of $Bx^p + Cy^p = z^2$ and $Bx^p + 2^r y^p = 2z^2$ over K.

Consider the equation

$$Bx^p + Cy^p = z^2, \tag{4.1}$$

where B is an odd integer, C is either an odd integer or a power of 2.

- (Trivial solution)
  $abc = 0$

4. On the solutions of $Bx^p + Cy^p = z^2$ and $Bx^p + 2^r y^p = 2z^2$ over K.

Consider the equation

$$Bx^p + Cy^p = z^2, \tag{4.1}$$

where B is an odd integer, C is either an odd integer or a power of 2.

- (Trivial solution)
  $abc = 0$

Definition

Let W_K be the set of all non-trivial primitive solutions $(a, b, c) \in \mathcal{O}_K^3$ to the equation (4.1) with $\mathfrak{P} | bc$ for every $\mathfrak{P} \in S_K$.

4. On the solutions of $Bx^p + Cy^p = z^2$ and $Bx^p + 2^r y^p = 2z^2$ over K.

Consider the equation

$$Bx^p + Cy^p = z^2, \tag{4.1}$$

where B is an odd integer, C is either an odd integer or a power of 2.

- (Trivial solution)
  $abc = 0$

Definition

Let W_K be the set of all non-trivial primitive solutions $(a, b, c) \in \mathcal{O}_K^3$ to the equation (4.1) with $\mathfrak{P} | bc$ for every $\mathfrak{P} \in S_K$.

- For any set $S \subseteq P$, let $\mathrm{Cl}_S(K) := \mathrm{Cl}(K)/\langle[\mathfrak{P}]\rangle_{\mathfrak{P}\in S}$ and $\mathrm{Cl}_S(K)[n]$ be its n-torsion points.

4. On the solutions of $Bx^p + Cy^p = z^2$ and $Bx^p + 2^r y^p = 2z^2$ over K.

Consider the equation

$$Bx^p + Cy^p = z^2, \tag{4.1}$$

where B is an odd integer, C is either an odd integer or a power of 2.

- (Trivial solution)
  $abc = 0$

Definition

Let W_K be the set of all non-trivial primitive solutions $(a, b, c) \in \mathcal{O}_K^3$ to the equation (4.1) with $\mathfrak{P} | bc$ for every $\mathfrak{P} \in S_K$.

- For any set $S \subseteq P$, let $\mathrm{Cl}_S(K) := \mathrm{Cl}(K)/\langle [\mathfrak{P}] \rangle_{\mathfrak{P} \in S}$ and $\mathrm{Cl}_S(K)[n]$ be its n-torsion points.
- Let $S'_K := \{\mathfrak{P} \in P : \mathfrak{P} | 2BC\}$.

4. On the solutions of $Bx^p + Cy^p = z^2$ and $Bx^p + 2^r y^p = 2z^2$ over K.

Consider the equation

$$Bx^p + Cy^p = z^2, \tag{4.1}$$

where B is an odd integer, C is either an odd integer or a power of 2.

- (Trivial solution)
  $abc = 0$

Definition

Let W_K be the set of all non-trivial primitive solutions $(a, b, c) \in \mathcal{O}_K^3$ to the equation (4.1) with $\mathfrak{P} | bc$ for every $\mathfrak{P} \in S_K$.

- For any set $S \subseteq P$, let $\mathrm{Cl}_S(K) := \mathrm{Cl}(K)/\langle[\mathfrak{P}]\rangle_{\mathfrak{P}\in S}$ and $\mathrm{Cl}_S(K)[n]$ be its n-torsion points.
- Let $S'_K := \{\mathfrak{P} \in P : \ \mathfrak{P} | 2BC\}$.
- For $r \in \mathbb{N}$, let S_r be the set consisting of elements $(\pm\sqrt{2^r + B}, 1, 1)$, $(\pm\sqrt{2^r - B}, -1, 1)$, $(\pm\sqrt{-2^r + B}, 1, -1)$, $(\pm\sqrt{-2^r - B}, 1, 1)$.

Main results.

Main results.

