On the pseudonullity of Fine Selmer groups over function fields

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1. Iwasawa's $\mu = 0$ conjecture.
2. Fine Selmer Group.
3. Pseudonullity.
4. Conjectures A and B over function fields ($\ell \neq p$ and $\ell = p$).

- p: an odd prime.
- $\mathbb{Z}_p$: the ring of p-adic integers and $\mathbb{Q}_p$: quotient field of $\mathbb{Z}_p$.
- R: ring, M: left R module.
  $M[p^r] := \{m \in M | p^r m = 0\}, M(p) := \bigcup_{r \geq 1} M[p^r]$.
- K: a number field.
- $Cl(K)$: the ideal class group.

- ζ_{p^n}: primitive p^n-th root of unity in $\bar{\mathbb{Q}}$.
- Let $\mathbb{Q}(\zeta_{p^\infty}) := \bigcup_{n\geq 1} \mathbb{Q}(\zeta_{p^n})$.
- $\mathbb{Q}_{\mathrm{cyc}}$: $\mathbb{Q}_{\mathrm{cyc}} \subset \mathbb{Q}(\zeta_{p^\infty})$ s.t. $\Gamma := \mathrm{Gal}(\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}) \cong \mathbb{Z}_p$ (cyclotomic $\mathbb{Z}_p$ extension).
- Set $K_{\mathrm{cyc}} := K\mathbb{Q}_{\mathrm{cyc}}$. Then $\mathrm{Gal}(K_{\mathrm{cyc}}/K) \cong \mathbb{Z}_p$ and $\exists\ K \subset K_n \subset K_{\mathrm{cyc}}$ with $\mathrm{Gal}(K_n/K) \cong \mathbb{Z}/p^n\mathbb{Z}$.

Theorem (Iwasawa (1959))

There exists $\lambda, \mu, \nu \geq 0$ such that $|Cl(K_n)(p)| = p^{n\lambda + \mu p^n + \nu}$ for $n >> 0$.

- Iwasawa further conjectured that $\mu = 0$.
- Conjecture $\mu = 0$ holds if $\mathrm{Gal}(K/\mathbb{Q})$ is abelian. (Ferrero-Washington 1979, Sinnot 1984)
- Open for a general number field.

- Let E/K be an elliptic curve over a field K.

Theorem (Mordell-Weil)

Let K be a number field. Then, $E(K)$ is a fin. gen. abelian group.

- The n-Selmer group of an elliptic curve E/K is defined as:

$$S_n(E/K) := \ker(H^1(G_K, E(\bar{K})[n]) \longrightarrow \prod_{\text{all prime v}} H^1(G_{K_v}, E(\bar{K}_v))).$$

- In fact, we have an exact sequence:

$$0 \longrightarrow E(K)/nE(K) \longrightarrow S_n(E/K) \longrightarrow Ш(E/K)[n] \longrightarrow 0$$

where $Ш(E/K) := \ker(H^1(G_K, E(\bar{K})) \longrightarrow \prod_v H^1(G_{K_v}, E(\bar{K}_v)))$.

- $S(E/K) := S_{p^\infty}(E/K) := \varinjlim_n S_{p^n}(E/K)$.
- $S-$ finite set of places of K containing primes above p, the infinite places and the primes of bad reduction of E/K.
- K_S- maximal algebraic extension of K unramified outside S.
- $$S(E/K) \cong \ker(H^1(K_S/K, E_{p^\infty}) \longrightarrow \bigoplus_{v \in S} H^1(K_v, E)).$$

Fine Selmer group

$$R(E/K) := R_{p^\infty}(E/K) := \ker(H^1(K_S/K, E_{p^\infty}) \longrightarrow \bigoplus_{v \in S} H^1(K_v, E_{p^\infty}))$$
$$\cong \ker(S_{p^\infty}(E/K) \longrightarrow \bigoplus_{v \mid p} E(K_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p).$$

- $R(E/K_{\mathrm{cyc}}) := \varinjlim_n R(E/K_n)$, where $K \subset K_n \subset K_{\mathrm{cyc}}$.

Conjecture A: (Coates-Sujatha (2005))

$R(E/K_{\mathrm{cyc}})^{\vee} := \mathrm{Hom}_{\mathsf{cont}}(R(E/K_{\mathrm{cyc}}), \mathbb{Q}_p/\mathbb{Z}_p)$ is a finitely generated $\mathbb{Z}_p$ module.

