Systems of many forms with differing degrees

Simon L. Rydin Myerson

2 September 2024

Warwick Mathematics Institute

Download these slides:

maths.fan/ICTS.pdf

Diophantine equations

Systems of polynomial equations to be solved in integers. Ubiquitous in mathematics; relatively few general results.

Notation (density of solutions)

- $\vec{f}(\vec{x}) \in \mathbb{Z}[x_1, \ldots, x_s]^R$ will be a system of R homogenous forms of degrees d_i in $s > \sum d_i$ variables with integer coefficients.
- We count solutions of $\vec{f} = \vec{0}$ in integers of size B, where B is big.

- $\vec{f}$ takes about $B^{\sum d_i}$ values; maybe it is zero about $\frac{1}{B^{\sum d_i}}$ of the time.
- That would mean about $B^{s - \sum d_i}$ solutions.
- Also need to consider the number of solutions modulo m for $m \in \mathbb{N}$.

Diophantine equations

Systems of polynomial equations to be solved in integers. Ubiquitous in mathematics; relatively few general results. No completely general result possible: Matiyasevich (1970), unsolvability of Hilbert's tenth problem.

Notation (density of solutions)

- $\vec{f}(\vec{x}) \in \mathbb{Z}[x_1, \ldots, x_s]^R$ will be a system of R homogenous forms of degrees d_i in $s > \sum d_i$ variables with integer coefficients.
- We count solutions of $\vec{f} = \vec{0}$ in integers of size B, where B is big.
- $\vec{\alpha} \cdot \vec{f} = \sum^R \alpha_i f_i$ is nonzero and indefinite for all $\vec{\alpha} \in \mathbb{R}^R \setminus \{\vec{0}\}$.
- We study $N_{\vec{f}}(B) = \#\{\vec{x} \in \mathbb{Z}^s \cap [-B, B]^s \setminus \{\vec{0}\} : \vec{f}(\vec{x}) = \vec{0}\}$.

- $\vec{f}$ takes about $B^{\sum d_i}$ values; maybe it is zero about $\frac{1}{B^{\sum d_i}}$ of the time.
- That would mean about $B^{s - \sum d_i}$ solutions.
- Also need to consider the number of solutions modulo m for $m \in \mathbb{N}$.
- Let's make this rigorous.

Diophantine equations

Systems of polynomial equations to be solved in integers. Ubiquitous in mathematics; relatively few general results. No completely general result possible: Matiyasevich (1970), unsolvability of Hilbert's tenth problem.

Notation (density of solutions)

- $\vec{f}(\vec{x}) \in \mathbb{Z}[x_1, \ldots, x_s]^R$ will be a system of R homogenous forms of degrees d_i in $s > \sum d_i$ variables with integer coefficients.
- We count solutions of $\vec{f} = \vec{0}$ in integers of size B, where B is big.
- $\vec{\alpha} \cdot \vec{f} = \sum^R \alpha_i f_i$ is nonzero and indefinite for all $\vec{\alpha} \in \mathbb{R}^R \setminus \{\vec{0}\}$.
- We study $N_{\vec{f}}(B) = \#\{\vec{x} \in \mathbb{Z}^s \cap [-B, B]^s \setminus \{\vec{0}\} : \vec{f}(\vec{x}) = \vec{0}\}$.

- $\vec{f}$ takes about $B^{\sum d_i}$ values; maybe it is zero about $\frac{1}{B^{\sum d_i}}$ of the time.
- That would mean about $B^{s - \sum d_i}$ solutions.
- Also need to consider the number of solutions modulo m for $m \in \mathbb{N}$.
- Let's make this rigorous.

Notation (density of solutions)

- $\vec{f}(\vec{x}) \in \mathbb{Z}[x_1, \ldots, x_s]^R$ will be a system of R homogenous forms of degrees d_i in $s > \sum d_i$ variables with integer coefficients.
- We count solutions of $\vec{f} = \vec{0}$ in integers of size B, where B is big.
- $\vec{\alpha} \cdot \vec{f} = \sum^R \alpha_i f_i$ is nonzero and indefinite for all $\vec{\alpha} \in \mathbb{R}^R \setminus \{\vec{0}\}$.
- We study $N_{\vec{f}}(B) = \#\{\vec{x} \in \mathbb{Z}^s \cap [-B, B]^s \setminus \{\vec{0}\} : \vec{f} = \vec{0}\} \leq (2B+1)^s$.

Heuristics

- Model $\vec{x}$ by a random real vector $\vec{X}$, and model $f_i(\vec{x})$ by $\lfloor f_i(\vec{X}) \rfloor$.
- That is, let $\vec{X}$ be a uniform random variable on $[-B, B]^s$. Maybe $N_{\vec{f}}(B) \asymp (2B)^s \cdot \mathbb{P}[\vec{f}(\vec{X}) \in [0,1)^R]$, which is typically $\sim \nu_{\vec{f}} B^{s - \sum d_i}$.

Notation (density of solutions)

- $\vec{f}(\vec{x}) \in \mathbb{Z}[x_1, \ldots, x_s]^R$ will be a system of R homogenous forms of degrees d_i in $s > \sum d_i$ variables with integer coefficients.
- We count solutions of $\vec{f} = \vec{0}$ in integers of size B, where B is big.
- $\vec{\alpha} \cdot \vec{f} = \sum^R \alpha_i f_i$ is nonzero and indefinite for all $\vec{\alpha} \in \mathbb{R}^R \setminus \{\vec{0}\}$.
- We study $N_{\vec{f}}(B) = \#\{\vec{x} \in \mathbb{Z}^s \cap [-B, B]^s \setminus \{\vec{0}\} : \vec{f} = \vec{0}\} \leq (2B+1)^s$.

