# Quantum state smoothing cannot be assumed classical

## even when the quantum filtering and retrofiltering equations are classical

Kiarn Laverick[1], Prahlad Warszawski, Areeya Chantasri[1,2], & Howard Wiseman[1]





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# Outline

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# Types of classical estimation

Consider estimating a variable $x$ at time $\tau$.

Many types of estimate are possible e.g. mean, mode, mode ....

The most powerful tool for this is the probability distribution $\wp(\xi) = \Pr(x = \xi)$. From this we can determine any type of estimate.




(Adapted from a diagram of Tsang, PRA 2009 [3])

**Filtering** (F):
Conditioning $\xi$ on past measurement record:

$$\wp_{\rm F}(\xi) \equiv \wp(\xi|\overleftarrow{O})$$

**Retrofiltering** (R):
Conditioning future record on $\xi$:

$$E_{\rm R}(\xi) \equiv \wp(\overrightarrow{O}|\xi)$$

Assume "Markov": $\wp(\overrightarrow{O}|\xi) = \wp(\overrightarrow{O}|\xi, \overleftarrow{O})$.

**Smoothing** (S):
Conditioning $\xi$ on entire record:

$$\wp_{\rm S}(\xi) \equiv \wp(\xi|\overleftrightarrow{O}) \propto \wp_{\rm F}(\xi)E_{\rm R}(\xi)$$

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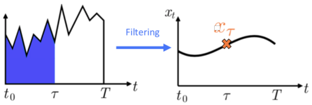


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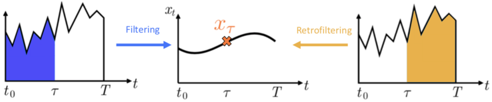


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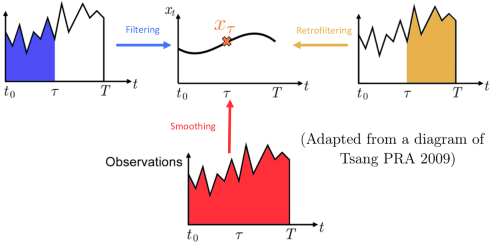


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# Quantum case: Two state (vector) formalism

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Y. Aharonov & D. Rohrlich (1990).

Filtering = pure state preparation at time $t_0 < \tau$:

$$\rho_{\rm F} = \hat{U}_{t_0}^{\tau} |\psi\rangle\langle\psi| \hat{U}_{t_0}^{\tau\,\dagger}.$$

Retrofiltering = pure state projection at time $T > \tau$:

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PRL **111,** 160401 (2013) PHYSICAL REVIEW LETTERS week ending 18 OCTOBER 2013

**Past Quantum States of a Monitored System**

Søren Gammelmark, Brian Julsgaard, and Klaus Mølmer*

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(Received 3 May 2013; revised manuscript received 22 August 2013; published 15 October 2013)

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$$\mathrm{Tr}\left[\hat{A}\rho_{\mathrm{S}}^{\mathrm{naive}}\right] = \frac{\langle\phi|\hat{U}_{\tau}^{T}\hat{A}\hat{U}_{t_0}^{T}|\psi\rangle}{\langle\phi|\hat{U}_{t_0}^{T}|\psi\rangle}$$

for the simple (2SVF) case where

$$\rho_{\mathrm{F}} = \hat{U}_{t_0}^{\tau}|\psi\rangle\langle\psi|\hat{U}_{t_0}^{\tau\,\dagger},\ \hat{E}_{\mathrm{R}} = \hat{U}_{\tau}^{T\,\dagger}|\phi\rangle\langle\phi|\hat{U}_{\tau}^{T}.$$

Naively, following the classical example,

$$\rho_{\mathrm{S}}^{\mathrm{naive}} = \hat{E}_{\mathrm{R}}\rho_{\mathrm{F}}/\mathrm{Tr}\,[\text{this}]$$

**However**, in general, this operator is not Hermitian, and, even if symmetrized, not positive. Nevertheless, as shown in

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# Application of Classical Smoothing to Quantum Systems

If $[\hat{E}_{\rm R}, \rho_{\rm F}] = 0$ then $\exists$ a basis $\{|\xi\rangle : \xi\}$: $\hat{E}_{\rm R} = \sum_\xi E^{\rm cl}_{\rm R}(\xi)|\xi\rangle\langle\xi|$ and $\rho_{\rm F} = \sum_\xi \wp^{\rm cl}_{\rm F}(\xi)|\xi\rangle\langle\xi|$, and

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PRL **103,** 173601 (2009) PHYSICAL REVIEW LETTERS week ending 23 OCTOBER 2009

**Spontaneous Dressed-State Polarization in the Strong Driving Regime of Cavity QED**

Michael A. Armen,[1,2] Anthony E. Miller,[1] and Hideo Mabuchi[1]
[1]*Edward L. Ginzton Laboratory, Stanford University, Stanford, California 94305, USA*
[2]*Physical Measurement and Control 266-33, California Institute of Technology, Pasadena, California 91125, USA*
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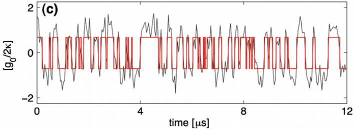

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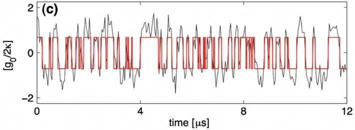


PHYSICAL REVIEW A **89**, 043839 (2014)

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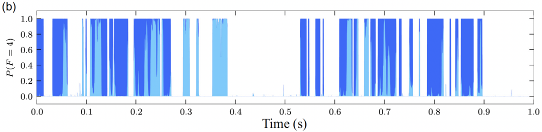

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PHYSICAL REVIEW A **91**, 062116 (2015)

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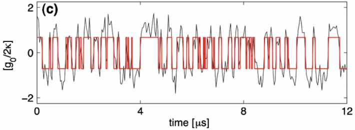