Theorem (Coates-Sujatha (2005))

Let E/K be an elliptic curve. Assume $K(E[p])/K$ is a finite p-extension. Then, $\mu = 0$ holds for K_{cyc} if and only if Conjecture A holds for $R(E/K_{\mathrm{cyc}})^{\vee}$.

- For example this theorem holds if $E[p](\bar{\mathbb{Q}}) \subset K$.

- G: a compact p-adic Lie group without any element of order p, with a closed normal subgroup H s.t. $\Gamma := G/H \cong \mathbb{Z}_p$.
- Define the Iwasawa algebra of G over $\mathbb{Z}_p$ as

$$\mathbb{Z}_p[[G]] := \varprojlim_{U} \mathbb{Z}_p[G/U],$$

  where U varies over open normal subgroups of G.
- If M is a fin. gen. $\mathbb{Z}_p[[G]]$-module, then M is said to be a pseudo-null $\mathbb{Z}_p[[G]]$-module if $\mathrm{Ext}^i_{\mathbb{Z}_p[[G]]}(M, \mathbb{Z}_p[[G]]) = 0$ for $i = 0, 1$. (Venjakob 2003)

Conjecture B (Coates-Sujatha, 2005)

Assume that the Conjecture A holds for E over F_{cyc}. Let F_∞ be an *admissible* p-adic Lie extension of F such that $G = \mathrm{Gal}(F_\infty/F)$ has dimension at least 2 as a p-adic Lie group. Then $R(E/F_\infty)^\vee$ is a pseudonull $\mathbb{Z}_p[[G]]$-module.

- Fix an integer prime $\ell \neq p$.
- $\mathbb{F}$: a finite field of char p. Set $K = \mathbb{F}(t)$.
- Let $K_\infty := K\mathbb{F}^{(\ell)}$, where $\mathbb{F}^{(\ell)} \subset \overline{\mathbb{F}}$ be the unique (unramified) $\mathbb{Z}_\ell$ extension of $\mathbb{F}$.
- The ℓ^∞-fine Selmer group of E/K is defined as:

$$R(E/K) := \ker(H^1(K_S/K, E_{\ell^\infty}) \to \bigoplus_{v \in S} H^1(K_v, E_{\ell^\infty}))$$

- In the function field case $\ell \neq p$, the analogue of Conjecture A is known [Witte, 2020].
- We give an explicit example using results of [A.Bandini, M.Valentino, (2014)] to show that the analogue of Conjecture B does not hold [G-Jha-Shekhar (2022), Preprint].

- $K = \mathbb{F}(t)$, a function field of char p.
- S- the set of places of K containing all the bad primes of bad reduction of E.
- The S-fine Selmer group of E/K is defined as:

$$R^S(E/K) := \ker\left(H^1_{\mathrm{fl}}(K, E_{p^\infty}) \longrightarrow \prod_{v \in S}(H^1_{fl}(K_v, E_{p^\infty}))\prod_{v \notin S}(H^1_{fl}(K_v, E)[p^\infty])\right).$$

- We consider two $\mathbb{Z}_p$ extensions over K:
  K_∞: the unramified $\mathbb{Z}_p$ extension over K and
  K'_∞: $\mathbb{Z}_p$ extension constructed from Carlitz module.
- We showed that the analogue of Conjecture A is true for K_∞ and K'_∞ [G-Jha-Shekhar (2022), Preprint].

- Corresponding to each prime $\mathfrak{P}$ of K, we have a $\mathbb{Z}_p^d$-extension, $K_d^{\mathfrak{P}}$ for $d \geq 1$.

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Let $F_\infty = K_\infty K_d^{\mathfrak{P}}$. Let ν_r be the unique prime of K that ramifies in F_∞. Assume E/K be an ordinary elliptic curve that has good reduction outside ν_r.
Then $R^S(E/F_\infty)^\vee$ is a pseudonull $\mathbb{Z}_p[[G]]$-module, where $G = \mathrm{Gal}(F_\infty/F)$ and S is any set of primes of K containing the prime ν_r.

S.Ghosh, S. Jha, and S. Shekhar, Iwasawa theory of fine Selmer groups over global fields, http://arxiv.org/abs/2201.01751.

S.Ghosh, Some cases in Pseudonullity of Fine Selmer groups over global fields, https://arxiv.org/abs/2201.01751.

Thank You.