Heuristics

- Model $\vec{x}$ by a random real vector $\vec{X}$, and model $f_i(\vec{x})$ by $\lfloor f_i(\vec{X}) \rfloor$.
- That is, let $\vec{X}$ be a uniform random variable on $[-B, B]^s$. Maybe $N_{\vec{f}}(B) \asymp (2B)^s \cdot \mathbb{P}[\vec{f}(\vec{X}) \in [0,1)^R]$, which is typically $\sim \nu_{\vec{f}} B^{s - \sum d_i}$.
- But: if $f(\vec{x}) = x_1^2 + x_2^2 - 3x_3^2$, then $N_f(B) = 0$ as $\vec{x} = \vec{0}$ "mod 2^∞".
- Fix: let $\vec{X}_p$ be uniformly distributed on $\mathbb{Z}_p^s$. Predict

$$N_{\vec{f}}(B) = (1 + o(1))(2B)^s \mathbb{P}[\vec{f}(\vec{X}) \in [0,1)^R] \cdot \prod_p \lim_{N \to \infty} p^{NR} \mathbb{P}[p^N \mid \vec{f}(\vec{X}_p)].$$

Density of solutions

- Let $\vec{X}$ be a uniform random variable on $[-B, B]^s$. Maybe $N_{\vec{f}}(B) \asymp (2B)^s \cdot \mathbb{P}[\vec{f}(\vec{X}) \in [0,1)^R]$, which is typically $\sim \nu_{\vec{f}} B^{s-\sum d_i}$.
- But: if $R = 1$, $f(\vec{x}) = x_1^2 + x_2^2 - 3x_3^2$, then $N_{\vec{f}}(B) = 0$.
- Fix: let $\vec{X}_p$ be uniformly distributed on $\mathbb{Z}_p^s$. Perhaps

$$N_{\vec{f}}(B) = (1 + o(1))(2B)^s \cdot \mathbb{P}[\vec{f}(\vec{X}) \in [0,1)^R] \prod_p \lim_{N \to \infty} p^{NR} \mathbb{P}[p^N \mid \vec{f}(\vec{X}_p)].$$

Notation

- $\vec{f}(\vec{x}) \in \mathbb{Z}[x_1, \ldots, x_s]^R$ is a system of R forms in $s > \sum d_i$ variables with integer coefficients, and $N_{\vec{f}}(B) = \sum_{\vec{x} \in \mathbb{Z}^s \cap [-B,B]^s \setminus \{\vec{0}\}, \vec{f}(\vec{x}) = \vec{0}} 1$.
- We say $a = O(b)$ iff $a \ll b$ iff $|a| < Cb$ for some constant C. Also write $a \asymp b$ iff $a \ll b \ll a$. And put $a \sim b$ iff $a/b \to 1$ as $B \to \infty$. And put $a = o(b)$ iff $a/b \to 0$ as $B \to \infty$.

- Let $\vec{X}$ be a uniform random variable on $[-B,B]^s$. Maybe $N_{\vec{f}}(B) \asymp (2B)^s \cdot \mathbb{P}[\vec{f}(\vec{X}) \in [0,1)^R]$, which is typically $\sim \nu_{\vec{f}} B^{s-\sum d_i}$.
- But: if $R=1$, $f(\vec{x}) = x_1^2 + x_2^2 - 3x_3^2$, then $N_{\vec{f}}(B) = 0$.
- Fix: let $\vec{X}_p$ be uniformly distributed on $\mathbb{Z}_p^s$. Perhaps

$$N_{\vec{f}}(B) = (1+o(1))(2B)^s \cdot \mathbb{P}[\vec{f}(\vec{X}) \in [0,1)^R] \prod_p \lim_{N\to\infty} p^{NR}\mathbb{P}[p^N \mid \vec{f}(\vec{X}_p)].$$

- This is the *analytic Hasse principle*; the *Manin-Peyre conjecture* is a more sophisticated version needed for $s \le 2\sum d_i$ or $\vec{f}$ singular.

Notation

- $\vec{f}(\vec{x}) \in \mathbb{Z}[x_1,\ldots,x_s]^R$ is a system of R forms in $s > \sum d_i$ variables with integer coefficients, and $N_{\vec{f}}(B) = \sum_{\vec{x}\in\mathbb{Z}^s\cap[-B,B]^s\setminus\{\vec{0}\},\vec{f}(\vec{x})=\vec{0}} 1$.
- We say $a = O(b)$ iff $a \ll b$ iff $|a| < Cb$ for some constant C. Also write $a \asymp b$ iff $a \ll b \ll a$. And put $a \sim b$ iff $a/b \to 1$ as $B \to \infty$. And put $a = o(b)$ iff $a/b \to 0$ as $B \to \infty$.

- Let $\vec{X}$ be a uniform random variable on $[-B,B]^s$. Maybe $N_{\vec{f}}(B) \asymp (2B)^s \cdot \mathbb{P}[\vec{f}(\vec{X}) \in [0,1)^R]$, which is typically $\sim \nu_{\vec{f}} B^{s-\sum d_i}$.
- But: if $R=1$, $f(\vec{x}) = x_1^2 + x_2^2 - 3x_3^2$, then $N_{\vec{f}}(B) = 0$.
- Fix: let $\vec{X}_p$ be uniformly distributed on $\mathbb{Z}_p^s$. Perhaps

$$N_{\vec{f}}(B) = (1+o(1))(2B)^s \cdot \mathbb{P}[\vec{f}(\vec{X}) \in [0,1)^R] \prod_p \lim_{N\to\infty} p^{NR}\mathbb{P}[p^N \mid \vec{f}(\vec{X}_p)].$$

- This is the *analytic Hasse principle*; the *Manin-Peyre conjecture* is a more sophisticated version needed for $s \leq 2\sum d_i$ or $\vec{f}$ singular.

Notation

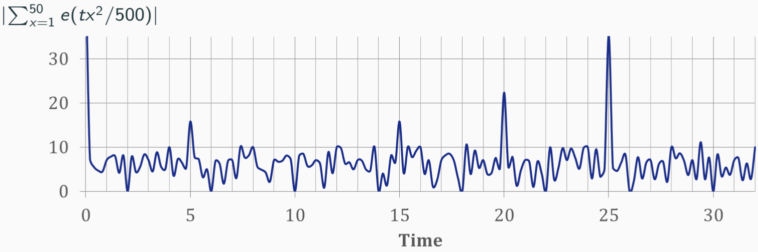
- $\vec{f}(\vec{x}) \in \mathbb{Z}[x_1,\ldots,x_s]^R$ is a system of R forms in $s > \sum d_i$ variables with integer coefficients, and $N_{\vec{f}}(B) = \sum_{\vec{x}\in\mathbb{Z}^s\cap[-B,B]^s\setminus\{\vec{0}\},\vec{f}(\vec{x})=\vec{0}} 1$.
- We say $a = O(b)$ iff $a \ll b$ iff $|a| < Cb$ for some constant C. Also write $a \asymp b$ iff $a \ll b \ll a$. And put $a \sim b$ iff $a/b \to 1$ as $B \to \infty$. And put $a = o(b)$ iff $a/b \to 0$ as $B \to \infty$.