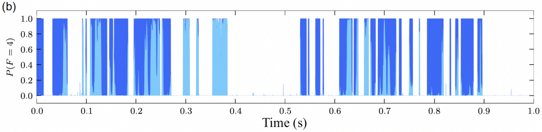


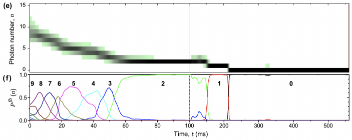

# Not the end of the story (in fact, only the beginning)

If $\hat{E}_{\rm R} = \sum_\xi \wp(\overrightarrow{O}|\xi)|\xi\rangle\langle\xi|$ and $\rho_{\rm F} = \sum_\xi \wp(\xi|\overleftarrow{O})|\xi\rangle\langle\xi|$, then

$$\rho_{\rm S}^{\rm cl} \propto \sum_\xi \wp(\overrightarrow{O}|\xi)\wp(\xi|\overleftarrow{O})|\xi\rangle\langle\xi|$$

is certainly **a** smoothed quantum state:

1. $\rho_{\rm S}$ is a single state, just as the classical theory gives $\wp_{\rm S}$, not a pair of states.
2. $\rho_{\rm S} \equiv \rho_{\overleftrightarrow{O}}$ such that $\int d\mu(\overrightarrow{O}|\rho_\tau = \rho_{\overleftarrow{O}}) \times \rho_{\overleftrightarrow{O}} = \rho_{\overleftarrow{O}} \equiv \rho_{\rm F}$.
3. $\rho_{\rm S}$ is a genuine state (positive and Hermitian).

But is it **the** smoothed quantum state under this condition? No!

- There is a more general way to define a smoothed quantum state $\rho_{\rm S}$, that satisfies Conditions 1–3 above, and is an optimal* estimate of the "true" quantum state.
- The more general $\rho_{\rm S}$ reduces to $\propto \sum_\xi \wp(\overrightarrow{O}|\xi)\wp(\xi|\overleftarrow{O})|\xi\rangle\langle\xi|$ only with an *extra assumption*: the 'true' quantum state is always an element of $\{|\xi\rangle : \xi\}$.

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3. $\rho_{\rm S}$ is a genuine state (positive and Hermitian).

But is it **the** smoothed quantum state under this condition? **No!**

- There is a more general way to define a smoothed quantum state $\rho_{\rm S}$, that satisfies Conditions **1–3** above, and is an optimal* estimate of the "true" quantum state.
- The more general $\rho_{\rm S}$ reduces to $\propto \sum_\xi \wp(\overrightarrow{O}|\xi)\wp(\xi|\overleftarrow{O})|\xi\rangle\langle\xi|$ only with an *extra assumption*: the 'true' quantum state is always an element of $\{|\xi\rangle : \xi\}$.

# Outline

# Background: Quantum Trajectory Theory = Quantum Filtering

- A master equation is derived by ignoring (tracing over) the bath.

$$\dot\rho(t) = \mathcal{L}\rho(t) \equiv -i[\hat H, \rho] + \sum_{\ell=1}^{L} \mathcal{D}[\hat c_\ell]\rho.$$

- It is not always appropriate to ignore the bath — under a strong Markov assumption, the bath can be measured continuously *without invalidating the ME on average*.
- This monitoring yields information about the system, so in any individual 'run' the conditioned system state $\rho_{\rm F}(t)$ will differ from the ME solution, and typically be purer.
- This $\rho_{\rm F}(t)$ is a function of the **past** measurement record and so evolves stochastically (*e.g.* quantum jumps or quantum diffusion).
- The ensemble of such "quantum trajectories" is an "unravelling" of the ME:

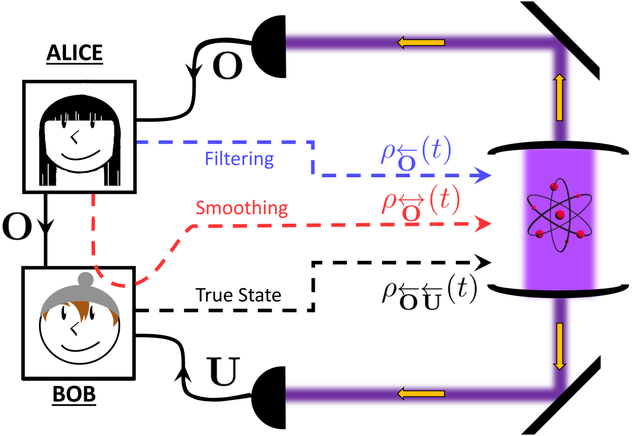
$$\mathrm{E}[\rho_{\rm F}(t)] = \rho(t) = \exp[\mathcal{L}(t-t_0)]\rho(t_0).$$

- Different ways of measuring the bath give different types of unravellings, for fixed $\mathcal{L}$.

PRL 115, 180407 (2015) PHYSICAL REVIEW LETTERS week ending 30 OCTOBER 2015

# Quantum State Smoothing

Ivonne Guevara and Howard Wiseman




$$\rho_{\mathrm{F}}(t) = \rho_{\overleftarrow{O}}(t) = \mathrm{E}_{\overleftarrow{U}|\overleftarrow{O}}[\rho_{\overleftarrow{O},\overleftarrow{U}}(t)].$$

$$\rho_{\mathrm{S}}(t) = \rho_{\overleftrightarrow{O}}(t) = \mathrm{E}_{\overleftarrow{U}|\overleftrightarrow{O}}[\rho_{\overleftarrow{O},\overleftarrow{U}}(t)].$$

is, on average, closer* than $\rho_{\mathrm{F}}(t)$ to

$$\rho_{\mathrm{T}} = \rho_{\overleftarrow{O},\overleftarrow{U}}(t).$$

# Partial Observation and Filtering

- Alice partially / imperfectly monitors (some of) the bath(s) to which the system is coupled, yielding a record $O$ (observed).
- Whatever (quantum) information Alice misses is seen by Bob, yielding a record $U^{\text{true}}$, unseen by Alice.
- Say for simplicity that Bob also knows Alice's record. Thus Bob's conditioned state is the 'true' state $\rho_{\text{T}}(t) = \rho_{\overleftarrow{O},\overleftarrow{U}^{\text{true}}}(t)$, which can be assumed *pure*.
- Alice wants to know the mind of Bob (*i.e.* know Bob's state) at all times $t$.
- If she uses only $\overleftarrow{O}$, she should* guess

$$\rho_{\text{F}}^{\text{Alice}}(t) = \int d\mu(\overleftarrow{U}|\overleftarrow{O}) \times \rho_{\overleftarrow{O},\overleftarrow{U}}(t), \text{ given } \rho_{\emptyset}(t_0).$$