Theorem (4.1) (Kumar-Sahoo)

Let K be a totally real field with $\mathrm{Cl}_{S'_K}(K)[2] = 1$.

Main results.

Theorem (4.1) (Kumar-Sahoo)

Let K be a totally real field with $\mathrm{Cl}_{S'_K}(K)[2] = 1$. Suppose for every solution (α, β, γ) to the equation

$$\alpha + \beta = \gamma^2, \text{ where } \alpha, \beta \in \mathcal{O}^*_{S'_K}, \ \gamma \in \mathcal{O}_{S'_K}. \tag{4.2}$$

there exists $\mathfrak{P} \in S_K$ that satisfies

Main results.

Theorem (4.1) (Kumar-Sahoo)

Let K be a totally real field with $\mathrm{Cl}_{S'_K}(K)[2] = 1$. Suppose for every solution (α, β, γ) to the equation

$$\alpha + \beta = \gamma^2, \text{ where } \alpha, \beta \in \mathcal{O}^*_{S'_K}, \ \gamma \in \mathcal{O}_{S'_K}. \tag{4.2}$$

there exists $\mathfrak{P} \in S_K$ that satisfies

$$\left| v_{\mathfrak{P}}\left(\alpha\beta^{-1}\right) \right| \leq 6 v_{\mathfrak{P}}(2). \tag{4.3}$$

Main results.

Theorem (4.1) (Kumar-Sahoo)

Let K be a totally real field with $\mathrm{Cl}_{S'_K}(K)[2] = 1$. Suppose for every solution (α, β, γ) to the equation

$$\alpha + \beta = \gamma^2, \text{ where } \alpha, \beta \in \mathcal{O}^*_{S'_K}, \ \gamma \in \mathcal{O}_{S'_K}. \tag{4.2}$$

there exists $\mathfrak{P} \in S_K$ that satisfies

$$\left| v_{\mathfrak{P}} \left(\alpha \beta^{-1} \right) \right| \leq 6 v_{\mathfrak{P}}(2). \tag{4.3}$$

Then, the equation $Bx^p + Cy^p = z^2$ (resp., $Bx^p + 2^r y^p = 2z^2$) has no asymptotic solution in W_K (resp., $\mathcal{O}_K^3$).

Main results.

Theorem (4.1) (Kumar-Sahoo)

Let K be a totally real field with $\mathrm{Cl}_{S'_K}(K)[2] = 1$. Suppose for every solution (α, β, γ) to the equation

$$\alpha + \beta = \gamma^2, \text{ where } \alpha, \beta \in \mathcal{O}^*_{S'_K}, \ \gamma \in \mathcal{O}_{S'_K}. \tag{4.2}$$

there exists $\mathfrak{P} \in S_K$ that satisfies

$$\left| v_{\mathfrak{P}} \left(\alpha \beta^{-1} \right) \right| \leq 6 v_{\mathfrak{P}}(2). \tag{4.3}$$

Then, the equation $Bx^p + Cy^p = z^2$ (resp., $Bx^p + 2^r y^p = 2z^2$) has no asymptotic solution in W_K (resp., $\mathcal{O}_K^3$).

Theorem (4.2) (Kumar-Sahoo)

Let K be a totally real field satisfying (ES) with $\mathrm{Cl}_{S'_K}(K)[2] = 1$.

Main results.

Theorem (4.1) (Kumar-Sahoo)

Let K be a totally real field with $\mathrm{Cl}_{S'_K}(K)[2] = 1$. Suppose for every solution (α, β, γ) to the equation

$$\alpha + \beta = \gamma^2, \text{ where } \alpha, \beta \in \mathcal{O}^*_{S'_K},\ \gamma \in \mathcal{O}_{S'_K}. \tag{4.2}$$

there exists $\mathfrak{P} \in S_K$ that satisfies

$$\left| v_{\mathfrak{P}}\left(\alpha\beta^{-1}\right)\right| \leq 6 v_{\mathfrak{P}}(2). \tag{4.3}$$

Then, the equation $Bx^p + Cy^p = z^2$ (resp., $Bx^p + 2^r y^p = 2z^2$) has no asymptotic solution in W_K (resp., $\mathcal{O}^3_K$).