The circle method

Notation

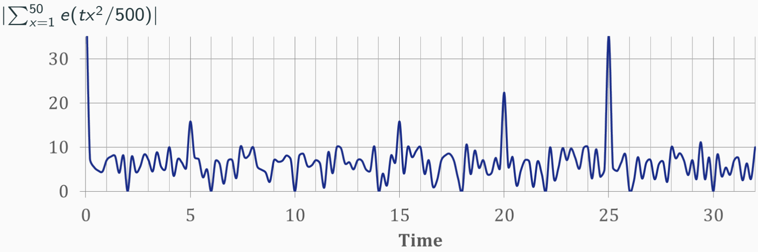
- $\vec{f}(\vec{x}) \in \mathbb{Z}[\vec{x}]^R$ is a system of R forms in s variables of degrees $d_i \geq 2$.
- Set $N_{\vec{f}}(B) := \#\{\vec{x} \in \mathbb{Z}^s : \vec{f}(\vec{x}) = \vec{0}, \ \|\vec{x}\| \leq B\}$, $\vec{\alpha} \cdot \vec{f} = \sum^R \alpha_i f_i$.

$$N_{\vec{f}}(B) = \int_0^1 \cdots \int_0^1 \sum_{\substack{\vec{x} \in \mathbb{Z}^s \\ \|\vec{x}\| \leq B}} e^{2\pi i \vec{t} \cdot \vec{f}(\vec{x})} \, d\vec{t}$$



The circle method

$$N_{\vec{f}}(B) = \int_0^1 \cdots \int_0^1 \sum_{\substack{\vec{x}\in\mathbb{Z}^s \\ \|\vec{x}\|\leq B}} e^{2\pi i \vec{t}\cdot\vec{f}(\vec{x})}\, d\vec{t}$$



Some peaks bigger than 12 or so. Random noise ≤ 12.

Repulsion: pick points $t, t+\beta$. If both are at peaks, $|\beta| < 1$ or $|\beta| > 4$.

So each peak has width 1, and they are at least 4 apart. Consequently the measure of t lying on peaks is small.

Notation

- $\vec{f}(\vec{x}) \in \mathbb{Z}[\vec{x}]^R$ is a system of R forms in s variables of degrees $d_i \geq 2$.
- We set $N_{\vec{f}}(B) := \#\{\vec{x} \in \mathbb{Z}^s : \vec{f}(\vec{x}) = \vec{0},\ \|\vec{x}\| \leq B\}$.
- $\vec{f}$ is smooth if $\vec{f} = 0$ defines a smooth $s - R$ dimensional complex manifold in $\mathbb{C}^s \setminus \{\vec{0}\}$(away from the origin).

Theorem (Birch 1962)

We have $N_{\vec{f}}(B) \sim c_{\vec{f}} B^{s-dR}$ as above if $\vec{f}$ smooth, $d_1 = \cdots = d_R = d$,

$$s \geq (d-1)2^{d-1}R(R+1) + R.$$

Notation

- $\vec{f}(\vec{x}) \in \mathbb{Z}[\vec{x}]^R$ is a system of R forms in s variables of degrees $d_i \geq 2$.
- We set $N_{\vec{f}}(B) := \#\{\vec{x} \in \mathbb{Z}^s : \vec{f}(\vec{x}) = \vec{0}, \|\vec{x}\| \leq B\}$.
- $\vec{f}$ is smooth if $\vec{f} = 0$ defines a smooth $s - R$ dimensional complex manifold in $\mathbb{C}^s \setminus \{\vec{0}\}$(away from the origin).

Theorem (Birch 1962)

We have $N_{\vec{f}}(B) \sim c_{\vec{f}} B^{s-dR}$ as above if $\vec{f}$ smooth, $d_1 = \cdots = d_R = d$,

$$s \geq (d-1)2^{d-1}R(R+1) + R.$$

Hope for $s > 2dR$. Much work on the range for s if $R = 1$. For $R \geq 2$:

- $(d, R, s) = (2, 2, 11)$ by Munshi (2015) - $s = 10$, Li-RM-Vishe, soon!
- $d = 2, s \geq 9R$, RM (2018);
- $d = 3, s \geq 25R$, RM (2019);
- $(d, R, s) = (3, 2, 39)$ by Northey and Vishe (2024).

The Birch-Davenport circle method

Notation

- $\vec{f}(\vec{x}) \in \mathbb{Z}[\vec{x}]^R$ is a system of R forms in s variables of degrees $d_i \geq 2$.
- We set $N_{\vec{f}}(B) := \#\{\vec{x} \in \mathbb{Z}^s : \vec{f}(\vec{x}) = \vec{0},\ \|\vec{x}\| \leq B\}$.
- smooth if $\vec{f} = 0$ defines an $s - R$ dimensional manifold in $\mathbb{C}^s \setminus \{\vec{0}\}$.

Theorem (Birch 1962)

We have $N_{\vec{f}}(B) \sim c_{\vec{f}} B^{s-dR}$ as above if $\vec{f}$ smooth, $d_1 = \cdots = d_R = d$,

$$s \geq (d-1)2^{d-1}R(R+1) + R.$$

Hope for $s > 2dR$. Much work on the range for s if $R = 1$. For $R \geq 2$:

- $(d, R, s) = (2, 2, 11)$ by Munshi (2015) - $s = 10$, Li-RM-Vishe, soon!
- $d = 2, s \geq 9R$, RM (2018);
- $d = 3, s \geq 25R$, RM (2019);
- $(d, R, s) = (3, 2, 39)$ by Northey and Vishe (2024).

For certain $\vec{f}$: $(d, R, s) = (2, 2, 10)$ Heath-Brown–Pierce 2015; $(2, 3, 20)$ Pierce-Schindler-Wood 2016; $(2, R, 6R)$ Browning-Pierce-Schindler 2024.

The circle method: differing degrees

Notation

- $\vec{f}(\vec{x}) \in \mathbb{Z}[x_1, \ldots, x_s]^R$ is a system of R forms in $s > \sum d_i$ variables with integer coefficients, and $N_{\vec{f}}(B) = \sum_{\vec{x} \in \mathbb{Z}^s \cap [-B,B]^s \setminus \{\vec{0}\}, \vec{f}(\vec{x}) = \vec{0}} 1$.

Theorem (Birch 1962)

We have $N_{\vec{f}}(B) \sim c_{\vec{f}} B^{s-dR}$ as above if $\vec{f}$ smooth, $d_1 = \cdots = d_R = d$,

$$s \geq (d-1)2^{d-1}R(R+1) + R.$$

Generalisations to unequal d_i.

- Browning–Dietmann–Heath-Brown (2014), $d_1 = 2, d_2 = 3$, $s \geq 29$.
- If $d_1, \ldots, d_R \leq d$, Browning–Heath-Brown (2017) handle around $d^3 R^2 2^d$ variables, improving on Schmidt (1985).

The circle method: differing degrees

Notation

- $\vec{f}(\vec{x}) \in \mathbb{Z}[x_1, \ldots, x_s]^R$ is a system of R forms in $s > \sum d_i$ variables with integer coefficients, and $N_{\vec{f}}(B) = \sum_{\vec{x} \in \mathbb{Z}^s \cap [-B,B]^s \setminus \{\vec{0}\}, \vec{f}(\vec{x}) = \vec{0}} 1$.