- It turns out this is identical to Alice's usual filtered state $\rho_{\overleftarrow{O}}(t)$, and is *independent of how Bob monitors* (i.e. the type of unravelling he uses).

# Partial Observation and Smoothing

- Set up as before, but now Alice realises that to guess Bob's state at time $t$ she might do better to use $\overleftrightarrow{O}$. Now she should* guess

$$\rho_{\rm S}^{\rm Alice}(t) = \int d\mu(\overleftarrow{U}\,|\,\overleftrightarrow{O}) \times \rho_{\overleftarrow{O},\overleftarrow{U}}(t), \text{ given } \rho_{\emptyset}(t_0).$$

- On average it is a *better** estimate of $\rho_{\rm T}(t)$, and is more pure, and

$$\mathrm{E}[\mathrm{Purity}(\rho_{\rm C})] = \mathrm{E}[\mathrm{Fidelity}(\rho_{\rm C}, \rho_{\rm T})].$$

- In this case, $\rho_{\rm S}(t)$ **does** depend on how Bob monitors his bath(s).
- Note that $\rho_{\rm S}^{\rm Alice}(t) \neq \rho_{\rm F}^{\rm Alice}(t)$ if and only if Alice's measurement does not capture all the information, so that $\rho_{\rm F}^{\rm Alice}(t)$ is not pure. This is also the case for classical smoothing.

# Main Results

PRX QUANTUM **4**, 040340 (2023)

**Quantum State Smoothing Cannot Be Assumed Classical Even When the Filtering and Retrofiltering Are Classical**

Kiarn T. Laverick,[1,*] Prahlad Warszawski,[2] Areeya Chantasri,[1,3] and Howard M. Wiseman[1,4,†]

Say $\exists$ a basis $\{|\xi\rangle : \xi\}$ such that $\hat{E}_{\rm R} = \sum_\xi \wp(\overrightarrow{O}|\xi)|\xi\rangle\langle\xi|$ and $\rho_{\rm F} = \sum_\xi \wp(\xi|\overleftarrow{O})|\xi\rangle\langle\xi|$. Then:

0 If, in each run, $\exists\ \xi$: $\rho_{\rm T} = |\xi\rangle\langle\xi|$ then $\rho_{\rm S} = \rho_{\rm S}^{\rm cl} :\propto \sum_\xi \wp(\overrightarrow{O}|\xi)\wp(\xi|\overleftarrow{O})|\xi\rangle\langle\xi|$.

1 If this condition does not hold, then it can be that $\rho_{\rm S} \neq \rho_{\rm S}^{\rm cl}$.

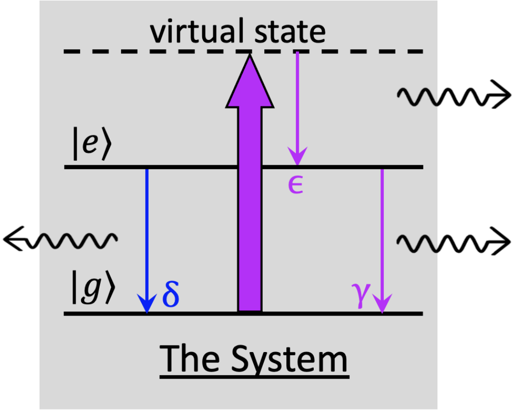
2 In fact, it can be that $\rho_{\rm S} \neq \sum_\xi \wp(\xi)|\xi\rangle\langle\xi|$ for any $\wp(\xi)$.

3 It is not even the case that the classical case (where, in each run, $\exists\ \xi$: $\rho_{\rm T} \propto |\xi\rangle\langle\xi|$ ) allows the *best** best* estimate of $\rho_{\rm T}$ at all times.

We show all of these results with a simple system, a qubit.

# Outline

# The Open Quantum System



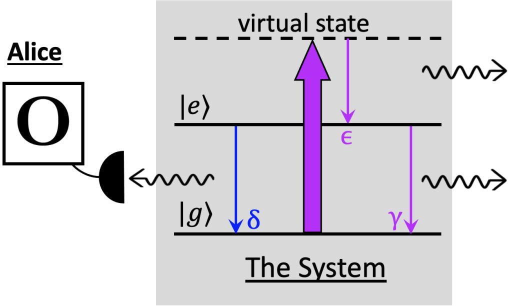

- Adiabatically eliminating the virtual level gives this Lindblad master equation:

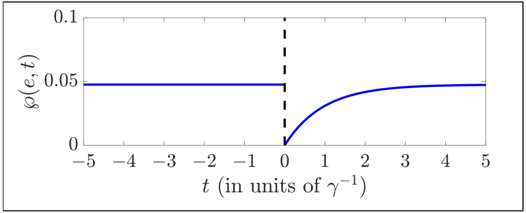
$$\dot{\rho} = (\delta + \gamma)\,\mathcal{D}[\hat{\sigma}_-]\rho + \epsilon\,\mathcal{D}[\hat{\sigma}_+]\rho.$$

- The left-going field goes to Alice, the right-going fields go to Bob.
- We will always consider the case $\delta \ll \epsilon = \frac{1}{20}\gamma$, so

$$\rho_{\rm ss} \approx \frac{20}{21}|g\rangle\langle g| + \frac{1}{21}|e\rangle\langle e|.$$