Theorem (4.2) (Kumar-Sahoo)

Let K be a totally real field satisfying (ES) with $\mathrm{Cl}_{S'_K}(K)[2] = 1$. Suppose for every solution (α, β, γ) to the equation (4.2), there exists $\mathfrak{P} \in U_K$ that satisfies

$$\left| v_{\mathfrak{P}}(\alpha\beta^{-1})\right| \leq 6 v_{\mathfrak{P}}(2) \text{ and } v_{\mathfrak{P}}\left(\alpha\beta^{-1}\right) \equiv 0 \pmod 3. \tag{4.4}$$

Main results.

Theorem (4.1) (Kumar-Sahoo)

Let K be a totally real field with $\mathrm{Cl}_{S'_K}(K)[2] = 1$. Suppose for every solution (α, β, γ) to the equation

$$\alpha + \beta = \gamma^2, \text{ where } \alpha, \beta \in \mathcal{O}^*_{S'_K},\ \gamma \in \mathcal{O}_{S'_K}. \tag{4.2}$$

there exists $\mathfrak{P} \in S_K$ that satisfies

$$\left| v_{\mathfrak{P}} \left(\alpha\beta^{-1} \right) \right| \leq 6 v_{\mathfrak{P}}(2). \tag{4.3}$$

Then, the equation $Bx^p + Cy^p = z^2$ (resp., $Bx^p + 2^r y^p = 2z^2$) has no asymptotic solution in W_K (resp., $\mathcal{O}_K^3$).

Theorem (4.2) (Kumar-Sahoo)

Let K be a totally real field satisfying (ES) with $\mathrm{Cl}_{S'_K}(K)[2] = 1$. Suppose for every solution (α, β, γ) to the equation (4.2), there exists $\mathfrak{P} \in U_K$ that satisfies

$$\left| v_{\mathfrak{P}}(\alpha\beta^{-1}) \right| \leq 6 v_{\mathfrak{P}}(2) \text{ and } v_{\mathfrak{P}} \left(\alpha\beta^{-1} \right) \equiv 0 \pmod 3. \tag{4.4}$$

Then the equation $Bx^p + 2^r y^p = z^2$ with $r \in \{1, 2, 4, 5\}$ has no asymptotic solution in $\mathcal{O}_K^3 \setminus S_r$.

Main results.

Theorem (4.1) (Kumar-Sahoo)

Let K be a totally real field with $\mathrm{Cl}_{S'_K}(K)[2] = 1$. Suppose for every solution (α, β, γ) to the equation

$$\alpha + \beta = \gamma^2, \text{ where } \alpha, \beta \in \mathcal{O}^*_{S'_K}, \ \gamma \in \mathcal{O}_{S'_K}. \tag{4.2}$$

there exists $\mathfrak{P} \in S_K$ that satisfies

$$\left| v_{\mathfrak{P}} \left(\alpha\beta^{-1} \right) \right| \leq 6 v_{\mathfrak{P}}(2). \tag{4.3}$$

Then, the equation $Bx^p + Cy^p = z^2$ (resp., $Bx^p + 2^r y^p = 2z^2$) has no asymptotic solution in W_K (resp., $\mathcal{O}_K^3$).

Theorem (4.2) (Kumar-Sahoo)

Let K be a totally real field satisfying (ES) with $\mathrm{Cl}_{S'_K}(K)[2] = 1$. Suppose for every solution (α, β, γ) to the equation (4.2), there exists $\mathfrak{P} \in U_K$ that satisfies

$$\left| v_{\mathfrak{P}}(\alpha\beta^{-1}) \right| \leq 6 v_{\mathfrak{P}}(2) \text{ and } v_{\mathfrak{P}} \left(\alpha\beta^{-1} \right) \equiv 0 \pmod 3. \tag{4.4}$$

Then the equation $Bx^p + 2^r y^p = z^2$ with $r \in \{1, 2, 4, 5\}$ has no asymptotic solution in $\mathcal{O}_K^3 \setminus S_r$. In particular, if $[K : \mathbb{Q}]$ is odd, then $S_r = \phi$ for $r = 2, 4, 5$.