Theorem (Birch 1962)

We have $N_{\vec{f}}(B) \sim c_{\vec{f}} B^{s-dR}$ as above if $\vec{f}$ smooth, $d_1 = \cdots = d_R = d$,

$$s \geq (d-1)2^{d-1}R(R+1) + R.$$

Generalisations to unequal d_i.

- Browning–Dietmann–Heath-Brown (2014), $d_1 = 2, d_2 = 3$, $s \geq 29$.
- If $d_1, \ldots, d_R \leq d$, Browning–Heath-Brown (2017) handle around $d^3 R^2 2^d$ variables, improving on Schmidt (1985).

Theorem (RM 2024)

$N_{\vec{f}}(B) \sim c_{\vec{f}} B^{s-dR}$ if $\vec{f}$ smooth and $s \geq R + \sum_{i=1}^{R} d_i 2^{d_i} 3^{2d(d-d_i)}$.

Typically this is about $3^{2d^2} R$ variables; when $d_1 = \cdots = d_R = d$ it is $(1 + d2^d)R$ variables. Improve both Birch's result and, in big R, BHB.

The circle method: differing degrees

Notation

- $\vec{f}(\vec{x}) \in \mathbb{Z}[x_1, \ldots, x_s]^R$ is a system of R forms in $s > \sum d_i$ variables with integer coefficients, and $N_{\vec{f}}(B) = \sum_{\vec{x} \in \mathbb{Z}^s \cap [-B,B]^s \setminus \{\vec{0}\}, \vec{f}(\vec{x}) = \vec{0}} 1$.

Theorem (Birch 1962)

We have $N_{\vec{f}}(B) \sim c_{\vec{f}} B^{s-dR}$ as above if $\vec{f}$ smooth, $d_1 = \cdots = d_R = d$,

$$s \geq (d-1)2^{d-1}R(R+1) + R.$$

Generalisations to unequal d_i.

- Browning–Dietmann–Heath-Brown (2014), $d_1 = 2, d_2 = 3$, $s \geq 29$.
- If $d_1, \ldots, d_R \leq d$, Browning–Heath-Brown (2017) handle around $d^3 R^2 2^d$ variables, improving on Schmidt (1985).

Theorem (RM 2024)

$N_{\vec{f}}(B) \sim c_{\vec{f}} B^{s-dR}$ *if $\vec{f}$ smooth and* $s \geq R + \sum_{i=1}^{R} d_i 2^{d_i} 3^{2d(d-d_i)}$.

Typically this is about $3^{2d^2} R$ variables; when $d_1 = \cdots = d_R = d$ it is $(1 + d2^d)R$ variables. Improve both Birch's result and, in big R, BHB.

The circle method: differing degrees

Notation

- $\vec{f}(\vec{x}) \in \mathbb{Z}[x_1, \dots, x_s]^R$ is a system of R forms in $s > \sum d_i$ variables with integer coefficients, and $N_{\vec{f}}(B) = \sum_{\vec{x} \in \mathbb{Z}^s \cap [-B,B]^s \setminus \{\vec{0}\}, \vec{f}(\vec{x}) = \vec{0}} 1$.

Theorem (Birch 1962)

We have $N_{\vec{f}}(B) \sim c_{\vec{f}} B^{s-dR}$ as above if $\vec{f}$ smooth, $d_1 = \dots = d_R = d$,

$$s \geq (d-1)2^{d-1}R(R+1) + R.$$

- If $d_1, \dots, d_R \leq d$, Browning–Heath-Brown (2017) handle around $d^3 R^2 2^d$ variables, improving on Schmidt (1985).

Theorem (RM 2024)

$N_{\vec{f}}(B) \sim c_{\vec{f}} B^{s-dR}$ *if $\vec{f}$ smooth and* $s \geq R + \sum_{i=1}^{R} d_i 2^{d_i} 3^{2d(d-d_i)}$.

Typically this is about $3^{2d^2} R$ variables; when $d_1 = \dots = d_R = d$ it is $(1 + d2^d)R$ variables. Improve both Birch's result and, in big R, BHB.

$d_1 = 2, d_3 = 3, s \geq 5858$, worse than BDHB $s \geq 29$! Key ideas: p-adic repulsion and a way to extract lower-order terms in exponential sums.

Notation

- $\vec{g}(\vec{x}) \in \mathbb{R}[\vec{x}]^R$ is a system of R forms in s variables of degrees $d_i \geq 2$.
- The function $M(B) := \#\{\vec{x} \in \mathbb{Z}^s \setminus \{\vec{0}\} : \|\vec{g}(\vec{x})\| \leq 1, \|\vec{x}\| \leq B\}$.
- $\vec{g}$ is nonsingular if the $R \times s$ Jacobian matrix $(\partial g_i(\vec{x})/\partial x_j)_{ij}$ has rank R at every nontrivial complex solution $\vec{x}$ to $\vec{g}(\vec{x}) = \vec{0}$.

- If $\vec{g}$ is nonsingular and the number of variables s is very large, can we estimate $M(B)$?
- In the case $R = 1$, $d = 2$ the breakthrough work of Margulis, which introduced ideas from ergodic theory, led to:

Notation

- $\vec{g}(\vec{x}) \in \mathbb{R}[\vec{x}]^R$ is a system of R forms in s variables of degrees $d_i \geq 2$.
- The function $M(B) := \#\{\vec{x} \in \mathbb{Z}^s \setminus \{\vec{0}\} : \|\vec{g}(\vec{x})\| \leq 1, \|\vec{x}\| \leq B\}$.
- $\vec{g}$ is nonsingular if the $R \times s$ Jacobian matrix $(\partial g_i(\vec{x})/\partial x_j)_{ij}$ has rank R at every nontrivial complex solution $\vec{x}$ to $\vec{g}(\vec{x}) = \vec{0}$.

- If $\vec{g}$ is nonsingular and the number of variables s is very large, can we estimate $M(B)$?
- In the case $R = 1$, $d = 2$ the breakthrough work of Margulis, which introduced ideas from ergodic theory, led to:

Theorem (Eskin, Margulis and Mozes 1998)

Let g be a single quadratic form. Suppose g is nonsingular, and not a multiple of a form with rational coefficients.

If $s \geq 5$, then $M(B) \sim \nu_g B^{s-2}$ as $B \to \infty$ for some $\nu \geq 0$.

If $\vec{X}$ is a uniform RV on $[-B, B]^s$, then $\nu_g B^{s-2} \sim B^s \mathbb{P}[|g(\vec{X})| \leq 1]$.