# Alice's Observation and Filtering



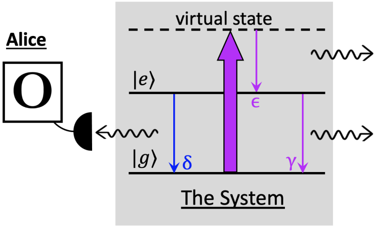

- We will always take Alice to perform photodetection (counting photons).
- Because $\delta \ll \epsilon = \frac{1}{20}\gamma$, Alice very rarely gets detections.
- We will consider an interval $[-5\gamma^{-1}, 5\gamma^{-1}]$ around a rare Alice-detection at $t = 0$.
- For $t < 0$ her filtered state is given by

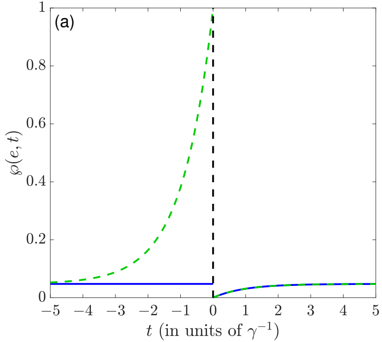
$$\rho_{\rm F} \approx \rho_{\rm ss} \approx \frac{20}{21}|g\rangle\langle g| + \frac{1}{21}|e\rangle\langle e|$$

and for $t \geq 0$ (following a detection):

$$\rho_{\rm F} \approx |g\rangle\langle g|{\rm e}^{-(\gamma+\epsilon)t} + \rho_{\rm ss}(1 - {\rm e}^{-(\gamma+\epsilon)t}).$$

# Alice's Naive (Classical) Smoothing ...




- Now take the limit $\delta \to 0^+$. Alice's filtered state is

$$\rho_F(t) = \rho_{ss} = \frac{20}{21}|g\rangle\langle g| + \frac{1}{21}|e\rangle\langle e| \qquad \text{for } t < 0$$

$$\rho_F(t) = |g\rangle\langle g|e^{-(\gamma+\epsilon)t} + \rho_{ss}(1 - e^{-(\gamma+\epsilon)t}) \qquad \text{for } t \geq 0$$

- Similarly, her *retrofiltered effect* is

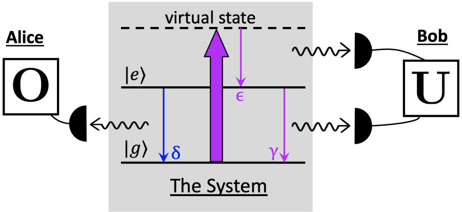
$$\hat{E}_R(t) \propto |e\rangle\langle e|e^{(\gamma+\epsilon)t} + (I/2)(1 - e^{(\gamma+\epsilon)t}) \qquad \text{for } t \leq 0$$

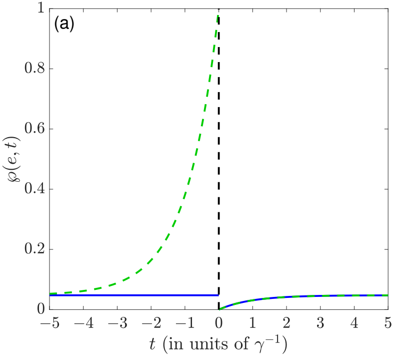
$$\hat{E}_R(t) \propto I/2 \qquad \text{for } t > 0$$

- Thus the naive smoothed state is also diagonal:

$$\rho_S^{naive}(t) = \hat{E}_R(t)\rho_F(t)/\mathrm{Tr}\,[\text{this}]$$
$$= \wp_S^{cl}(e,t)|e\rangle\langle e| + [1 - \wp_S^{cl}(e,t)]|g\rangle\langle g|.$$

# ... can be derived from QSS if Bob performs *Photodetection* (0)




- If Bob also counts photons then

$$\forall t,\ \rho_{\overleftarrow{O},\overleftarrow{U}} = |\psi_{\rm T}(t)\rangle\langle\psi_{\rm T}(t)| \in \{|e\rangle\langle e|, |g\rangle\langle g|\}\,.$$

- Alice's *knowledge* of the true state is thus described by $\wp^{\rm cl}_{O}(e,t)$, where $O$ is

$$\overleftarrow{O}(t) = \begin{cases} \text{"no click so far"} & \text{for } t < 0 \\ \text{"click at time zero"} & \text{for } t \geq 0 \end{cases}$$

$$\overleftrightarrow{O}(t) = \text{"click at time zero"}$$

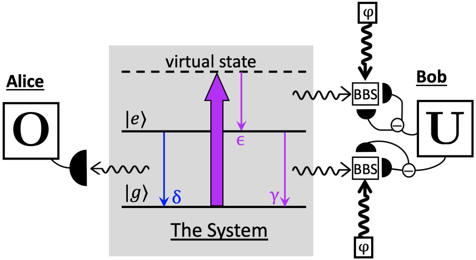
- It is easy to verify that in this case

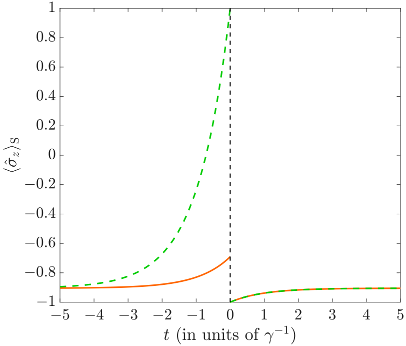
$$\rho_{\rm S}^{\rm Alice}(t) \equiv \int d\mu(\overleftarrow{U}|\overleftrightarrow{O}) \times \rho_{\overleftarrow{O},\overleftarrow{U}}(t)$$

$$= \wp^{\rm cl}_{\rm S}(e,t)|e\rangle\langle e| + [1 - \wp^{\rm cl}_{\rm S}(e,t)]|g\rangle\langle g|.$$

# Outline

# Now consider QSS if Bob performs *Homodyne detection* (1)




- If Bob does $\varphi = 0$ homodyne, then for $t < 0$

$$d|\tilde\psi_{\rm T}(t)\rangle = \Big[ -\frac{\gamma}{2}|e\rangle\langle e|dt - \frac{\epsilon}{2}|g\rangle\langle g|dt + \sqrt{\gamma}\hat\sigma_- dW_\gamma(t) + \sqrt{\epsilon}\hat\sigma_+ dW_\epsilon(t)\Big]|\tilde\psi_{\rm T}(t)\rangle,$$

  *i.e.*, quantum state diffusion.