Methodology.

Here, we will explain the methodology that is common in the proof of Theorem (4.1) and Theorem (4.2).

Methodology.

Here, we will explain the methodology that is common in the proof of Theorem (4.1) and Theorem (4.2).

- For any solution $(a, b, c) \in \mathcal{O}_K^3$ to the equation $Bx^p + Cy^p = z^2$,

Methodology.

Here, we will explain the methodology that is common in the proof of Theorem (4.1) and Theorem (4.2).

- For any solution $(a, b, c) \in \mathcal{O}_K^3$ to the equation $Bx^p + Cy^p = z^2$, consider the Frey elliptic curve $E := E_{a,b,c}$ as

$$E : Y^2 = X(X^2 + 2aX + Bb^p), \tag{4.5}$$

with $c_4 = 2^4(Bb^p + 4Cc^p)$, $\Delta_E = 2^6(B^2C)(b^2c)^p$ and $j_E = 2^6 \frac{(Bb^p+4Cc^p)^3}{B^2C(b^2c)^p}$.

Methodology.

Here, we will explain the methodology that is common in the proof of Theorem (4.1) and Theorem (4.2).

- For any solution $(a, b, c) \in \mathcal{O}_K^3$ to the equation $Bx^p + Cy^p = z^2$, consider the Frey elliptic curve $E := E_{a,b,c}$ as

$$E : Y^2 = X(X^2 + 2aX + Bb^p), \tag{4.5}$$

  with $c_4 = 2^4(Bb^p + 4Cc^p)$, $\Delta_E = 2^6(B^2C)(b^2c)^p$ and $j_E = 2^6\frac{(Bb^p+4Cc^p)^3}{B^2C(b^2c)^p}$.
- There exists a constant $A := A_{K,B,C} > 0$ (depending on K, B, C) such that for primes $p > A$, the Frey elliptic curve E/K is modular.

Methodology.

Here, we will explain the methodology that is common in the proof of Theorem (4.1) and Theorem (4.2).

- For any solution $(a, b, c) \in \mathcal{O}_K^3$ to the equation $Bx^p + Cy^p = z^2$, consider the Frey elliptic curve $E := E_{a,b,c}$ as

$$E : Y^2 = X(X^2 + 2aX + Bb^p), \tag{4.5}$$

  with $c_4 = 2^4(Bb^p + 4Cc^p)$, $\Delta_E = 2^6(B^2C)(b^2c)^p$ and $j_E = 2^6 \frac{(Bb^p+4Cc^p)^3}{B^2C(b^2c)^p}$.
- There exists a constant $A := A_{K,B,C} > 0$ (depending on K, B, C) such that for primes $p > A$, the Frey elliptic curve E/K is modular.
- For $p \gg 0$, the residual representation $\bar{\rho}_{E,p}$ is irreducible.

Methodology.

Here, we will explain the methodology that is common in the proof of Theorem (4.1) and Theorem (4.2).

- For any solution $(a, b, c) \in \mathcal{O}_K^3$ to the equation $Bx^p + Cy^p = z^2$, consider the Frey elliptic curve $E := E_{a,b,c}$ as

$$E : Y^2 = X(X^2 + 2aX + Bb^p), \tag{4.5}$$

  with $c_4 = 2^4(Bb^p + 4Cc^p)$, $\Delta_E = 2^6(B^2C)(b^2c)^p$ and $j_E = 2^6 \frac{(Bb^p+4Cc^p)^3}{B^2C(b^2c)^p}$.
- There exists a constant $A := A_{K,B,C} > 0$ (depending on K, B, C) such that for primes $p > A$, the Frey elliptic curve E/K is modular.
- For $p \gg 0$, the residual representation $\bar{\rho}_{E,p}$ is irreducible.
- The Frey elliptic curve E has semi-stable reduction away from S_K' and satisfies $p | v_{\mathfrak{q}}(\Delta_E)$ for $\mathfrak{q} \notin S_K'$.