Forms with real coefficients

Notation

- $\vec{g}(\vec{x}) \in \mathbb{R}[\vec{x}]^R$ is a system of R forms in s variables of degrees $d_i \geq 2$.
- The function $M(B) := \#\{\vec{x} \in \mathbb{Z}^s \setminus \{\vec{0}\} : \|\vec{g}(\vec{x})\| \leq 1, \|\vec{x}\| \leq B\}$.
- $\vec{g}$ is nonsingular if the $R \times s$ Jacobian matrix $(\partial g_i(\vec{x})/\partial x_j)_{ij}$ has rank R at every nontrivial complex solution $\vec{x}$ to $\vec{g}(\vec{x}) = \vec{0}$.

- If $\vec{g}$ is nonsingular and the number of variables s is very large, can we estimate $M(B)$?
- In the case $R = 1$, $d = 2$ the breakthrough work of Margulis, which introduced ideas from ergodic theory, led to:

Theorem (Eskin, Margulis and Mozes 1998)

Let g be a single quadratic form. Suppose g is nonsingular, and not a multiple of a form with rational coefficients.

If $s \geq 5$, then $M(B) \sim \nu_g B^{s-2}$ as $B \to \infty$ for some $\nu \geq 0$.

If $\vec{X}$ is a uniform RV on $[-B, B]^s$, then $\nu_g B^{s-2} \sim B^s \mathbb{P}[|g(\vec{X})| \leq 1]$.

Notation

- $\vec{g}(\vec{x}) \in \mathbb{R}[\vec{x}]^R$ is a system of R forms in s variables of degrees $d_i \geq 2$.
- The function $M(B) := \#\{\vec{x} \in \mathbb{Z}^s \setminus \{\vec{0}\} : \|\vec{g}(\vec{x})\| \leq 1,\ \|\vec{x}\| \leq B\}$.
- $\vec{g}$ is nonsingular if the $R \times s$ Jacobian matrix $(\partial g_i(\vec{x})/\partial x_j)_{ij}$ has rank R at every nontrivial complex solution $\vec{x}$ to $\vec{g}(\vec{x}) = \vec{0}$.
- $\vec{g}(\vec{x})$ is irrational if no $\vec{\alpha} \in \mathbb{R}^R \setminus \{\vec{0}\}$ satisfies $\vec{\alpha} \cdot \vec{g}(\vec{x}) \in \mathbb{Z}[\vec{x}]$.

- Bentkus and Götze (1999) used the circle method to give a new proof of EMM's result $M(B) \sim \nu_g B^{s-2}$ when $s \geq 9$.

Quadratic forms with real coefficients

Notation

- $\vec{g}(\vec{x}) \in \mathbb{R}[\vec{x}]^R$ is a system of R forms in s variables of degrees $d_i \geq 2$.
- The function $M(B) := \#\{\vec{x} \in \mathbb{Z}^s \setminus \{\vec{0}\} : \|\vec{g}(\vec{x})\| \leq 1, \|\vec{x}\| \leq B\}$.
- $\vec{g}$ is nonsingular if the $R \times s$ Jacobian matrix $(\partial g_i(\vec{x})/\partial x_j)_{ij}$ has rank R at every nontrivial complex solution $\vec{x}$ to $\vec{g}(\vec{x}) = \vec{0}$.
- $\vec{g}(\vec{x})$ is irrational if no $\vec{\alpha} \in \mathbb{R}^R \setminus \{\vec{0}\}$ satisfies $\vec{\alpha} \cdot \vec{g}(\vec{x}) \in \mathbb{Z}[\vec{x}]$.

- Bentkus and Götze (1999) used the circle method to give a new proof of EMM's result $M(B) \sim \nu_g B^{s-2}$ when $s \geq 9$.

Theorem (Müller, 2008, slightly rephrased)

If $d_1 = \cdots = d_R = 2$, $\vec{g}$ is nonsingular and irrational, and $s \geq 9R$, then we have $M(B) \sim \nu_{\vec{g}} B^{s-2}$ as $P \to \infty$ for some $\nu \geq 0$.

- Freeman (2000, 2001): Standardised this form of the circle method.
- Buterus-Götze-Hille-Margulis (2022): EMM's $R = 1$, $s \geq 5$, explicit, by circle method! Exp sums $\ll$ nice functions on 1-param groups

Notation

- $\vec{g}(\vec{x}) \in \mathbb{R}[\vec{x}]^R$ is a system of R forms in s variables of degrees $d_i \geq 2$.
- The function $M(B) := \#\{\vec{x} \in \mathbb{Z}^s \setminus \{\vec{0}\} : \|\vec{g}(\vec{x})\| \leq 1, \|\vec{x}\| \leq B\}$.
- $\vec{g}(\vec{x})$ is irrational if no $\vec{\alpha} \in \mathbb{R}^R \setminus \{\vec{0}\}$ satisfies $\vec{\alpha} \cdot \vec{g}(\vec{x}) \in \mathbb{Z}[\vec{x}]$.

- Diagonal case, $d_i = d$, indefinite: $M(B) > 0$ for $R = 1$ and $s \gg cd \log d$ by Davenport-Heilbronn-Roth-. . . ; $M(B) > 0$ for $s \geq R\lceil Rd^2 \log 3Rd\rceil$ by Nadesalingam-Pitman (1989).
- Schmidt (1980) gave a lower bound for $M(B)$ when s is (very) large.
- Cubic case: Freeman (2004) got $M(B) > 0$ for $s > (10R)^{(10R)^5}$. Chow (2014) took $R = 1$ and $s \geq 358\,823\,708$.
- Asymptotics? Diagonal case: Freeman ('03), Wooley ('03).
- Ergodic (EMM-like) methods? Special cases only, see overview of Yukie (arxiv:9710214).

Diophantine inequalities: $d > 2$

- Diagonal case, $d_i = d$, indefinite: $M(B) > 0$ for $R = 1$ and $s \gg cd \log d$ by Davenport-Heilbronn-Roth-...; $M(B) > 0$ for $s \geq R\lceil Rd^2 \log 3Rd \rceil$ by Nadesalingam-Pitman (1989).
- Schmidt (1980) gave a lower bound for $M(B)$ when s is (very) large.
- Cubic case: Freeman (2004) got $M(B) > 0$ for $s > (10R)^{(10R)^5}$. Chow (2014) took $R = 1$ and $s \geq 358\,823\,708$.
- Asymptotics? Diagonal case: Freeman ('03), Wooley ('03).
- Ergodic (EMM-like) methods? Special cases only, see overview of Yukie (arxiv:9710214).