- Now $|\tilde\psi_{\rm T}(t)\rangle \notin \{|e\rangle, |g\rangle\}$. Instead it, can be anywhere on the $y = 0$ great circle.
- Alice *knows* this, but her click only reveals information about $z$, not $x$, so

$$\rho_{\rm S}^{\rm Alice}(t) \equiv \int d\mu(\overleftarrow{U}|\overleftrightarrow{O}) \times \rho_{\overleftarrow{O},\overleftarrow{U}}(t)$$
$$= \wp_{\rm S}^{\mathbf{1}}(e,t)|e\rangle\langle e| + [1 - \wp_{\rm S}^{\mathbf{1}}(e,t)]|g\rangle\langle g|.$$

- But $\wp_{\rm S}^{\mathbf{1}}(e,t) \neq \wp_{\rm S}^{\mathbf{0}}(e,t) = \wp_{\rm S}^{\rm cl}(e,t)$.

# Result 1

## Theorem (1)

*The commutativity of the filtered quantum state and the retrofiltered quantum effect does* **not** *imply that the smoothed quantum state is given by their product:*

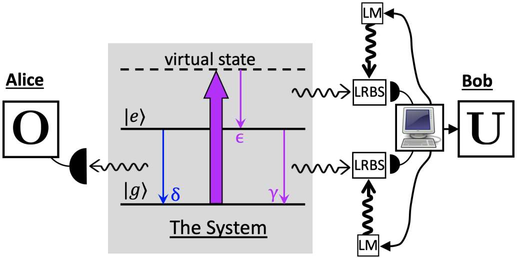
$$[\hat{E}_R, \rho_F] = 0 \;\not\Longrightarrow\; \rho_S \propto \hat{E}_R \rho_F.$$

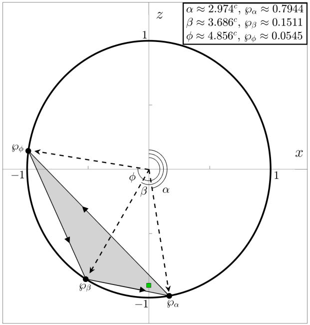
*Put another way, the existence of an orthonormal basis $\{|\xi\rangle : \xi\}$ such that the filtering and retrofiltering have classical descriptions does* **not** *imply that classical smoothing gives the smoothed quantum state:*

$$\rho_F = \sum_{\xi} \wp_F^{cl}(\xi)|\xi\rangle\langle\xi| \;\; \textit{and} \;\; \hat{E}_R = \sum_{\xi} E_R^{cl}(\xi)|\xi\rangle\langle\xi| \;\not\Longrightarrow\; \rho_S \propto \sum_{\xi} \wp_R^{cl}(\xi)\wp_F^{cl}(\xi)|\xi\rangle\langle\xi|.$$

# Outline

# Now Bob performs *Adaptive interferometric detection* ...




- Now Bob uses photodetection with two *weak* local oscillators, with amplitudes and phases set by *light modulators*, controlled by *feedback* from his past record of clicks $\overleftarrow{U}$. [Karasik & Wiseman, PRL (2011).]

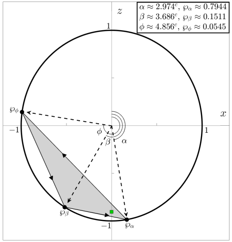
- With suitable feedback control, $|\tilde{\psi}_{\rm T}(t)\rangle$ is again confined to the $y = 0$ great circle, and more particularly, after transients,

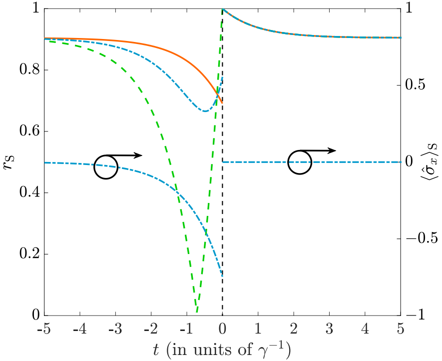
$$\forall t < 0, \ |\tilde{\psi}_{\rm T}(t)\rangle \in \{|\alpha\rangle, |\beta\rangle, |\phi\rangle\},$$

  jumping cyclically between these three states whenever Bob gets a click. [Warszawski & Wiseman, NJP (2019).]

- This is not true for $t \geq 0$, but that's transient ($t \lesssim \gamma^{-1}$) and not relevant for smoothing.

# ... giving rise to a non-diagonal Smoothed State




- Just as with scheme **1** (homodyne detection), the true state $\rho_{\overleftarrow{O},\overleftarrow{U}}(t)$ has both $x$ and $z$ components, but Alice's click only reveals information about $z$.
- But now, with scheme **2** (this particular adaptive detection), the *sign* of $x$ in the true state is *correlated* with $z$.
- Thus the smoothed state

$$\rho_{\mathrm{S}}^{\mathrm{Alice}}(t) \equiv \int d\mu(\overleftarrow{U}|\overleftrightarrow{O}) \times \rho_{\overleftarrow{O},\overleftarrow{U}}(t)$$

is **not diagonal** in the $\{|e\rangle, |g\rangle\}$ basis.