- For any solution $(a, b, c) \in W_K$ to the equation $Bx^p + Cy^p = z^2$ with exponent $p \gg 0$, we get $p | \#\bar{\rho}_{E,p}(I_{\mathfrak{P}})$ for $\mathfrak{P} \in S_K$.

- For any solution $(a, b, c) \in W_K$ to the equation $Bx^p + Cy^p = z^2$ with exponent $p \gg 0$, we get $p | \# \bar{\rho}_{E,p}(I_{\mathfrak{P}})$ for $\mathfrak{P} \in S_K$.
- Then there exists an elliptic curve E'/K having a non-trivial 2-torsion, E' has good reduction away from S'_K such that $\bar{\rho}_{E,p} \sim \bar{\rho}_{E',p}$ and $v_{\mathfrak{P}}(j_{E'}) < 0$ for $\mathfrak{P} \in S_K$.

- For any solution $(a, b, c) \in W_K$ to the equation $Bx^p + Cy^p = z^2$ with exponent $p \gg 0$, we get $p | \# \bar{\rho}_{E,p}(I_{\mathfrak{P}})$ for $\mathfrak{P} \in S_K$.
- Then there exists an elliptic curve E'/K having a non-trivial 2-torsion, E' has good reduction away from S'_K such that $\bar{\rho}_{E,p} \sim \bar{\rho}_{E',p}$ and $v_{\mathfrak{P}}(j_{E'}) < 0$ for $\mathfrak{P} \in S_K$.
- Finally using a nice technique of Mocanu, we relate $j_{E'}$ in terms of solution $(\alpha, \beta, \gamma) \in (\mathcal{O}^*_{S'_K}, \mathcal{O}^*_{S'_K}, \mathcal{O}_{S'_K})$ to the equation $\alpha + \beta = \gamma^2$ along with the condition $\left| v_{\mathfrak{P}} \left(\alpha \beta^{-1} \right) \right| \leq 6 v_{\mathfrak{P}}(2)$ to get $v_{\mathfrak{P}}(j_{E'}) \geq 0$ for some $\mathfrak{P} \in S_K$, which is a contradiction.

- For any solution $(a,b,c) \in W_K$ to the equation $Bx^p + Cy^p = z^2$ with exponent $p \gg 0$, we get $p | \#\bar{\rho}_{E,p}(I_{\mathfrak{P}})$ for $\mathfrak{P} \in S_K$.
- Then there exists an elliptic curve E'/K having a non-trivial 2-torsion, E' has good reduction away from S'_K such that $\bar{\rho}_{E,p} \sim \bar{\rho}_{E',p}$ and $v_{\mathfrak{P}}(j_{E'}) < 0$ for $\mathfrak{P} \in S_K$.
- Finally using a nice technique of Mocanu, we relate $j_{E'}$ in terms of solution $(\alpha, \beta, \gamma) \in (\mathcal{O}^*_{S'_K}, \mathcal{O}^*_{S'_K}, \mathcal{O}_{S'_K})$ to the equation $\alpha + \beta = \gamma^2$ along with the condition $\left| v_{\mathfrak{P}}\left(\alpha\beta^{-1}\right)\right| \leq 6 v_{\mathfrak{P}}(2)$ to get $v_{\mathfrak{P}}(j_{E'}) \geq 0$ for some $\mathfrak{P} \in S_K$, which is a contradiction.
- A similar idea also works for any solution $(a,b,c) \in \mathcal{O}_K^3 \setminus S_r$ to the equation $Bx^p + 2^r y^p = z^2$ (with $r = 1,2,4,5$) but in this case we have either $p | \#\bar{\rho}_{E,p}(I_{\mathfrak{P}})$ or $3 | \#\bar{\rho}_{E,p}(I_{\mathfrak{P}})$ for $\mathfrak{P} \in U_K$.