Theorem (RM 2024)

Let $d_i \leq d$. Suppose that $\vec{g}$ is nonsingular and irrational, and that

$$s \geq R + \sum_{i=1}^{R} d_i \max\{d_i - 2, 1\} 2^{d_i} 3^{2d(d-d_i)}.$$

Then $M(B) \sim \nu_{\vec{g}} B^{n-\sum d_i}$ where $\nu_{\vec{g}} = \lim B^{\sum d_i} \mathbb{P}[\vec{g}(\vec{X}) \in [0,1)^R]$.

Notation

- $\vec{g}(\vec{x}) \in \mathbb{R}[\vec{x}]^R$ is a system of R forms in s variables of degrees $d_i \geq 2$.
- The function $M(B) := \#\{\vec{x} \in \mathbb{Z}^s \setminus \{\vec{0}\} : \|\vec{g}(\vec{x})\| \leq 1, \|\vec{x}\| \leq B\}$.
- $\vec{g}(\vec{x})$ is irrational if no $\vec{\alpha} \in \mathbb{R}^R \setminus \{\vec{0}\}$ satisfies $\vec{\alpha} \cdot \vec{g}(\vec{x}) \in \mathbb{Z}[\vec{x}]$.

Theorem (RM 2024)

Let $d_i \leq d$. Suppose that $\vec{g}$ is nonsingular and irrational, and that

$$s \geq R + \sum_{i=1}^{R} d_i \max\{d_i - 2, 1\} 2^{d_i} 3^{2d(d-d_i)}.$$

Then $M(B) \sim \nu_{\vec{g}} B^{n - \sum d_i}$ where $\nu_{\vec{g}} = \lim B^{\sum d_i} \mathbb{P}[\vec{g}(\vec{X}) \in [0,1)^R]$.

Setup for repulsion

- $e(t) = 2^{2\pi i t}$, $\Delta_{\vec{h}} f(\vec{x}) = f(\vec{x}+\vec{h}) - f(\vec{x})$, $\Delta_{\vec{h}_1,\ldots,\vec{h}_r} = \Delta_{\vec{h}_1}\cdots\Delta_{\vec{h}_r}$
- $f^{[k]}$ is the degree k part of f, with $|f^{[k]}| = |\text{biggest coefficient}|$.
- *NB* $\Delta_{\vec{h}_1,\ldots,\vec{h}_k} f^{[k]}$ *is a multilinear form in the* $\vec{h}_i$, *independent of* $\vec{x}$
- $\vec{m}^{f,k}_{\vec{h}_1,\ldots,\vec{h}_{k-1}}$ is the vector of coefficients of $\Delta_{\vec{h}_1,\ldots,\vec{h}_{k-1},(\,\cdot\,)} f^{[k]}$

For $w \in C_c^\infty(\mathbb{R}^s)$, $f, g \in \mathbb{R}[\vec{x}]$ with degree $\le k$, and $1 \le P \le B$ we have

$$\Big|\sum_{\vec{x}\in\mathbb{Z}^s} B^{-s} w(\vec{x}/B) e\left(g(\vec{x})\right) \sum_{\vec{x}\in\mathbb{Z}^s} B^{-s} w(\vec{x}/B) e\left(g(\vec{x}) + f(\vec{x})\right)\Big|^{2^{k-1}}$$

$$\le \Big|\sum_{\vec{h}\in\mathbb{Z}^s}\sum_{\vec{x}\in\mathbb{Z}^s} B^{-2s} w(\vec{x}+\vec{h}/B)\overline{w}(\vec{x}/B) e\left(f(\vec{x}) - \Delta_{\vec{h}} g(\vec{x})\right)\Big|^{2^{k-1}}$$

$$\ll B^{-(k-1)s} \#\{(\vec{h}_1,\ldots,\vec{h}_{k-1},\vec{u}) \in \mathbb{Z}^{ks} : |\vec{h}_i| \le B, |\vec{m}^{f,k}_{\vec{h}_1,\ldots,\vec{h}_{k-1}} - \vec{u}| \le B^{-1}\}$$

$$\ll \frac{1}{P^{(k-1)s}} \#\{(\vec{h}_1,\ldots,\vec{h}_{k-1},\vec{u}) \in \mathbb{Z}^{ks} : |\vec{h}_i| \le P, |\vec{m}^{f,k}_{\vec{h}_1,\ldots,\vec{h}_{k-1}} - \vec{u}| \le \frac{P^{k-1}}{B^k}\}$$

If $PB^{-k} \le |f^{[k]}| \le P^{1-k}$, this is

$$< \frac{1}{P^{(k-1)s}} \#\{(\vec{h}_1,\ldots,\vec{h}_{k-1}) \in \mathbb{Z}^{(k-1)s} : |\vec{h}_i| \le P, |\vec{m}^{f,k}_{\vec{h}_1,\ldots,\vec{h}_{k-1}}| \le |f^{[k]}| P^{k-2}\}.$$

Setup for repulsion

- $e(t) = 2^{2\pi it}$, $\Delta_{\vec{h}} f(\vec{x}) = f(\vec{x}+\vec{h}) - f(\vec{x})$, $\Delta_{\vec{h}_1,\dots,\vec{h}_r} = \Delta_{\vec{h}_1} \cdots \Delta_{\vec{h}_r}$
- $f^{[k]}$ is the degree k part of f, with $|f^{[k]}| = |\text{biggest coefficient}|$.
- $\vec{m}^{f,k}_{\vec{h}_1,\dots,\vec{h}_{k-1}}$ is the vector of coefficients of $\Delta_{\vec{h}_1,\dots,\vec{h}_{k-1},(\,\cdot\,)} f^{[k]}$
- The *normalised* sum $S_w(B;f) = \sum_{\vec{x}\in\mathbb{Z}^s} B^{-s} w(\vec{x}/B) e\left(f(\vec{x})\right)$

For $f, g \in \mathbb{R}[\vec{x}]$ with degree $\leq k$, and $PB^{-k} \ll |f^{[k]}| \ll P^{1-k}$ we have

$$|S_w(B;f)S_w(B;g+f)|^{2^{k-1}} \ll P^{-(k-1)s} \cdot \#\{(\vec{h}_1,\dots,\vec{h}_{k-1}) \in \mathbb{Z}^{(k-1)s} : |\vec{h}_i| \leq P, |\vec{m}^{f,k}_{\vec{h}_1,\dots,\vec{h}_{k-1}}| \leq P^{k-2}|f^{[k]}|\}.$$

If $k=3$, $f^{[3]}$ smooth get $\ll P^{-s}$ by ideas of Davenport; else, pigeonhole.