# Result 2

**Theorem (2)**

*The commutativity of the filtered quantum state and the retrofiltered quantum effect does* **not** *imply that the smoothed quantum state commutes with them:*

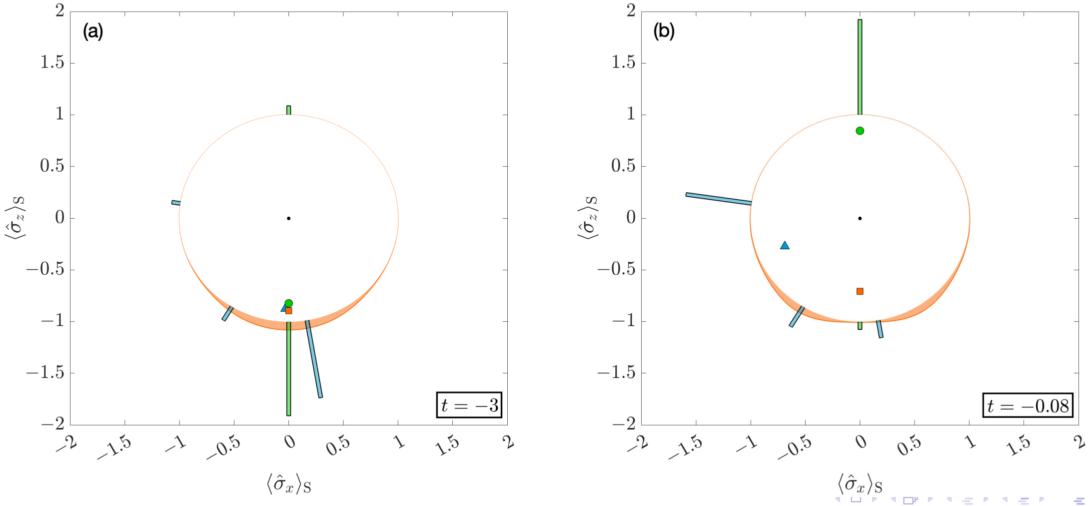
$$[\hat{E}_R, \rho_F] = 0 \;\not\Longrightarrow\; [\rho_S, \hat{E}_R] = [\rho_S, \rho_F] = 0.$$

*Put another way, the existence of an orthonormal basis* $\{|\xi\rangle : \xi\}$ *such that the filtering and retrofiltering have classical descriptions does* **not** *imply that the smoothed quantum state is diagonal in the same basis:*

$$\rho_F = \sum_\xi \wp_F^{cl}(\xi)|\xi\rangle\langle\xi| \;\; \textit{and} \;\; \hat{E}_R = \sum_\xi \wp_R^{cl}(\xi)|\xi\rangle\langle\xi| \;\not\Longrightarrow\; \rho_S = \sum_\xi \wp(\xi)|\xi\rangle\langle\xi|.$$

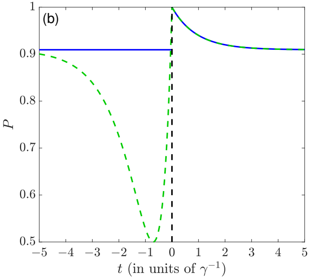
$$\text{For } t<0,\ \rho_{\rm S}^{\rm Alice}(t) = \int d\mu(\overleftarrow{U}\,|\,\overleftrightarrow{O}) \times \rho_{\overleftarrow{O},\overleftarrow{U}}(t) \propto \int d\mu_{\rm ss}^{\mathbf{M}}(|\psi\rangle)\, \langle\psi|\hat{E}_{\rm R}(t)|\psi\rangle\, |\psi\rangle\langle\psi|$$

where $\text{Supp}(d\mu_{\rm ss}^{\mathbf{0}}) = \{|e\rangle, |g\rangle\}$, $\text{Supp}(d\mu_{\rm ss}^{\mathbf{1}})$ = pure rebit manifold, $\text{Supp}(d\mu_{\rm ss}^{\mathbf{2}}) = \{|\alpha\rangle, |\beta\rangle, |\phi\rangle\}$.



# Outline

# Return to case 0 — Bob performs *photodetection*




- Recall, for $t < 0$, Alice's filtered state $= \rho_{\rm ss}$, while her smoothed state goes smoothly from $\rho_{\rm ss}$ to $|e\rangle\langle e|$.
- Hence, $P$urity$[\rho_{\rm F}(t)] > P$urity$[\rho_{\rm S}^{\mathbf{0}}(t)]$, even though [Chantasri & *al.*, Phys. Rep. (2021)], for $O = \overleftarrow{O}$ or $\overleftrightarrow{O}$,

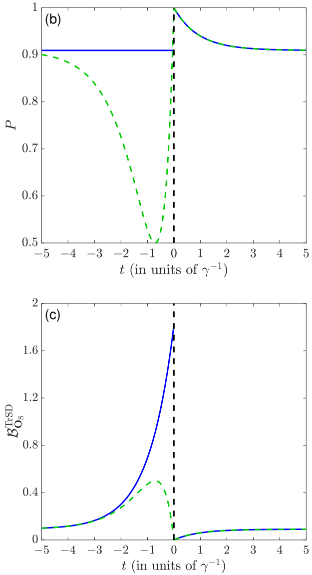
$$P[\rho_O] = {\rm E}_{\overleftarrow{U}|O}\left[{\rm Fidelity}(\rho_O, \rho_{\overleftarrow{O},\overleftarrow{U}})\right]$$

- Is the smoothed state a worse estimate?!
- No, because the ***cost function which all these estimates minimize*** is not the infidelity, but

$$\mathcal{B}_O^{\rm TrSD} := {\rm E}_{\overleftarrow{U}|O}{\rm Tr}\left[(\rho_O - \rho_{\overleftarrow{O},\overleftarrow{U}})^2\right].$$

- As expected, $\mathcal{B}^{\rm TrSD}_{\overleftrightarrow{O}} < \mathcal{B}^{\rm TrSD}_{\overleftarrow{O}}$, here for $\overleftarrow{U}$ arising from Bob's photodetection.