- For any solution $(a, b, c) \in W_K$ to the equation $Bx^p + Cy^p = z^2$ with exponent $p \gg 0$, we get $p | \#\bar{\rho}_{E,p}(I_{\mathfrak{P}})$ for $\mathfrak{P} \in S_K$.
- Then there exists an elliptic curve E'/K having a non-trivial 2-torsion, E' has good reduction away from S'_K such that $\bar{\rho}_{E,p} \sim \bar{\rho}_{E',p}$ and $v_{\mathfrak{P}}(j_{E'}) < 0$ for $\mathfrak{P} \in S_K$.
- Finally using a nice technique of Mocanu, we relate $j_{E'}$ in terms of solution $(\alpha, \beta, \gamma) \in (\mathcal{O}^*_{S'_K}, \mathcal{O}^*_{S'_K}, \mathcal{O}_{S'_K})$ to the equation $\alpha + \beta = \gamma^2$ along with the condition $\left| v_{\mathfrak{P}} \left(\alpha \beta^{-1} \right) \right| \leq 6 v_{\mathfrak{P}}(2)$ to get $v_{\mathfrak{P}}(j_{E'}) \geq 0$ for some $\mathfrak{P} \in S_K$, which is a contradiction.
- A similar idea also works for any solution $(a, b, c) \in \mathcal{O}_K^3 \setminus S_r$ to the equation $Bx^p + 2^r y^p = z^2$ (with $r = 1, 2, 4, 5$) but in this case we have either $p | \#\bar{\rho}_{E,p}(I_{\mathfrak{P}})$ or $3 | \#\bar{\rho}_{E,p}(I_{\mathfrak{P}})$ for $\mathfrak{P} \in U_K$.
- Similar to $x^p + y^p = 2^r z^p$ case, we can also give local criteria of K for the solutions of $Bx^p + Cy^p = z^2$ and $Bx^p + 2^r y^p = 2z^2$.

References I

- Deconinck, Heline. On the generalized Fermat equation over totally real fields. Acta Arith. 173 (2016), no. 3, 225–237.
- Darmon, Henri; Merel, Loïc. Winding quotients and some variants of Fermat's last theorem. J. Reine Angew. Math. 490 (1997), 81–100.
- Freitas, Nuno; Siksek, Samir. The asymptotic Fermat's last theorem for five-sixths of real quadratic fields. Compos. Math. 151 (2015), no. 8, 1395–1415.
- Ivorra, Wilfrid. Sur les équations $x^p + 2^\beta y^p = z^2$ et $x^p + 2^\beta y^p = 2z^2$. (French) [[On the equations $x^p + 2^\beta y^p = z^2$ and $x^p + 2^\beta y^p = 2z^2$]] Acta Arith. 108 (2003), no. 4, 327–338.
- Işik, Erman; Kara, Yasemin; Ozman, Ekin. On ternary Diophantine equations of signature $(p, p, 2)$ over number fields. Turkish J. Math. 44 (2020), no. 4, 1197–1211.
- Kumar, Narasimha; Sahoo, Satyabrat. On the solutions of $x^p + y^p = 2^r z^p, x^p + y^p = z^2$ over totally real fields. To appear in Acta Arithmetica.
- Kumar, Narasimha; Sahoo, Satyabrat. On the solutions of $x^2 = By^p + Cz^p$ and $2x^2 = By^p + Cz^p$ over totally real fields. https://arxiv.org/abs/2301.09263.

References II

- Mocanu, Diana. Asymptotic Fermat for signatures $(p, p, 2)$ and $(p, p, 3)$ over totally real fields. Mathematika 68 (2022), no. 4, 1233–1257.
- Ribet, Kenneth A. On the equation $a^p + 2^\alpha b^p + c^p = 0$. Acta Arith. 79 (1997), no. 1, 7–16.
- Siksek, Samir. On the Diophantine equation $x^2 = y^p + 2^k z^p$. J. Théor. Nombres Bordeaux 15 (2003), no. 3, 839–846.
- Silverman, Joseph H. The arithmetic of elliptic curves. Second edition. Graduate Texts in Mathematics, 106. Springer, Dordrecht, 2009.
- Wiles, Andrew. Modular forms, elliptic curves, and Fermat's last theorem. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 243–245, Birkhäuser, Basel, 1995.