Repulsion: suppose that, whenever $PB^{-k} \leq |\vec{\beta}\cdot\vec{f}^{[k]}| \leq P^{1-k}$, we have

$$S_w(B;\vec{\alpha}\cdot\vec{f})S_w\left(B;\left(\vec{\alpha}+\vec{\beta}\right)\cdot\vec{f}\right) \ll P^{-2E(k)},$$

then $\text{meas}\{\vec{\alpha}\in[0,1)^R : |S_w(B;\vec{\alpha}\cdot\vec{f})| \asymp B^{-A}\} \ll B^{\sum^R d_i(A/E(d_i)-1)}$ $(A>0)$

$\bigcirc$: If $\sum^R d_i/E(d_i) < 1$ then $\begin{cases} N_{\vec{f}}(B) \sim c_{\vec{f}} B^{s-\sum^R d_i} & \text{if } \vec{f}\in\mathbb{Z}[\vec{x}]^R, \text{ or} \\ M(B) \sim \nu_{\vec{f}} B^{s-\sum^R d_i} & \text{if } \vec{f} \text{ is irrational.} \end{cases}$

Setup for p-adic repulsion

- $e(t) = 2^{2\pi i t}$, $\Delta_{\vec{h}} f(\vec{x}) = f(\vec{x}+\vec{h}) - f(\vec{x})$, $\Delta_{\vec{h}_1,\ldots,\vec{h}_r} = \Delta_{\vec{h}_1}\cdots\Delta_{\vec{h}_r}$
- $f^{[k]}$ is the degree k part of f, with $|f^{[k]}| = |\text{biggest coefficient}|$.
- $\vec{m}^{f,k}_{\vec{h}_1,\ldots,\vec{h}_{k-1}}$ is the vector of coefficients of $\Delta_{\vec{h}_1,\ldots,\vec{h}_{k-1},(\,\cdot\,)} f^{[k]}$
- The *normalised* sum $S_w(B;f) = \sum_{\vec{x}\in\mathbb{Z}^s} B^{-s} w(\vec{x}/B) e\left(f(\vec{x})\right)$

For $f \in \mathbb{Z}[\vec{x}], g \in \mathbb{R}[\vec{x}]$ with degree $\leq k$, and P prime with $P^M \ll B^k$, such that $P^{1-M} \leq |f^{[k]}|_P \leq P^{1-k}$ we have

$$|S_w(B;f)S_w(B;g+\tfrac{1}{P^M}f)|^{2^{k-1}} \ll P^{-(k-1)s}$$
$$\cdot \#\{(\vec{h}_1,\ldots,\vec{h}_{k-1}) \in \mathbb{Z}^{(k-1)s} : |\vec{h}_i| \leq P, |\vec{m}^{f,k}_{\vec{h}_1,\ldots,\vec{h}_{k-1}}|_P \leq P^{-1}|f^{[k]}|_P\}.$$

For $f^{[k]}$ smooth this is $\ll P^{-s}$ ($\mathbb{F}_p$-points on varieties).

Repulsion: suppose that, whenever $P^{1-M} \leq |\vec{b}\cdot\vec{f}^{[k]}|_P \leq P^{1-k}$, we have

$$S_w(B;\vec{\alpha}\cdot\vec{f})S_w\left(B;(\vec{\alpha}+\tfrac{\vec{b}}{P^M})\cdot\vec{f}\right) \ll P^{-2E(k)},$$

then $\text{meas}\{\vec{\alpha}\in[0,1)^R : |S_w(B;\vec{\alpha}\cdot\vec{f})| \asymp B^{-A}\} \ll B^{\sum^R d_i(A/E(d_i)-1)}$ $(A>0)$

◯: If $\sum^R d_i/E(d_i) < 1$ then $N_{\vec{f}}(B) \sim cB^{s-\sum^R d_i}$. Smooth: $E(d) = \frac{n-R+1}{2^d}$.

Measures by repulsion

- $e(t) = 2^{2\pi it}$, $\Delta_{\vec{h}} f(\vec{x}) = f(\vec{x}+\vec{h}) - f(\vec{x})$, $\Delta_{\vec{h}_1,\dots,\vec{h}_r} = \Delta_{\vec{h}_1} \cdots \Delta_{\vec{h}_r}$
- $f^{[k]}$ is the degree k part of f, with $|f^{[k]}| = |\text{biggest coefficient}|$.
- $\vec{m}^{f,k}_{\vec{h}_1,\dots,\vec{h}_{k-1}}$ is the vector of coefficients of $\Delta_{\vec{h}_1,\dots,\vec{h}_{k-1},(\cdot)} f^{[k]}$
- The *normalised* sum $S_w(B; f) = \sum_{\vec{x}\in\mathbb{Z}^s} B^{-s} w(\vec{x}/B) e\left(f(\vec{x})\right)$
- $\vec{f}(\vec{x}) \in \mathbb{Z}[x_1,\dots,x_s]^R$ is a system of R forms in $s > \sum d_i$ variables.

Repulsion: suppose that, whenever $PB^{-k} \le |\vec{\beta}\cdot\vec{f}^{[k]}| \le P^{1-k}$, we have

$$S_w(B;\vec{\alpha}\cdot\vec{f}) S_w\left(B;(\vec{\alpha}+\vec{\beta})\cdot\vec{f}\right) \ll P^{-2E(k)},$$

then $\text{meas}\{\vec{\alpha}\in[0,1)^R : |S_w(B;\vec{\alpha}\cdot\vec{f})| \asymp B^{-A}\} \ll B^{\sum^R d_i(A/E(d_i)-1)}$ $(A>0)$

Proof idea: Let $P \gg B^{A/E(k)}$. Let $\vec{\alpha}, \vec{\alpha}+\frac{\vec{b}}{P^M}$ belong to the set. Then

$$|\vec{\beta}\cdot\vec{f}^{[k]}| < PB^{-k}, \text{ or } |\vec{\beta}\cdot\vec{f}^{[k]}| > P^{1-k}.$$

It follows that the $\vec{\alpha}\cdot\vec{f}^{[k]}$ are contained in a few infrequent regions (peaks) of diameter $\le PB^{-k}$, separated by gaps of size $\ge P^{1-k}$, hence with total measure $\le (P/B)^{kR_k}$ where there are R_k forms of degree k.