# Return to case 0 — Bob performs *photodetection*




- Recall, for $t < 0$, Alice's filtered state $= \rho_{\rm ss}$, while her smoothed state goes smoothly from $\rho_{\rm ss}$ to $|e\rangle\langle e|$.
- Hence, $P$urity$[\rho_{\rm F}(t)] > P$urity$[\rho^{\bf 0}_{\rm S}(t)]$, even though [Chantasri & *al.*, Phys. Rep. (2021)], for $O = \overleftarrow{O}$ or $\overleftrightarrow{O}$,

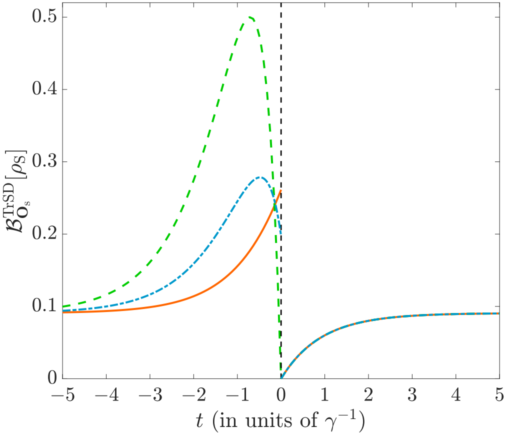
$$P[\rho_O] = \mathrm{E}_{\overleftarrow{U}|O}\left[\mathrm{Fidelity}(\rho_O, \rho_{\overleftarrow{O},\overleftarrow{U}})\right]$$

- Is the smoothed state a worse estimate?!
- No, because the ***cost function which all these estimates minimize*** is not the infidelity, but

$$\mathcal{B}^{\mathrm{TrSD}}_O := \mathrm{E}_{\overleftarrow{U}|O}\mathrm{Tr}\left[(\rho_O - \rho_{\overleftarrow{O},\overleftarrow{U}})^2\right].$$

- As expected, $\mathcal{B}^{\mathrm{TrSD}}_{\overleftrightarrow{O}} < \mathcal{B}^{\mathrm{TrSD}}_{\overleftarrow{O}}$, here for $\overleftarrow{U}$ arising from Bob's photodetection.

# Result 3: Comparing Costs for the Smoothed State




- In all cases, $\rho_{\mathrm{S}}$ are **optimal Bayesian estimates** in that they minimize

$$\mathcal{B}^{\mathrm{TrSD}}_{\overleftrightarrow{O}} := \mathrm{E}_{\overleftarrow{U}|\overleftrightarrow{O}} Tr \left[ (\rho_{\mathrm{S}} - \rho_{\overleftarrow{O},\overleftarrow{U}})^2 \right] .$$

- They differ because of the different nature of $\rho_{\overleftarrow{O},\overleftarrow{U}}$, under different measurement schemes for Bob, *even though this doesn't affect* $\rho_{\mathrm{F}}$ *or* $\hat{E}_{\mathrm{R}}$.
- One might think the most classical, photodetection, where $\rho_{\mathrm{S}}^{\mathbf{0}} \propto \hat{E}_{\mathrm{R}}\rho_{\mathrm{F}}$, would have the lowest expected cost.
- In fact, for most of the time, $\mathcal{B}^{\mathrm{TrSD},\mathbf{0}}_{\overleftrightarrow{O}}$ is higher than for homodyne and adaptive.

# Outline

# Summary

- Classically, there is no great conceptual difference between states obtained by filtering $\wp_{\rm F}(\xi;t) = \wp_{\overleftarrow{O}}(\xi;t)$ and smoothing $\wp_{\rm S}(\xi;t) = \wp_{\overleftrightarrow{O}}(\xi;t)$.
- The latter is just $\wp_{\rm F}(\xi;t)$ times the retrofiltered "effect": $\wp_{\rm S}(\xi;t) \propto \wp_{\rm F}(\xi;t)\wp(\overrightarrow{O}|\xi;t)$.
- The QM the analogues are the usual conditioned quantum state $\rho_{\overleftarrow{O}}$ and effect $E_{\overrightarrow{O}}$.
- But in QM, the obvious analogue of smoothing does not work when $[\rho_{\overleftarrow{O}}, E_{\overrightarrow{O}}] \neq 0$.
- $\rho_{\rm S} \propto E_{\overrightarrow{O}}\rho_{\overleftarrow{O}}$ does "work" when $[\rho_{\overleftarrow{O}}, E_{\overrightarrow{O}}] = 0$ ...
    0. and it can be derived from Quantum State Smoothing theory $\rho_{\rm S}(t) = {\rm E}_{\overleftarrow{U}|\overleftrightarrow{O}}[\rho_{\overleftarrow{O},\overleftarrow{U}}(t)]$ when the true state $\rho_{\overleftarrow{O},\overleftarrow{U}}$ is pure and commutes with $E_{\overrightarrow{O}}$ and $\rho_{\overleftarrow{O}}$.
    1. However, if $\rho_{\overleftarrow{O},\overleftarrow{U}}$ doesn't commute with $E_{\overrightarrow{O}}$ and $\rho_{\overleftarrow{O}}$ then $\rho_{\rm S}(t) \not\propto E_{\overrightarrow{O}}\rho_{\overleftarrow{O}}$,
    2. and in fact $\rho_{\rm S}$ need not even by co-diagonal with $E_{\overrightarrow{O}}$ and $\rho_{\overleftarrow{O}}$.
    3. Moreover, the commuting-$\rho_{\overleftarrow{O},\overleftarrow{U}}$ case is not even best for minimizing the optimality- defining cost function, the trace-mean-square-deviation of $\rho_{\rm S}$ from the true state $\rho_{\overleftarrow{O},\overleftarrow{U}}$.

# Some Other Past and Future Work on Quantum State Smoothing


- Kiarn T. Laverick, Areeya Chantasri, and Howard M. Wiseman, *Quantum State Smoothing for Linear Gaussian Systems* **Phys. Rev. Lett.** (2019).
- Areeya Chantasri, Ivonne Guevara, and Howard M. Wiseman, *Quantum state smoothing: Why the types of observed and unobserved measurements matter* **New J. Phys.** (2019).
- Kiarn T. Laverick, Areeya Chantasri, and Howard M. Wiseman, *General criteria for quantum state smoothing* ... **Quantum Stud.: Math. Found.** (2020).
- Kiarn T. Laverick, Areeya Chantasri, and Howard M. Wiseman, *Linear Gaussian quantum state smoothing: Understanding the optimal unravelings for Alice to estimate Bob's state* **Phys. Rev. A** (2021).
- Areeya Chantasri, Ivonne Guevara, Kiarn T. Laverick, and Howard M. Wiseman, *Unifying theory of quantum state estimation using past and future information* **Physics Reports** (2021).
- Kiarn T. Laverick, Ivonne Guevara, and Howard M. Wiseman, *Quantum state smoothing as an optimal Bayesian estimation problem with three different cost functions* **Phys. Rev. A** (2021).

- In various stages of preparation: 2 experimental papers, 4 theory papers.

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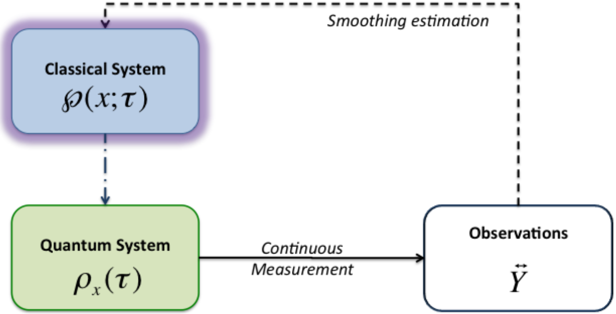
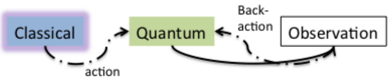
# Q. (Parameter) Smoothing [Tsang, PRL (2009)]

[adapted from a diagram of Tsang, PRA 2009.]




Tsang, PRL (2009)

# Applications of this Quantum Smoothing

PRL **104,** 093601 (2010) PHYSICAL REVIEW LETTERS week ending 5 MARCH 2010

**Adaptive Optical Phase Estimation Using Time-Symmetric Quantum Smoothing**

T. A. Wheatley,[1,2,3] D. W. Berry,[4] H. Yonezawa,[3] D. Nakane,[3] H. Arao,[3] D. T. Pope,[5] T. C. Ralph,[1,6,*] H. M. Wiseman,[1,7,†] A. Furusawa,[3,‡] and E. H. Huntington[1,2,§]

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**Fundamental Quantum Limit to Waveform Estimation**

Mankei Tsang,[1,*] Howard M. Wiseman,[2] and Carlton M. Caves[1]

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**Quantum-Enhanced Optical-Phase Tracking**

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# $\nu$: Bayesian State Estimation Revisited

- Recall that, given a set of data $\mathbf{Y}$, the *Bayesian* state is

$$\wp(\mathbf{x}) = \wp_{\mathbf{Y}}^{\text{Bayes}}(\mathbf{x}) \propto P(\mathbf{Y}^{\text{true}} = \mathbf{Y}|\mathbf{x}^{\text{true}} = \mathbf{x})\wp_{\emptyset}(\mathbf{x}).$$

- Why this?

0 $\wp_{\mathbf{Y}}^{\text{Bayes}}(\mathbf{x}) = P(\mathbf{x}^{\text{true}} = \mathbf{x}|\mathbf{Y})$.

1 to *predict* any property $\Lambda(\mathbf{x})$, with minimum Mean-Square-Error (mMSE). That is,

$$\Lambda_{\text{est}} = \sum_{\mathbf{x}} \wp_{\mathbf{Y}}^{\text{Bayes}}(\mathbf{x})\Lambda(\mathbf{x}) \text{ minimizes } R_{\Lambda} = \sum_{\mathbf{x}} P(\mathbf{x}^{\text{true}} = \mathbf{x}|\mathbf{Y})[\Lambda_{\text{est}} - \Lambda(\mathbf{x})]^2.$$

2 to *estimate*, with mM$\sum$SE, the *true state* $\wp^{\text{true}}(\mathbf{x}) = \delta(\mathbf{x}, \mathbf{x}^{\text{true}})$. That is,

$$\wp = \wp_{\mathbf{Y}}^{\text{Bayes}} \text{ minimizes } R(\wp) = \sum_{\mathbf{x}} P(\mathbf{x}^{\text{true}} = \mathbf{x}|\mathbf{Y}) \sum_{\mathbf{x}'} [\wp(\mathbf{x}') - \wp^{\text{true}}(\mathbf{x}')]^2.$$

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# $\nu$: Quantum State Filtering Revisited

- Recall: if Alice wants to guess Bob's state at all times $\tau$, from $\overleftarrow{O}$, she *should* guess

$$\rho = \rho^{\text{Bayes}}_{\overleftarrow{O}}(\tau) \equiv \sum_{\overleftarrow{U}} P(\overleftarrow{U} = \overleftarrow{U}^{\text{true}}|\overleftarrow{O}) \times \rho_{\overleftarrow{O},\overleftarrow{U}}(\tau), \text{ given } \rho_{\emptyset}(t_0).$$

- But why *should* she do this?

0 $\rho^{\text{Bayes}}_{\overleftarrow{O}}(\tau) = \rho^{\text{Alice}}_{\text{F}}(\tau)$ from quantum measurement theory.

1 To *predict* the minimum Mean-Square-Error (mMSE) value of a measurement of any observable $\hat{\Lambda}(\tau+)$, as $\text{Tr}[\rho_{\overleftarrow{O}}(\tau)\hat{\Lambda}]$.

2 To *estimate*, with mMTrSE, the true state (Bob's state), $\rho^{\text{true}}(\tau) = \rho_{\overleftarrow{O},\overleftarrow{U}^{\text{true}}}(\tau)$. That is,

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