Measures by p-adic repulsion

- $e(t) = 2^{2\pi i t}$, $\Delta_{\vec{h}} f(\vec{x}) = f(\vec{x} + \vec{h}) - f(\vec{x})$, $\Delta_{\vec{h}_1,\dots,\vec{h}_r} = \Delta_{\vec{h}_1} \cdots \Delta_{\vec{h}_r}$
- $f^{[k]}$ is the degree k part of f, with $|f^{[k]}| = |\text{biggest coefficient}|$.
- $\vec{m}^{f,k}_{\vec{h}_1,\dots,\vec{h}_{k-1}}$ is the vector of coefficients of $\Delta_{\vec{h}_1,\dots,\vec{h}_{k-1},(\cdot)} f^{[k]}$
- The *normalised* sum $S_w(B; f) = \sum_{\vec{x}\in\mathbb{Z}^s} B^{-s} w(\vec{x}/B) e\left(f(\vec{x})\right)$
- $\vec{f}(\vec{x}) \in \mathbb{Z}[x_1, \dots, x_s]^R$ is a system of R forms in $s > \sum d_i$ variables.

Repulsion: suppose that, whenever $P^{1-M} \leq |\vec{b} \cdot \vec{f}^{[k]}|_P \leq P^{1-k}$, we have

$$S_w(B; \vec{\alpha} \cdot \vec{f}) S_w\left(B; (\vec{\alpha} + \tfrac{\vec{b}}{P^M})\right) \cdot \vec{f}) \ll P^{-2E(k)},$$

then $\text{meas}\{\vec{\alpha} \in [0,1)^R : |S_w(B; \vec{\alpha} \cdot \vec{f})| \asymp B^{-A}\} \ll B^{\sum^R d_i(A/E(d_i)-1)}$ $(A > 0)$

Proof idea: Let $P \gg B^{A/E(k)}$ prime, $P^M \ll B^k$. Let $\vec{\alpha}, \vec{\alpha} + \frac{\vec{b}}{P^M}$ belong to the set. Either $\vec{\beta} = 0$ or $\vec{\beta} \in P^{k-2-M}\mathbb{Z}$, because

$$|\vec{b} \cdot \vec{f}^{[k]}|_P < P^{1-M}, \text{ or } |\vec{b} \cdot \vec{f}^{[k]}|_P > P^{1-k}.$$

Hence each lattice $\vec{\alpha}_0 + \frac{1}{P^{k-1-M}}\mathbb{Z}^R$ contains at most one element of the set, hence it has total measure $\leq P^{(k-1-M)R_k}$ where there are R_k forms of degree k, this is $\leq (P/B)^{kR_k}$ if we choose M so $B^{-k} \gg P^{-M-1}$.

Accessing the lower-degree part

Notation

- $e(t) = 2^{2\pi it}$, $\Delta_{\vec{h}} f(\vec{x}) = f(\vec{x}+\vec{h}) - f(\vec{x})$, $\Delta_{\vec{h}_1,\ldots,\vec{h}_r} = \Delta_{\vec{h}_1} \cdots \Delta_{\vec{h}_r}$
- NB $\Delta_{\vec{h}_1,\ldots,\vec{h}_k} f^{[k]}$ is a multilinear form in the $\vec{h}_i$, independent of $\vec{x}$
- $\vec{m}^{f,k}_{\vec{h}_1,\ldots,\vec{h}_{k-1}}$ is the vector of coefficients of $\Delta_{\vec{h}_1,\ldots,\vec{h}_{k-1},(\,\cdot\,)} f^{[k]}$

Proposition

Given $w \in C_c^\infty(\mathbb{R}^s)$ there is $\tilde{w} \in C_c^\infty(\mathbb{R}^{ks})$ as follows. For any $f \in \mathbb{R}[\vec{x}]$ of degree $\leq d$, and any $1 < k < d$,

$$\left| \sum_{\vec{x}\in\mathbb{Z}^s} \frac{1}{B^s} w\left(\frac{\vec{x}}{B}\right) e\left(f(\vec{x})\right) \right|^{(2^{d-1}+1)\cdots(2^k+1)2^{k-1}3^{(k+1)(d-k-1)+1}}$$

$$\leq B^{-ks} \sum_{\vec{h}_1,\ldots,\vec{h}_{k-1}\in\mathbb{Z}^s} \left| \sum_{\vec{h}_k\in\mathbb{Z}^s} \tilde{w}\left(\frac{\vec{h}_1}{B},\ldots,\frac{\vec{h}_k}{B}\right) e\left(\Delta_{\vec{h}_1,\ldots,\vec{h}_k} f^{[k]}\right) \right|$$

$$\ll \frac{1}{B^{(k-1)s}} \#\{(\vec{h}_1,\ldots,\vec{h}_{k-1},\vec{u}) \in \mathbb{Z}^{ks} : |\vec{h}_i| \leq P, |\vec{m}^{f,k}_{\vec{h}_1,\ldots,\vec{h}_{k-1}} - \vec{u}| \leq \frac{1}{B}\}$$

Lemma

Given $w \in C_c^\infty(\mathbb{R}^s)$ there is $\tilde{w} \in C_c^\infty(\mathbb{R}^{(d+1)s})$ as follows. For any $f \in \mathbb{R}[\vec{x}]$ of degree $\leq d$,

$$\left|\sum_{\vec{x}\in\mathbb{Z}^s} \frac{1}{B^s} w\left(\frac{\vec{x}}{B}\right) e\left(f(\vec{x})\right)\right|^{2^{d-1}+1} \leq \left|\sum_{\vec{x}_1,\dots,\vec{x}_d} \sum_{\vec{x}} \tilde{w}\left(\frac{\vec{x}_d}{B},\dots,\frac{\vec{x}_1}{B},\frac{\vec{x}}{B}\right) e\left(f^{[<d]} + F\left(\vec{x};\vec{x}_1,\dots,\vec{x}_d\right)\right)\right|$$

with $F = f^{[d]} - \Delta_{\vec{x}+\vec{x}_1,\dots,\vec{x}+\vec{x}_d} f^{[d]}$ of degree $< d$ in $\vec{x}$ and ≤ 1 in each $\vec{x}_i$.

Lemma

For $L \in \mathrm{GL}_s(\mathbb{R})$, $A := \{\|\vec{x}\| \leq B : \|L\vec{x} - \vec{u}\| \leq 1/B \text{ some } \vec{u} \in \mathbb{Z}^s\}$. Given $w \in C_c^\infty(\mathbb{R}^{2s})$ there are $\tilde{w}_{\vec{c}}^{L,B} \in C_c^\infty(\mathbb{R}^s)$, whose values, support and derivatives are bounded in terms of w only, such that for all real g,

$$\left|\sum_{\vec{x},\vec{y}} w\left(\frac{\vec{x}}{B},\frac{\vec{y}}{B}\right) e\left(\vec{y}\cdot L\vec{x} + g(\vec{x})\right)\right|^3 \leq \sum_{\vec{c}\in A-A} \sum_{\vec{x}} \tilde{w}_{\vec{c}}^{L,B}\left(\frac{\vec{x}}{B}\right) e\left(\Delta_{\vec{c}} g(\vec{x})\right).$$